Automatica 66 (2016) 39–47

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Stabilization with guaranteed safety using Control Lyapunov–Barrier Function[☆]

Muhammad Zakiyullah Romdlony[a,b,1], Bayu Jayawardhana[a]

[a] Engineering and Technology Institute Groningen, Faculty of Mathematics and Natural Science, University of Groningen, The Netherlands

[b] School of Electrical Engineering, Telkom University, Indonesia

ARTICLE INFO

Article history:
Received 22 December 2014
Received in revised form
24 July 2015
Accepted 23 November 2015
Available online 15 January 2016

Keywords:
Stabilization with guaranteed safety
Control Lyapunov Function
Control Barrier Function

ABSTRACT

We propose a novel nonlinear control method for solving the problem of stabilization with guaranteed safety for nonlinear systems. The design is based on the merging of the well-known Control Lyapunov Function (CLF) and the recent concept of Control Barrier Function (CBF). The proposed control method allows us to combine the design of a stabilizing feedback law based on CLF and the design of safety control based on CBF(s); both of which can be designed independently. Our proposed approach can also accommodate the case of multiple CBFs which correspond to multiple different sets of unsafe states. Lastly, the efficacy of the proposed approach is demonstrated in the simulation results on the stabilization of a nonlinear mechanical system and on the navigation of a mobile robot.

1. Introduction

With the recent surge of research interests in cyber-physical systems and in networked control systems, the safety of the process has become an integral part of the control design. Also, for safety-critical systems, such as, autonomous vehicles, chemical plant and robotic systems, where both human operator and the process itself might be at risk whenever certain unsafe states are reached, it is imperative to avoid unsafe states while controlling them. Consequently the design of feedback stabilizing controller must comply with state constraints, avoid unsafe states and adhere to input constraints.

There have been several control design methods proposed in the literature that deal with (non-)linear constraints for (non-)linear systems. For example, Model Predictive Control-based approach has been proposed in Bemporad, Borelli, and Morari (2002), Maciejowski (2002), Morari and Lee (1999) and the use of reference governor has been proposed in Bemporad (1998), Bemporad, Casavola, and Mosca (1997) and Gilbert and Kolmanovsky (2001). Both approaches lead to a high-level controller that generates admissible reference signals for the low-level controller, in order to avoid violating the constraints. Another control design approach for dealing with constraint is the invariance control principle proposed in Gilbert and Tan (1991) and Wolff and Buss (2005). An implicit assumption in these works is that time-scale separation can be applied to the stabilization (fast-time scale) and to the safety control (slow-time scale), i.e., safety is not considered as a time-critical issue. In this paper, we investigate the case where safety control is time-critical and propose a nonlinear control design that simultaneously stabilize the closed-loop systems and guarantee the safety of the systems.

One of the modern control design tools for the stabilization of affine nonlinear systems is the so-called Control Lyapunov Function (CLF) method. Artstein in Artstein (1983) has given necessary and sufficient conditions for the existence of such CLF, which has been used to design a universal control law for affine nonlinear systems in Sontag (1989). Recently, various Lyapunov-based control designs have been proposed using the same principle as CLF, such as, Passivity Based Control (Ortega, Loría, Nicklasson, & Sira-Ramírez, 1998; Ortega, van der Schaft, Maschke, & Escobar, 2002), backstepping (Krstic, Kanellakopoulos, & Kokotovic, 1995), stabilization via forwarding (Praly, Ortega, & Kaliora, 2001), and contraction-based method (Andrieu, Jayawardhana, & Praly, 2013).

Since CLFs can be designed to meet specific performance criteria, such as, optimality, transient behavior or robustness

[☆] The work of M.Z. Romdlony is supported by the Indonesian DIKTI scholarship program and the work of B. Jayawardhana is supported by the EU Interreg IV SmartBot project. The material in this paper was partially presented at the 53rd IEEE Conference on Decision and Control, December 15–17, 2014, Los Angeles, CA, USA. This paper was recommended for publication in revised form by Associate Editor Hiroshi Ito under the direction of Editor Andrew R. Teel.

E-mail addresses: m.z.romdlony@rug.nl, zakiyullah@telkomuniversity.ac.id (M.Z. Romdlony), b.jayawardhana@rug.nl (B. Jayawardhana).

[1] Tel.: +31 503633436.

http://dx.doi.org/10.1016/j.automatica.2015.12.011
0005-1098/

properties, the question on how to combine several CLFs for mixed performance objectives has been addressed, to name a few, in Andrieu and Prieur (2010), Balestrino, Crisostomi, and Caiti (2011), Clarke (2011), Grammatico, Blanchini, and Caiti (2014), Prieur (2001) and Prieur and Praly (1999). With the exception of combining/merging/uniting CLF approach proposed in Clarke (2011) that results in a non-smooth CLF, the synthesis of the combined (or merged) CLF is generally achieved by a convex combination of two CLFs where the weights can be state-dependent.

Akin to the CLF method, Wieland and Allgöwer in Wieland and Allgöwer (2007) have proposed the construction of Control Barrier Functions (CBF), where the Lyapunov function is interchanged with the Barrier certificate studied in Prajna (2005) and Prajna and Jadbabaie (2004). Using a CBF as in Wieland and Allgöwer (2007), one can design a universal feedback law for steering the states from the set of initial conditions to the set of terminal conditions, without visiting the set of unsafe states.

In order to combine the stabilization property of CLF with the safety aspect from the CBF, we study in this paper a simple control design procedure where we merge a CLF with a CBF. Some previous relevant works, where a barrier function is incorporated explicitly in the CLF control design method, have been proposed in Ngo et al. (2005) and Tee, Ge, and Tay (2009). In these papers, a stabilization control problem with state saturation is considered which is solved by incorporating explicitly a "barrier function" in the design of a CLF. The resulting CLF has a strong property of being unbounded on the boundary of the state's domain. While in this paper, we consider a more general problem where the unsafe set can be any form of open and bounded set in the domain of the state. It is solved by combining a CLF and CBF that results in a Control Lyapunov–Barrier Function (CLBF) control design method which does not impose unboundedness condition on the boundary of the unsafe set. Hence we admit a larger class of functions than the former approaches.

As mentioned earlier, there are various results in literature on combining several CLFs for improving control performances, which include the use of convex combination as pursued in Andrieu and Prieur (2010) or Grammatico et al. (2014). Based on these works, one can intuitively consider to merge or to unite the CLF and CBF for solving the stabilization with guaranteed safety. However, such an approach may not solve the problem. Note that the important features of the CLF for stabilizing the origin are the (local-) convexity and global minimum at the origin. Hence, the merged CLF (as a result of merging multiple CLFs) has these properties and they are inherited from the original CLFs. On the other hand, the important characteristic of the CBF is that it is (locally-)concave with the level-set of zero belongs to the safe domain. Moreover, CBF may not have a global minimum at all. As a result, CBF and CLF cannot be merged using the same principle of merging multiple CLFs. It may shift the desired equilibrium point (away from the origin) and the merged CLF–CBF may not be proper (i.e., the level-set may not be compact). A recent paper on the uniting of CLF and CBF has also appeared in Ames, Grizzle, and Tabuada (2014) that uses a quadratic programming approach to combine the Lyapunov inequality and Barrier certificate inequality.

Another related control problem in the literature is the obstacle avoidance control problem (Dimarogonas, Loizou, Kyriakopoulos, & Zavlanos, 2006), where the systems are described by a single integrator and the proposed control law is based on a gradient of a particular potential function. Similar works in the context of avoidance control problem for multi-agent systems are Do (2007) and Stipanovic et al. (2007). One important characteristic of the potential function in such method is that it grows unbounded as it reaches the boundary of the obstacle (or the set of unsafe state), akin to the works in Ngo et al. (2005) and Tee et al. (2009) which is generally complicated and difficult to construct.

A preliminary version of this paper has appeared in Romdlony and Jayawardhana (2014). In this paper, we extend the results in Romdlony and Jayawardhana (2014) by including the merging of multiple CBFs into a single CBF and the uniting of multiple CBFs with a CLF. The latter case is relevant to applications that involve multiple constraints and multiple sets of unsafe states.

Recently, we have developed a control design technique that combine our results in this paper with the idea of the Interconnection-and-Damping Assignment Passivity-Based Control (IDA-PBC) (see, for example, Ortega et al., 2002) in Romdlony and Jayawardhana (2015). Using existing numerical tools for implementing the classical IDA-PBC, our results in Romdlony and Jayawardhana (2015) enable further development of numerical tools for implementing our control approach.

In Section 2, we review briefly the concept of Control Lyapunov Function, of Control Barrier Function, and of universal control law which are based on Sontag (1989) and Wieland and Allgöwer (2007). In Section 3, we discuss the problem of stabilization with guaranteed safety and the concept of Control Lyapunov–Barrier function. In Section 4 we propose control design methods that merges a CBF and a CLF. In Section 5, we discuss the extension of the proposed method to the multiple CBFs case. In Section 6 we provide numerical simulations where in one example we present the design of a stabilizer with guaranteed safety for a nonlinear system and in the other one, we present an example of merging multiple CBFs with a single CLF for the navigation of mobile robots.

2. Preliminaries

Notations. In this section, we consider a nonlinear affine system in the form of

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^p$ denote the state and the control input of the system, respectively. We assume also that the functions $f(x)$ and $g(x)$ are smooth, $f(0) = 0$, and $g(x) \in \mathbb{R}^{n\times p}$ is full rank[2] for all x. As usual, we define $L_f V(x)$ and $L_g V(x)$ by $L_f V(x) := \frac{\partial V(x)}{\partial x} f(x)$ and $L_g V(x) := \frac{\partial V(x)}{\partial x} g(x)$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called *proper* if the set $\{x | V(x) \le c\}$ is compact for all constant $c \in \mathbb{R}$, or equivalently, V is radially unbounded. The space $\mathcal{C}^1(\mathbb{R}^l, \mathbb{R}^p)$ consists of all the continuously differentiable functions $F : \mathbb{R}^l \to \mathbb{R}^p$.

Let $\mathcal{X}_0 \subset \mathbb{R}^n$ be the set of initial conditions, and an open set $\mathcal{D} \subset \mathbb{R}^n$ be the set of unsafe states, where $\mathcal{D} \cap \mathcal{X}_0 = \emptyset$. Since we consider also the stabilization problem, we assume that $0 \in \mathcal{X}_0$. For a given set $\mathcal{D} \subset \mathbb{R}^n$, we denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$ and the closure of $\mathcal{D}$ by $\overline{\mathcal{D}}$. We denote by $\mathbb{R}_+$ the set $[0, \infty)$ and use the notation $\overline{\mathbb{R}}_+$ to denote the closure of $\mathbb{R}_+$, i.e., $\overline{\mathbb{R}}_+ := [0, \infty]$. For a given $\delta > 0$ and $\epsilon \in \mathbb{R}^n$, we define an open ball centered at ϵ with radius of δ by $\mathbb{B}_\delta(\epsilon) := \{x | \, \|x - \epsilon\| < \delta\}$. For brevity of notation, we denote an open ball centered at the origin by $\mathbb{B}_\delta := \mathbb{B}_\delta(0)$.

In the following, let us recall some basic results relating to Control Lyapunov Functions and its universal control laws (see also Sontag, 1989).

A proper, positive-definite function $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ that satisfies

$$L_f V(x) < 0 \quad \forall x \in \{z \in \mathbb{R}^n \backslash \{0\} \mid L_g V(z) = 0\} \tag{2}$$

is called a *Control Lyapunov Function* (CLF).

Given a CLF $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$, the system (1) has the *Small Control Property*(SCP) with respect to V if for every $\varepsilon > 0$ there

[2] Here, the rank of matrix function $g(x)$ is defined as the number of linearly independent rows/columns in $g(x)$.

exists a $\delta > 0$ such that for every $x \in \mathbb{B}_\delta$

$\exists u \in \mathbb{R}^p$ such that $\|u\| < \varepsilon$

and $L_f V(x) + L_g V(x)u < 0$.

We define a function $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ by

$$k(\gamma, a, b) = \begin{cases} -\dfrac{a + \sqrt{a^{2+\gamma\|b\|^4}}}{b^T b} b & \text{if } b \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Using the notions of CLF and small-control property, Sontag in Sontag (1989) has proposed a universal control law as summarized in the following theorem.

Theorem 1. *Assume that the nonlinear system (1) has a CLF $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ and satisfies the small-control property w.r.t. V. Then the feedback law*

$$u = k(\gamma, L_f V(x), (L_g V(x))^T) \quad \gamma > 0, \tag{4}$$

is continuous at the origin and ensures that the closed-loop system is globally-asymptotically stable.

In order to incorporate the safety aspect into the control design, we modify the safety definition as used in Wieland and Allgöwer (2007) as follows.

Definition 1 (*Safety*)**.** Given an autonomous system

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathcal{X}_0, \tag{5}$$

where $x(t) \in \mathbb{R}^n$, the system is called *safe* if for all $x_0 \in \mathcal{X}_0$ and for all $t \in \overline{\mathbb{R}}_+$, $x(t) \notin \overline{\mathcal{D}}$.

In the definition of safety as in Wieland and Allgöwer (2007), the safety of any trajectory $x(t)$ is only evaluated in a finite-time interval $[0, T]$ where $T > 0$. If this condition holds for arbitrary $T > 0$, it does not immediately imply that the state trajectory $x(t)$ will not converge to $\partial\mathcal{D}$ as $t \to \infty$. Therefore we add the asymptotic behavior condition to the definition of safety above for excluding such case.

Using this safety definition, the control problem that is considered in Wieland and Allgöwer (2007) is given as follows (see also *Problem* 5 in Wieland & Allgöwer, 2007).

Safety control problem: Given the system (1) with a given initial condition $\mathcal{X}_0$ and a given set of unsafe states $\mathcal{D} \subset \mathbb{R}^n$, design a feedback law $u = \alpha(x)$ s.t. the closed loop system $\dot{x} = f(x) + g(x)\alpha(x)$, $x(0) = x_0 \in \mathcal{X}_0$ is safe.

In order to solve the above problem and motivated by universal control law based on CLF, Wieland and Allgöwer have recently proposed the concept of Control Barrier Function in Wieland and Allgöwer (2007). Let us recall the basic definition of a Control Barrier Function as in Wieland and Allgöwer (2007).

Given a set of unsafe states $\mathcal{D} \subset \mathbb{R}^n$, the function $B \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ satisfying

$$B(x) > 0 \quad \forall x \in \mathcal{D} \tag{6a}$$
$$L_f B(x) \leq 0 \quad \forall x \in \{z \in \mathbb{R}^n \setminus \mathcal{D} | L_g B(z) = 0\} \tag{6b}$$
$$\mathcal{U} := \{x \in \mathbb{R}^n | \, B(x) \leq 0\} \neq \emptyset \tag{6c}$$

is called a *Control Barrier Function* (CBF).

In the following theorem, we present the safety control design method which generalizes the result in Wieland and Allgöwer (2007).

Theorem 2. *Assume that the nonlinear system (1) has a CBF $B \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ with a given set of unsafe states $\mathcal{D} \subset \mathbb{R}^n$, then the feedback law*

$$u = k(\gamma, L_f B(x), (L_g B(x))^T) \quad \gamma > 0, \tag{7}$$

solves the safety control problem, i.e. the closed-loop system is safe with admissible initial condition $\mathcal{X}_0 = \mathcal{U}$ with $\mathcal{U}$ be as in (6c).

Additionally if

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset \tag{8}$$

holds then the closed-loop system is globally safe with $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$.

In comparison to Theorem 7 in Wieland and Allgöwer (2007), in Theorem 2 we allow the possibility of having an initial state x_0 such that $B(x_0) > 0$ with $x_0 \notin \mathcal{D}$; in particular, in Wieland and Allgöwer (2007) it is assumed that $\mathcal{X}_0 \subset \mathcal{U}$. For completeness, we provide the proof to Theorem 2 below.

Proof. The proof of the first claim follows the same line as in the proof of Theorem 7 in Wieland and Allgöwer (2007). Note that the closed-loop system is given by

$$\dot{x} = f(x) + g(x)k(\gamma, L_f B(x), (L_g B(x))^T) =: F_B(x) \tag{9}$$

and it follows from (6a)–(6c) that the time-derivative of B along the solution of (9) satisfies

$$\frac{\partial B(x)}{\partial x} F_B(x) \leq 0 \quad \forall x \in \mathbb{R}^n \setminus \mathcal{D}, \tag{10}$$

which implies that B is non-increasing along the trajectory x satisfying (9).

For proving the first claim, we consider the case $\mathcal{X}_0 = \mathcal{U}$ such that $B(x(0)) \leq 0$ for all $x(0) \in \mathcal{X}_0$. By using (10), we also have that B satisfies

$$B(x(t)) - B(x(0)) \leq 0 \quad \forall t \in \mathbb{R}_+. \tag{11}$$

Therefore $x(t) \in \mathcal{U}$ for all $t \in \mathbb{R}_+$. This proves the first claim since $\mathcal{D} \cap \mathcal{U} = \emptyset$.

We will now prove the second claim where $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$. When $x(0) \in \mathcal{U}$, it has been shown before that $x(t) \in \mathcal{U}$ for all $t \in \mathbb{R}_+$. It remains now to show that for all $x(0) \in \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$, we have $x(t) \notin \mathcal{D}$ for all $t \in \mathbb{R}_+$. In this case, we note that $B(x(0)) \geq 0$ and, as before, B is non-increasing along the trajectory of x for all t.

Since the set $\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})}$ does not intersect with the set $\overline{\mathcal{D}}$, it implies that the trajectory $x(t)$ which starts in $\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$ will not enter $\mathcal{D}$ before it reaches first the boundary of $\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$ (modulo the infinity), in which case, $B(x) = 0$. Once the trajectory $x(t)$ is on the boundary of $\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$, the inequality (11) implies that $x(t)$ will remain in $\mathcal{U}$ thereafter. □

Remark 1. If (6a) and (6c) hold, then the condition (8) implies that $B(x) = 0$ for all $x \in \partial\mathcal{D}$. Indeed, this can be shown by contradiction. Suppose that there exists $x^* \in \partial\mathcal{D}$ such that $B(x^*) \neq 0$ and (8) holds. It follows from (6a) that $B(x^*) > 0$. Hence $x^* \in (\mathbb{R}^n \setminus \mathcal{D}) \cap (\mathbb{R}^n \setminus \mathcal{U}) = \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U}) \subset \overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})}$. Since x^* is also in $\overline{\mathcal{D}}$, we have a contradiction.

However the converse is not true. Fig. 1 shows graphical illustration of a counter-example to this claim (i.e., $B(x) = 0$ for all $x \in \partial\mathcal{D} \not\Rightarrow (8)$). In this counter-example, the sets $\overline{\mathcal{D}}$ and $\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})}$ intersect at a single point x^*, which implies that (8) does not hold but we have $B(x^*) = 0$ according to (6c). One such numerical example of B is given by

$$B(x) = \begin{cases} \text{dist}(x, \partial\mathcal{D}) & \forall x \in \mathcal{D} \\ -\text{dist}(x, \partial\mathcal{D} \cup \partial\mathcal{U}) & \forall x \in \mathcal{U} \\ \text{dist}(x, \partial\mathcal{U}) & \forall x \in \mathbb{R}^2 \setminus \{\mathcal{D} \cup \mathcal{U}\}, \end{cases}$$

where $\mathcal{D} := \mathbb{B}_1\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$, $\mathcal{U} := \overline{\mathbb{B}}_4 \setminus \mathbb{B}_1\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$ and *dist* denotes the usual set distance. In this numerical example, $\partial\mathcal{D}$ and $\partial\mathcal{U}$ intersect only at $\begin{bmatrix}4\\0\end{bmatrix}$.

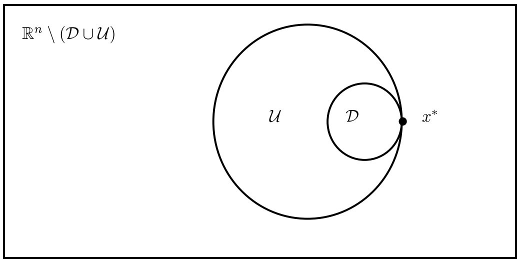

Fig. 1. A counter example where we have $B(x) = 0$ for all $x \in \partial\mathcal{D} \not\Rightarrow$ (8).

3. Stabilization with guaranteed safety

Let us now consider the incorporation of the safety aspect in the standard stabilization problem.

Stabilization with guaranteed safety control problem: Given the system (1) with a given set of initial conditions $\mathcal{X}_0$ and a given set of unsafe states $\mathcal{D}$, design a feedback law $u = \alpha(x)$ s.t. the closed loop system is safe and asymptotically stable, i.e. $\lim_{t\to\infty} \|x(t)\| = 0$. Moreover, when $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$ we call it the *global stabilization with guaranteed safety control problem*.

As briefly discussed in the Introduction, one can intuitively consider to merge or to unite the CLF and CBF by a convex combination a'la Andrieu and Prieur (2010) or Grammatico et al. (2014) for solving the above problem. However, such approach may not immediately guarantee the solvability of the problem. Firstly, the convex combination can lead to the shifting of the global minimum of the combined function which can result in the shifting of the equilibrium point away from the origin. This does not happen in the uniting/merging CLFs since each CLF has minimum at the origin. In the extreme case, when the function of $B(x)$ is not lower-bounded, the combined function may not even admit a global minimum. Secondly, we need a theoretical framework to combine the stability analysis via Lyapunov method and the safety analysis via Barrier Certificate. Motivated by the safety analysis using Barrier Certificate (see, for example Prajna & Jadbabaie, 2004 and Wisniewski & Sloth, 2013), we provide below a proposition on the stability with safety.

Proposition 1. *Consider an autonomous system*

$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{12}$$

with a set of unsafe state $\mathcal{D}$ which is open. Suppose that there exists a proper and lower-bounded function $W \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that

$$W(x) > 0 \quad \forall x \in \mathcal{D} \tag{13a}$$

$$L_f W(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \tag{13b}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n | W(x) \le 0\} \ne \emptyset \tag{13c}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset \tag{13d}$$

then the origin of (12) is asymptotically stable and the system (12) is safe with $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$.

Proof. We firstly prove that if $x_0 \in \mathcal{X}_0$, then the state trajectory x never enters $\mathcal{D}$, i.e., for all $t \ge 0, x(t) \notin \mathcal{D}$.

If $x_0 \in \mathcal{U}$ (i.e. $W(x(0)) \le 0$ by definition) then it follows from (13b), that $\dot{W} < 0$ thus $W(x(t)) - W(x(0)) < 0$ for all $t \in R_+$. Hence, it implies that $W(x(t)) < 0$ for all $t \in R_+$. In other words, the set $\mathcal{U}$ is forward invariant and $x(t) \notin \mathcal{D}$ for all $t \in R_+$ by (13a). Moreover, by the properness of W, the set $\mathcal{U}$ is compact. Note that by the compactness of $\mathcal{U}$, it holds that $\lim_{t\to\infty} x(t) \notin \mathcal{D}$. Now consider the other case when $x_0 \in \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$. By using the same argument as in the proof of the second claim of Theorem 2, the trajectory x will remain in $\mathcal{U}$ and will never enter $\mathcal{D}$.

We will now prove that if $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ then $x(t) \to 0$ as $t \to \infty$.

Let $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ which (according to the previous arguments) implies that the trajectory $x(t) \notin \mathcal{D}$ for all $t \ge 0$. Correspondingly, it follows from (13b) that

$$\begin{aligned} \frac{d}{dt} W(x(t)) < 0 \qquad & \forall x(t) \notin (\mathcal{D} \cup \{0\}) \\ \Rightarrow W(x(t)) < W(x(0)) < \infty \qquad & \forall t \ge 0. \end{aligned} \tag{14}$$

By the properness of W, the last inequality implies that the trajectory x is bounded, and thus it is pre-compact,[3] i.e., the closure of $\{x(t) | t \in [0, \infty)\}$ is compact. This implies that the ω-limit set $\Omega(x_0)$ is non-empty, compact, connected and $\lim_{t\to\infty} d(x(t), \Omega(x_0)) = 0$ where d defines the distance.[4]

Additionally, since the function $\mathcal{W} := W \circ x$ is an absolutely continuous function of t and bounded from below, (14) implies that $\mathcal{W}(t)$ is monotonically decreasing and it has a limit h as $t \to \infty$. On the other hand, for any point ξ in the ω-limit set $\Omega(x_0)$, there is a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to \xi$. By the continuity of W, $W(\xi) = \lim_n \mathcal{W}(t_n) = h$. Therefore, in the invariant set $\Omega(x_0)$, W is constant and is given by h. Using (13b), and the fact that $\mathcal{D} \not\subset \Omega(x(0))$, we have that W is constant only at $x = 0$ and thus $\Omega(x_0) = \{0\}$. Hence,

$$\lim_{t\to\infty} \|x(t)\| = 0. \quad \square$$

We will make a few remarks on the assumptions in Proposition 1. When we restrict the state space to $\mathcal{D} \cup \mathcal{U}$, the conditions in (13a)–(13c) are reminiscent of the conditions in Barrier Certificate theorem (cf. Prajna, 2005, Prop. 2.18).

On the other hand, the properness of W together with (13b) resembles the standard Lyapunov stability theorem (albeit, in this proposition, we do not impose positive-definiteness of W). The addition of condition (13d) is to ensure that the first entry point to the set of $\mathcal{D} \cup \mathcal{U}$ is the boundary of $\mathcal{D} \cup \mathcal{U}$, and not that of $\mathcal{D}$.

Obviously, one can observe from the condition (13b) and (13c) that the origin lies inside the set of $\mathcal{U}$. Indeed, we can prove this by contradiction. Suppose that $0 \notin \mathcal{U}$. Let $x_0 \in \mathcal{U}$ which implies that $x(t) \in \mathcal{U}$ for all t (following the same argument as in the proof of Proposition 1). By (13b), $W(x(t))$ is decreasing and converges to a constant. Similar to the last arguments in the proof of Proposition 1, the ω-limit set is a singleton $\{0\}$ which is a contradiction.

Let us now present a control design framework for solving the stabilization with guaranteed safety control problem. For this, we introduce the notion of Control Lyapunov–Barrier Function as follows.

Definition 2 (*CLBF*)**.** Given a set of unsafe state $\mathcal{D}$, a proper and lower-bounded function $W \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ satisfying

$$W(x) > 0 \quad \forall x \in \mathcal{D} \tag{15a}$$

$$L_f W(x) < 0 \quad \forall x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{15b}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n | W(x) \le 0\} \ne \emptyset \tag{15c}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset \tag{15d}$$

is called a Control Lyapunov–Barrier Function (CLBF).

Using this notion and Proposition 1, we can solve the problem in the following theorem.

[3] The trajectory x in $\mathcal{X}$ is *pre-compact* if it is bounded for all $t \in [0, \infty)$ and for any sequences (t_n) in $[0, \infty)$, the limit $\lim_{n\to\infty} x(t_n)$ exists and is in $\mathcal{X}$ (La Salle, 1976).

[4] For the concept of ω-limit set, we refer interested readers to Jayawardhana and Weiss (2009), La Salle (1976) and Logeman and Ryan (2004).

Theorem 3. *Assume that the system (1) admits a CLBF $W \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ with a given set of unsafe states $\mathcal{D}$ and satisfies the small-control property w.r.t. W, then the feedback law*

$$u = k(\gamma, L_f W(x), (L_g W(x))^T) \quad \gamma > 0, \tag{16}$$

is continuous at the origin and solves the global stabilization with guaranteed safety control problem.

Proof. We prove the theorem by showing that the conditions (13a)–(13d) in Proposition 1 hold for the closed-loop autonomous system

$$\dot{x} = F_W(x)$$

where $F_W(x) := f(x) + g(x)k(\gamma, L_f W(x), (L_g W(x))^T)$.

The conditions (13a), (13c) and (13d) follow trivially from (15a), (15c) and (15d), respectively. Now, for all $x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \mid L_g W(z) \neq 0\}$, we have that

$$\begin{aligned} L_F W(x) &= L_f W(x) + L_g W(x) k(\gamma, L_f W(x), (L_g W(x))^T) \\ &= -\sqrt{\|L_f W(x)\|^2 + \gamma \|L_g W(x)\|^4} \\ &< 0 \end{aligned}$$

holds. On the other hand, for all $x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \mid L_g W(z) = 0\}$, the condition (15b) implies that

$$L_F W(x) < 0.$$

These two inequalities show that (13b) also holds.

The continuity of the feedback law at the origin follows the same proof as in Sontag (1989). □

Using the same argument as in the proof of Proposition 1, it can be checked that the condition (15b) can be weakened by

$$L_f W(x) \le 0 \quad \forall x \in \mathcal{M},$$

where the CLBF function W is still assumed to be $\mathcal{C}^1$,

$$\mathcal{M} := \{z \in \mathbb{R}^n \setminus \mathcal{D} | L_g W(z) = 0\}$$

and the largest invariant set in $\mathcal{M}$ is $\{0\}$. This condition will be useful later in the simulation result. This is formalized in the following proposition.

Proposition 2. *Let $\mathcal{D}$ be a given set of unsafe states. Assume that the system in (1) has a proper and lower-bounded function $W \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ satisfying*

$$W(x) > 0 \quad \forall x \in \mathcal{D} \tag{17a}$$

$$L_f W(x) \le 0 \quad \forall x \in \mathcal{M} := \{z \in \mathbb{R}^n \setminus \mathcal{D} \mid L_g W(z) = 0\} \tag{17b}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n \mid W(x) \le 0\} \neq \emptyset \tag{17c}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset. \tag{17d}$$

Assume also that the system is zero-state detectable with respect to $L_g W(x)$, i.e., $L_g W(x(t)) = 0 \; \forall t \ge 0 \Rightarrow x(t) \to 0$. Suppose that the system in (1) has the small-control property w.r.t. W. Then the feedback law

$$u = k\left(\gamma, L_f W(x), (L_g W(x))^T\right) \quad \gamma > 0, \tag{18}$$

is continuous at the origin and solves the global stabilization with guaranteed safety control problem.

Proof. The proof is akin to the proof of Theorem 3 and Proposition 1. Similar to the proof of Proposition 1, if $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ then the trajectory x will never enter $\mathcal{D}$, i.e., $x(t) \in \mathbb{R}^n \setminus \mathcal{D}$ for all $t \ge 0$ and $\mathcal{D} \not\subseteq \Omega(x_0)$.

It remains to show that in the closed-loop system, for every $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ we have $\Omega(x_0) = \{0\}$. As in the proof of Theorem 3, the time-derivative of W satisfies

$$\begin{aligned} L_F W(x) &= -\sqrt{\|L_f W(x)\|^2 + \gamma \|L_g W(x)\|^4} \\ &\le -\sqrt{\gamma}\|L_g W(x)\|^2 \quad \forall x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{M}). \end{aligned}$$

On the other hand, for all $x \in \mathcal{M}$, the assumption (17b) implies that $L_F W(x) \le 0$. Hence, combining these two inequalities, we have that for all $x(t) \in \mathbb{R}^n \setminus \mathcal{D}$,

$$\dot{W}(x(t)) \le -\sqrt{\gamma}\|L_g W(x(t))\|^2.$$

This inequality implies that W converges to a constant and the trajectory x converges to the largest invariant set $\mathcal{N}$ contained in $\mathcal{M}$, i.e., $\Omega(x_0) \subset \mathcal{N} \subset \mathcal{M}$. By the zero-state detectability assumption with respect to $L_g W$, we have that the largest invariant set $\mathcal{N} = \{0\}$. Hence, $\Omega(x_0) = \mathcal{N} = \{0\}$, i.e., $\lim_{t\to\infty} \|x(t)\| = 0$. □

4. Constructive design of a CLBF

Equipped with Theorem 3 we can now present results on the construction of CLBF by uniting a CLF and a CBF. This will potentially allow us to separate the control design for achieving the asymptotic stability and safety by designing the CLF and CBF, independently, and then combine them together. In the following proposition, we assume first that B is lower-bounded.

Proposition 3. *Suppose that for system (1), with a given set of unsafe states $\mathcal{D}$ that is open, there exist a CLF $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ and a CBF $B \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ which satisfy*

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2 \quad \forall x \in \mathbb{R}^n \; c_2 > c_1 > 0, \tag{19}$$

and a compact and connected set $\mathcal{X}$ s.t.

$$\mathcal{D} \subset \mathcal{X}, \qquad 0 \notin \mathcal{X} \quad \text{and} \quad B(x) = -\varepsilon, \quad \varepsilon > 0 \, \forall x \in \mathbb{R}^n \setminus \mathcal{X}. \tag{20}$$

If

$$L_f W(x) < 0 \quad \forall x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{21}$$

where

$$W(x) = V(x) + \lambda B(x) + \kappa,$$

with $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$, $\kappa = -c_1 c_4$, $c_3 := \max_{x \in \partial\mathcal{X}} \|x\|^2$, $c_4 := \min_{x \in \partial\mathcal{D}} \|x\|^2$, then the feedback law (16) solves the stabilization with guaranteed safety control problem with the set of initial states $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}_{relaxed}$ where $\mathcal{D}_{relaxed} := \{x \in \mathcal{X} | W(x) > 0\} \supset \mathcal{D}$.

Moreover if (1) has the small-control property w.r.t. V then it has also the small-control property w.r.t. W. In which case, the feedback law (16) is continuous at the origin.

Proof. The proof of the proposition will be based on proving that $\mathcal{D} \subset \mathcal{D}_{relaxed}$ and (15a)–(15d) hold with $\mathcal{D}$ being replaced by $\mathcal{D}_{relaxed}$. Note that (15a) holds by the definition of $\mathcal{D}_{relaxed}$. A routine computation shows that for all $x \in \mathcal{D}$,

$$\begin{aligned} W(x) &= V(x) + \lambda B(x) - c_1 c_4 \\ &> c_1\|x\|^2 - c_1 c_4 \\ &> 0, \end{aligned} \tag{22}$$

since $\lambda > 0$, $B(x) > 0$ for all $x \in \mathcal{D}$ and $\|x\|^2 > c_4$ for all $x \in \mathcal{D}$.

Also for all $x \in \partial\mathcal{X}$,

$$\begin{aligned} W(x) &= V(x) + \lambda B(x) - c_1 c_4 \\ &= V(x) - \lambda\varepsilon - c_1 c_4 \\ &\le c_2\|x\|^2 - \lambda\varepsilon - c_1 c_4 \\ &< c_2 c_3 - (c_2 c_3 - c_1 c_4) - c_1 c_4 = 0, \end{aligned} \tag{23}$$

where the strict inequality is due to the hypotheses of $\lambda > \frac{c_2c_3-c_1c_4}{\varepsilon}$. Hence we have that (15c) holds. By the continuity of $W(x)$, the inequalities (22) and (23) imply that the open set $\mathcal{D}_{relaxed}$ is the interior of $\mathcal{X}$ and moreover $\mathcal{D} \subset \mathcal{D}_{relaxed}$. Hence $\partial\mathcal{X} \cap \partial\mathcal{D}_{relaxed} = \emptyset$ and we have

$$\mathcal{D} \subset \mathcal{D}_{relaxed} \subset \mathcal{X} \subset \mathcal{D}_{relaxed} \cup \mathcal{U}. \tag{24}$$

The last relation is due to the decomposition of $\mathcal{X} = \mathcal{D}_{relaxed} \cup \mathcal{X}_-$ where $\mathcal{X}_- := \{x \in \mathcal{X} | W(x) \le 0\} \subset \mathcal{U}$. Since $\mathcal{D} \subset \mathcal{D}_{relaxed}$, we have that (21) $\Longrightarrow$ (15b) (with $\mathcal{D}$ being replaced by $\mathcal{D}_{relaxed}$). Finally, since the boundary of $\mathcal{X}$ does not intersect with the boundary of $\mathcal{D}_{relaxed}$, (24) implies that $\overline{\mathbb{R}^n \setminus (\mathcal{D}_{relaxed} \cup \mathcal{U})} \cap \overline{\mathcal{D}_{relaxed}} = \emptyset$, i.e. (15d) holds.

The proof on the claim of SCP follows trivially from the hypothesis in (20). Indeed, since $0 \notin \mathcal{X}$ and $\mathcal{X}$ being compact, we can define a neighborhood $\mathbb{B}_\delta = \{x| \, \|x\| < \delta\}$ such that $\mathbb{B}_\delta \cap \mathcal{X} = \emptyset$. In $\mathbb{B}_\delta$ it holds that $L_fW(x) + L_gW(x) = L_fV(x) + L_gV(x)$ since B is constant outside $\mathcal{X}$. Thus if (1) has SCP w.r.t. V then it has also SCP w.r.t. W. □

We note that the condition (21) implies that the function W has a global minimum in $\mathbb{R}^n \setminus \mathcal{D}$ at 0. This can be shown by contradiction. Suppose that W admits another minimum $x^* \neq 0$ in $\mathbb{R}^n \setminus \mathcal{D}$ such that (21) holds. The point x^* being minimum implies that $\frac{\partial W(x^*)}{\partial x} = 0$ so that $L_fW(x^*) = 0$, which contradicts (21).

In Proposition 3, it is assumed that B is lower-bounded. In general, when the CBF $B(x)$ is not lower-bounded, we can always construct another CBF $\widetilde{B}(x)$ satisfying (20) based on $B(x)$ which satisfies (6a)–(6c). Hence Proposition 3 can still be applicable using this new CBF $\widetilde{B}(x)$.

Proposition 4. *Suppose that the set of unsafe states $\mathcal{D}$ is bounded and simply-connected. Assume that there exist a CBF $B \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and $\delta > 0$ such that $\mathcal{J} := \{x | B(x) \ge -\delta\}$ is simply-connected, contains $\mathcal{D}$ and B is strictly-concave on $\mathcal{J}$. Let $\rho : \mathbb{R} \to [0, 1]$ be a non-decreasing $\mathcal{C}^1$ function such that $\rho(z) = 0$ for all $z \le -\delta$ and $\rho(z) = 1$ for all $z \ge 0$. By using any arbitrary point $\omega \in \partial\mathcal{D}$, define the function $\widetilde{B}(x) \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ by*

$$\widetilde{B}(x) = \begin{cases} B(\omega) + \oint_\Gamma \rho(B(\sigma)) \frac{\partial B(\sigma)}{\partial x} d\sigma & \forall x \in \mathcal{J} \\ -\varepsilon & \text{otherwise}, \end{cases} \tag{25}$$

where Γ is any path from ω to $x \in \mathcal{J}$ and the constant ε is defined by $\varepsilon = -\widetilde{B}(\phi)$ where ϕ is any point on $\partial\mathcal{J}$, i.e.

$$\varepsilon = -B(\omega) - \oint_{\Gamma_{\omega\to\phi}} \rho(B(\sigma)) \frac{\partial B(\sigma)}{\partial x} d\sigma,$$

where $\Gamma_{\omega\to\phi}$ is any path from ω to ϕ. Then $\widetilde{B}$ is also a CBF satisfying the conditions (6a)–(6c) and also (20) with $\mathcal{X}$ be given by $\overline{\mathcal{J}}$.

Proof. We prove the proposition by showing (6a)–(6c) holds with the same $\mathcal{D}$. Notice that the integration in (25) is proper and $\widetilde{B}$ is a potential function. Indeed, it is trivial to check that the Hessian matrix of (25) is symmetric and hence, it defines a potential function.

Now, for every $x \in \mathcal{D}$, there exists a path Γ from ω to x since $\mathcal{D}$ is connected and it follows that

$$\widetilde{B}(x) = B(\omega) + \oint_\Gamma \frac{\partial B(\sigma)}{\partial x} d\sigma = B(x) > 0$$

where we have used the fact that $\rho(B(\sigma)) = 1$ for all $\sigma \in \Gamma$. Hence (6a) holds.

In order to show that (6b) holds with the new CBF $\widetilde{B}$, we first note that for all $x \in \mathcal{J}$, we have that $\frac{\partial \widetilde{B}}{\partial x} g(x) = 0 \Leftrightarrow \frac{\partial B}{\partial x} g(x) = 0$. Hence, for all $x \in \{z \in \mathcal{J} \setminus \mathcal{D} | L_g\widetilde{B}(z) = 0\}$ we have

$$\frac{\partial \widetilde{B}}{\partial x} f(x) = \rho(B(x)) \frac{\partial B}{\partial x} f(x) \le 0. \tag{26}$$

On the other hand, for all $x \in \mathbb{R}^n \setminus \mathcal{J}$, we have $\frac{\partial \widetilde{B}}{\partial x} = 0$ which implies that $L_f\widetilde{B}(x) = 0, \forall x \in \{z \in \mathbb{R}^n \setminus \mathcal{J} | L_g\widetilde{B}(z) = 0\}$. Together with (26), we have that (6b) holds.

Eq. (6c) follows trivially. Now we will prove (20), i.e., $\widetilde{B}(x)$ is a negative constant in $\mathbb{R}^n \setminus \mathcal{J}$. By using the concavity of $B(x)$ on $\mathcal{J}$, and since $\mathcal{J} \supset \mathcal{D}$, we have that for any point $\phi \in \partial\mathcal{J}$, $\widetilde{B}(\phi) = -\varepsilon < 0$, i.e., (20) holds with $\mathcal{X} = \overline{\mathcal{J}}$. □

One can show easily that the constant ε as calculated in Proposition 4 is less than or equal to δ.

As it was shown in the proof of Proposition 4, the set $\mathcal{X}$ is closely related to the parameter δ used to define ρ in (25). One can immediately check that for every enlargement of $\mathcal{D}$ with a radius of $\mu > 0$, i.e. $\mathcal{D} + \mathbb{B}_\mu$,[5] we can always find $\delta > 0$ such that the resulting $\mathcal{X}$ lies in the interior of $\mathcal{D} + \mathbb{B}_\mu$. This property will be useful later when we want to combine multiple CBFs with a single CLF.

Corollary 1. *For every $\mu > 0$ there exists $\delta > 0$ such that $\widetilde{B}(x)$ as constructed in (25) satisfies (20) with $\mathcal{X} \subset \mathcal{D} + \mathbb{B}_\mu$.*

Proof. By the continuity of B there exists a neighborhood Ω of $\mathcal{D}$ such that $\Omega \subset \mathcal{D} + \mathbb{B}_\mu$ and $B(\partial\Omega) = -\delta < 0$. The proof of the claim follows the same line as that of Proposition 4. Note that, here $\mathcal{J}$ (as used in the proposition) is given by Ω. □

5. Handling multiple sets of unsafe state

In the previous section, we dealt with the problem of combining a CLF with a CBF for designing a CLBF, i.e., it handles only a set of unsafe states $\mathcal{D}$.

For accommodating a general set of unsafe states $\mathcal{D}$, we present in this section a constructive method for combining multiple CBFs and a single CLF. The main assumption in this study is that we can decompose $\mathcal{D}$ into a finite number of disjoint simply-connected sets $\mathcal{D}_1, \mathcal{D}_2 \dots \mathcal{D}_N$, each of which admits a CBF. Our main result in Proposition 3 cannot directly be used in this case, even if there exists a CBF that covers the multiple sets of unsafe state $\mathcal{D}_1, \mathcal{D}_2 \dots \mathcal{D}_N$. Our proposed approach is based on combining the CBFs together to make a CBF which can then be merged with a CLF as before.

Let us assume that the set of unsafe states $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots \cup \mathcal{D}_N$ where $\overline{\mathcal{D}_i} \cap \overline{\mathcal{D}_j} = \emptyset$ for all $i \neq j$ and for every i, $\mathcal{D}_i$ is bounded and simply-connected. Suppose that for every i, there exists a CBF B_i for $\mathcal{D}_i$ such that (6a)–(6c) hold. Using these functions B_i, $i = 1, \dots, N$, we can construct a family of CBFs B for $\mathcal{D}$ as follows.

By the boundedness of $\mathcal{D}_i$ and since the sets $\mathcal{D}_i$, $i = 1, \dots, N$ are disjoint, there exist $\mu > 0$ such that the open sets $\mathcal{D}_i + \mathbb{B}_\mu$, $i = 1, \dots, N$ are also disjoint. Indeed, by the assumptions, the distance between the sets $\mathcal{D}_i$ and $\mathcal{D}_j$, $i \neq j$, is strictly positive. Hence by choosing $\mu > 0$ such that

$$\mu < \frac{1}{4} \min_{i,j} d(\mathcal{D}_i, \mathcal{D}_j), \tag{27}$$

it follows that the sets $\mathcal{D}_i + \mathbb{B}_\mu$ and $\mathcal{D}_j + \mathbb{B}_\mu$, for all $i \neq j$, are disjoint. By Corollary 1, for every i, there exist $\widetilde{B}_i(x)$ and $\delta_i > 0$

[5] Here, we use the Minkowski sum for the set addition.

(which is constructed using $B_i(x)$ and μ) such that (20) holds with $\mathcal{X}_i \subset \mathcal{D}_i + \mathbb{B}_\mu$ and $\varepsilon_i > 0$. Finally, a family of CBFs B for $\mathcal{D}$ is given by

$$B(x) = \sum_i \lambda_i \widetilde{B}_i(x) \tag{28}$$

where $\lambda_i \geq 0,\ i = 1, \ldots, N$ are design parameter that can be chosen appropriately when it is merged with a CLF.

In the following proposition, we present a slight modification to Proposition 3 where we merge B as in (28) with a proper CLF V.

Proposition 5. *Assume that for system (1), there exists a CLF $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ and CBFs $\widetilde{B}_i \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ which satisfy*

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \quad \forall x \in \mathbb{R}^n,\ c_2 > c_1 > 0. \tag{29}$$

If

$$L_f W(x) < 0 \quad \forall x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{30}$$

where

$$W(x) = V(x) + \sum_i \lambda_i \widetilde{B}_i(x) + \kappa$$

with λ_i and κ be chosen such that

$$\sum_{j \neq i} \lambda_j \varepsilon_j - c_1 c_{4i} < \kappa < \sum_i \lambda_i \varepsilon_i - c_2 c_{3i} \quad \forall i, \tag{31}$$

$c_{3i} := \max_{x \in \partial \mathcal{X}_i} \|x\|^2$, $c_{4i} := \min_{x \in \partial \mathcal{D}_i} \|x\|^2$, then the feedback law (16) solves the stabilization with guaranteed safety control problem with the set of initial states $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}_{relaxed}$ where $\mathcal{D}_{relaxed} := \{x \in \mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_N | W(x) > 0\}$.

Proof. The proof follows similar arguments as those in Proposition 3. The main differences are in the computation of $W(x) > 0$ for all $x \in \mathcal{D}_i$ and $W(x) < 0$ for all $x \in \partial \mathcal{X}_i$.

For all $x \in \mathcal{D}_i$, it can be checked that

$$\begin{aligned} W(x) &= V(x) + \lambda_i \widetilde{B}_i(x) - \sum_{j \neq i} \lambda_j \varepsilon_j + \kappa \\ &\geq c_1\|x\|^2 - \sum_{j \neq i} \lambda_j \varepsilon_j + \kappa \\ &\geq c_1 c_{4i} - \sum_{j \neq i} \lambda_j \varepsilon_j + \kappa > 0, \end{aligned} \tag{32}$$

thus (15a) holds. Eq. (15b) holds by the assumption of (30). Now it remains to verify (15c) and (15d).

Note that for all $x \in \partial \mathcal{X}_i$,

$$\begin{aligned} W(x) &\leq c_2\|x\|^2 + \sum_i \lambda_i \widetilde{B}_i(x) + \kappa \\ &= c_2\|x\|^2 - \sum_i \lambda_i \varepsilon_i + \kappa \\ &\leq c_2 c_{3i} - \sum_i \lambda_i \varepsilon_i + \kappa < 0, \end{aligned} \tag{33}$$

and hence (15c) holds. Similar to Proposition 3, (32) and (33) $\Longrightarrow$ $\mathcal{D}_{relaxed} \subset \mathcal{X}$ and $\partial \mathcal{D}_{relaxed} \cap \partial \mathcal{X} = \emptyset$, i.e., (15d) holds (with $\mathcal{X} = \mathcal{D}_{relaxed} \cup \mathcal{U}$). $\square$

Remark 2. It can be shown that the set of λ_i and κ that satisfy (31) is non-empty, i.e., the inequalities in (31) are solvable. The following algorithm provides a systematic way to design such λ_i and κ.

A1 For every i choose $\lambda_i > 0$ such that $\lambda_i > \frac{c_2 c_{3i} - c_1 c_{4i}}{\varepsilon_i}$.

A2 Choose κ such that $\kappa \in$

$$\left(\sum_i \lambda_i \varepsilon_i - \min_i \lambda_i \varepsilon_i - c_1 \min_i c_{4i},\ \sum_i \lambda_i \varepsilon_i - c_2 \max_i c_{3i} \right).$$

Indeed, if we choose λ_i and κ as in **(A1)** and **(A2)**, the conditions (31) hold.

6. Examples

In order to demonstrate the applicability of the developed methods, we will consider two numerical examples, which are described as follows.

6.1. Nonlinear mechanical system

Consider the system described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -s(x_2) - x_1 + u, \end{aligned} \tag{34}$$

where $x = [x_1\ x_2]^T \in \mathbb{R}^2$, and $u \in \mathbb{R}$. This example can represent a mechanical system where x_1 describe the displacement, x_2 describe the velocity. In this case, the mass is 1, the damping parameter is described by Stribeck friction model $s(x_2) = (0.8 + 0.2e^{-100|x_2|}) \tanh(10x_2) + x_2$ and spring constant is 1. For this system, $f(x) = \begin{bmatrix} x_2 \\ -s(x_2) - x_1 \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It can be checked that the system (34) admits $V(x) = x_1^2 + x_1 x_2 + x_2^2$ as a CLF, i.e. (2) holds and it has small control property. Also, the function

$$B(x) = \begin{cases} e^{-\left(\frac{1}{1-(x_1-2)^2} + \frac{1}{1-x_2^2}\right)} - e^{-4} & \forall x \in \mathcal{X} \\ -e^{-4} & \text{elsewhere}, \end{cases} \tag{35}$$

where $\mathcal{X} := (1, 3) \times (-1, 1)$, define a CBF for (34) with the set of unsafe states as $\mathcal{D} := \{x \in \mathcal{X} | \frac{1}{1-(x_1-2)^2} + \frac{1}{1-x_2^2} < 4\}$. Note that for all $x \in \mathcal{D}$, $B(x) > 0$.

Indeed, by direct evaluation, we have that for all $x \in \mathcal{X}$

$$\frac{\partial B}{\partial x} g(x) = 0 \Rightarrow x_2 = 0.$$

Hence the manifold $\{x | L_g B = 0\}$ is given by $\{x | x_2 = 0\}$, in which case

$$\left. \frac{\partial B}{\partial x} f(x) \right|_{x_2 = 0} = 0,$$

hence (6b) holds.

Let us now construct a CLBF $W(x)$ according to the construction as in Proposition 3. It is easy to see that the CLF $V(x)$ satisfies $\frac{1}{2}\|x\|^2 \leq V(x) \leq \frac{3}{2}\|x\|^2,\ \forall x \in \mathbb{R}^2$, i.e., (19) holds with $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

On the other hand, it can be checked that $c_3 = \max_{x \in \partial \mathcal{X}} \|x\|^2 = 10$, $c_4 = \min_{x \in \partial \mathcal{D}} \|x\|^2 = 1.4$ and $\varepsilon = e^{-4}$. Hence, by taking $\lambda = 1000$, the condition $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$ is satisfied. Also, as defined in Proposition 3, $\kappa = -c_1 c_4 = -0.7$. Using this constant λ and κ, the CLBF $W(x)$ is given by

$$W(x) = V(x) + \lambda B(x) + \kappa,$$

and the control law for solving the problem of stabilization with guaranteed safety is given by (16).

Fig. 2 shows the numerical simulation of the closed-loop system with the gain $\gamma = 2$. In this plot, eight trajectories are shown with eight different initial states (4 0), (2 2), (−4 0), (−2 2), (−2 −2), (3 0), (3 −1), and (1 2). It can be seen from this figure that all trajectories converge to zero and avoid the unsafe state $\mathcal{D}$.

Fig. 3 shows the resulting CLBF $W(x)$ where it is shown that for all $x \in \mathcal{D}$, $W(x) > 0$.

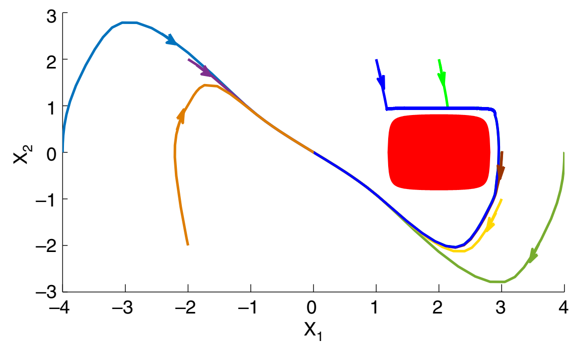

Fig. 2. The numerical simulation result of the closed-loop system using our proposed uniting CLBF method. The set of unsafe state $\mathcal{D}$ is shown in solid area and the plot of closed-loop trajectories is shown in solid lines based on eight different initial conditions.

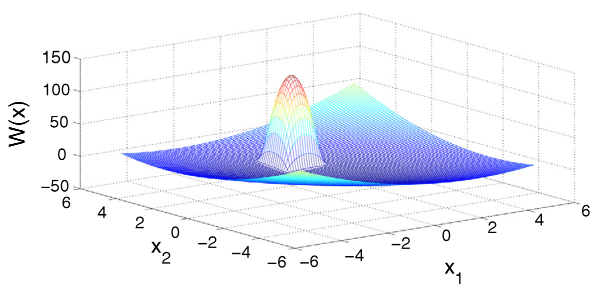

Fig. 3. The plot of the resulting Control Lyapunov–Barrier Function $W(x)$ as considered in the numerical simulation for Example 6.1.

6.2. Mobile robot

In this example, we consider a simple mobile robot navigation that can be described by the following equations

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2 \tag{36}$$

where x_1, x_2 are the positions in a 2D plane, and u_1, u_2 are their velocities, respectively.

This system admits a CLF $V(x) = x_1^2 + x_1x_2 + x_2^2$. We can choose $c_1 = 0.5$ and $c_1 = 2$ such that (19) is satisfied. Assume that we have two disjoint sets of unsafe states $\mathcal{D}_1 := \{x \in \mathcal{X}_1 | (x_1 - 3)^2 + x_2^2 < 4\}$ and $\mathcal{D}_2 := \{x \in \mathcal{X}_2 | x_1^2 + (x_2 - 5)^2 < 1\}$. It is easy to verify that the smallest distance between the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ is 2.83, thus according to (27), we can enlarge each unsafe sets $\mathcal{D}_1$ and $\mathcal{D}_2$ with the open ball with radius $\mu < 0.7$. According to definition of $\mathcal{D}_1$ and $\mathcal{D}_2$, and by letting $\mathcal{X}_1 = \mathcal{D}_1 + \mathbb{B}_\mu$, $\mathcal{X}_2 = \mathcal{D}_2 + \mathbb{B}_\mu$ with $\mu = 0.3$, we have $c_{31} = 31.36$, $c_{41} = 1$, $c_{32} = 43.56$, $c_{42} = 16$.

We can choose two CBFs $B_1(x) = -((x_1 - 3)^2 + x_2^2) + 8$ and $B_2(x) = -(x_1^2 + (x_2 - 5)^2) + 4$ for $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. It can be checked that $-B_1$ and $-B_2$ are locally-strictly-concave functions. Now, for constructing $\widetilde{B}_1(x)$ and $\widetilde{B}_2(x)$ as in Proposition 4, we can choose a $\mathcal{C}^1$ function ρ_i, $i = 1, 2$, that satisfies

$$\rho_i(z) = \begin{cases} 1 & \forall z \geq 0 \\ 0 & \forall z \leq -\delta_i \\ \dfrac{1}{2}\left(\cos\left(\dfrac{\pi}{\delta_i}z\right) + 1\right) & \forall z \in (-\delta_i, 0). \end{cases}$$

We choose the following parameters $\delta_1 = 1.24$, $\delta_2 = 1.44$ and $\lambda_1 = \lambda_2 = 100$ that satisfy $\lambda_i > \frac{c_2c_{3i} - c_1c_{4i}}{\varepsilon_i}$, with $\varepsilon_i \leq \delta_i$, and the feedback gain $\gamma = 3$. Thus by using the control law as in (16) with $W(x) = V(x) + \lambda_1\widetilde{B}_1(x) + \lambda_2\widetilde{B}_2(x) + \kappa$, with κ can be arbitrarily chosen as in Proposition 5. Fig. 4 shows the simulation results of the closed-loop system where it can be seen from this figure that all

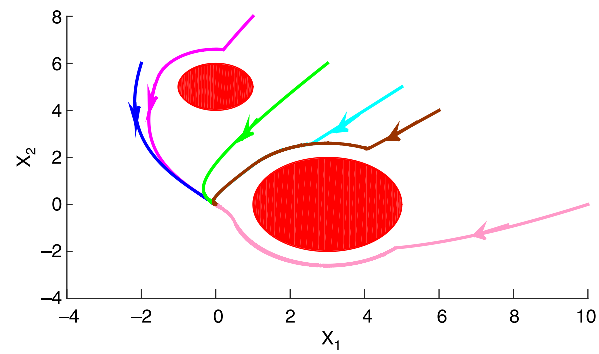

Fig. 4. The numerical simulation result of the closed-loop system using our proposed uniting CLF and CBFs method for the mobile robot example in Section 6.2. The sets of unsafe state $\mathcal{D}_1$ and $\mathcal{D}_2$ are shown in solid area and the plot of closed-loop trajectories is shown in solid lines based on six different initial conditions (1 8), (−2 6), (5 5), (3 6), (10 0), and (6 4).

state trajectories with different initial conditions avoid the unsafe sets $\mathcal{D}_1$ and $\mathcal{D}_2$, and all trajectories converge to zero, i.e., the closed loop system is safe and stable.

7. Conclusions and discussions

In this paper, we have proposed a method to combine a CLF and a CBF. Simulation results show the effectiveness of the control law based on the resulting Control Lyapunov–Barrier Function for solving the stabilization with guaranteed safety. Our proposed approach can simultaneously stabilize the closed-loop systems and guarantee its safety.

Acknowledgments

We thank Fan Zhang and J.J. Barradas Berglind for their constructive comments.

References

Ames, A.D., Grizzle, J.W., & Tabuada, P. (2014). Control barrier function based quadratic programs with application to adaptive cruise control. In *Proc. IEEE conf. dec. contr.*. Los Angeles.

Andrieu, V., Jayawardhana, B., & Praly, L. (2013). On transverse exponential stability and its use in incremental stability, observer and synchronization. In *Proc. IEEE conf. dec. contr.*. Florence.

Andrieu, V., & Prieur, C. (2010). Uniting two control Lyapunov functions for affine systems. *IEEE Transactions on Automatic Control*, *55*(8), 1923–1927.

Artstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Analysis*, *7*, 1163–1173.

Balestrino, A., Crisostomi, E., & Caiti, A. (2011). Logical composition of Lyapunov functions. *International Journal of Control*, *84*(3), 563–573.

Bemporad, A. (1998). Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *43*(3), 415–419.

Bemporad, A., Borelli, F., & Morari, M. (2002). Model predictive control based on linear programming—the explicit solution. *IEEE Transactions on Automatic Control*, *47*(12), 1974–1985.

Bemporad, A., Casavola, A., & Mosca, E. (1997). Nonlinear control of constrained linear systems via predictive reference management. *IEEE Transactions on Automatic Control*, *42*(3), 340–349.

Clarke, F. (2011). Lyapunov functions and discontinuous stabilizing feedback. *Annual Reviews in Control*, *35*, 13–33.

Dimarogonas, D. V., Loizou, S. G., Kyriakopoulos, K. J., & Zavlanos, M. M. (2006). A feedback stabilization and collision avoidance scheme for multiple independent non-point agents. *Automatica*, *42*(2), 229–243.

Do, K. D. (2007). Bounded controllers for formation stabilization of mobile agents with limited sensing ranges. *IEEE Transactions on Automatic Control*, *52*(3), 569–576.

Gilbert, E.G., & Kolmanovsky, I. (2001). A generalized reference governor for nonlinear systems. In *Proc. IEEE conf. on dec. and contr.* (pp. 4222–4227). Orlando.

Gilbert, E. G., & Tan, K. T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, *36*(9), 1008–1020.

Grammatico, S., Blanchini, F., & Caiti, A. (2014). Control-sharing and merging control Lyapunov functions. *IEEE Transactions on Automatic Control*, *59*(1), 107–119.

Jayawardhana, B., & Weiss, G. (2009). State convergence of passive nonlinear systems with an L^2 input. *IEEE Transactions on Automatic Control*, *54*(7), 1723–1727.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design*. Wiley.

La Salle, J. P. (1976). *The stability of dynamical systems*. Philadelphia, PA: SIAM, Appendix by Z. Artstein.

Logeman, H., & Ryan, E. P. (2004). Asymptotic behaviour of nonlinear systems. *American Mathematical Monthly*, *111*(10), 864–889.

Maciejowski, J. (2002). *Predictive control with constraints*. England: Prentice Hall.

Morari, M., & Lee, J. H. (1999). Model predictive control: past, present and future. *Computers and Chemical Engineering*, *23*, 667–682.

Ngo, K.B. et al. (2005). Integrator backstepping using barrier functions for systems with multiple state constraints. In *Proc. IEEE conf. dec. contr. & eur. contr. conf.* (pp. 8306–8312). Sevilla.

Ortega, R., Loría, A., Nicklasson, P. J., & Sira-Ramírez, H. (1998). *Passivity-based control of Euler–Lagrange systems*. London: Springer-Verlag.

Ortega, R., van der Schaft, A., Maschke, B., & Escobar, G. (2002). Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, *38*, 585–596.

Prajna, S. (2005). *Optimization-based methods for nonlinear and hybrid systems verification*. (PhD thesis), California Institute of Technology.

Prajna, S., & Jadbabaie, A. (2004). Safety verification of hybrid systems using Barrier certificates. In *Lecture notes in computer science*, *Hybrid systems: computation and control* (pp. 477–492). Springer-Verlag.

Praly, L., Ortega, R., & Kaliora, G. (2001). Stabilization of nonlinear systems via forwarding mod $\{L_g V\}$. *IEEE Transactions on Automatic Control*, *46*(9), 1461–1466.

Prieur, C. (2001). Uniting local and global controllers with robustness to vanishing noise. *Mathematics of Control, Signals, and Systems*, *14*, 143–172.

Prieur, C., & Praly, L. (1999). Uniting local and global controllers. In *IEEE conf. dec. contr.* (pp. 1214–1219).

Romdlony, M.Z., & Jayawardhana, B. (2014). Uniting control Lyapunov and Barrier function. In *Proc. IEEE conf. on dec. contr.*. Los Angeles.

Romdlony, M.Z., & Jayawardhana, B. (2015). Passivity-based control with guaranteed safety via interconnection and damping assignment. In *Proc. 5th IFAC conf. on anal. and des. of hybr. syst.*. Atlanta.

Sontag, E. D. (1989). A universal construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, *13*, 117–123.

Stipanovic, D. M., et al. (2007). Cooperative avoidance control for multiagent systems. *Journal of Dynamic Systems, Measurement, and Control*, *129*, 699–707.

Tee, K. P., Ge, S. S., & Tay, E. H. (2009). Barrier Lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, *45*(4), 918–927.

Wieland, P., & Allgöwer, F. (2007). Constructive safety using control barrier functions. In *Proc. IFAC symp. nonl. contr. syst.* (pp. 473–478). Pretoria.

Wisniewski, R., & Sloth, C. (2013). Converse Barrier certificate theorem. In *Proc. IEEE conf. dec. contr.* (pp. 4713–4718). Florence.

Wolff, J., & Buss, M. (2005). Invariance control design for constrained nonlinear systems. In *Proc. 16th IFAC world congr.*. Prague.

Muhammad Zakiyullah Romdlony was born in Tasikmalaya, Indonesia in 1986. He obtained his B.Sc. and M.Sc. degrees in Electrical Engineering from the Institut Teknologi Bandung, Indonesia in 2009 and 2012, respectively. Currently, he is a Ph.D. candidate at Discrete Technology and Production Automation, University of Groningen, The Netherlands. His research interest includes control of nonlinear systems.

Bayu Jayawardhana received bachelor degree in Electrical and Electronics Engineering from the Institut Teknologi Bandung, Indonesia in 2000, M.Eng. degree in Electrical and Electronics Engineering from the Nanyang Technological University, Singapore in 2003 and Ph.D. degree in Electrical and Electronics Engineering from Imperial College London, United Kingdom in 2006. He was with Bath University (Bath, United Kingdom) and with Manchester Interdisciplinary Biocentre, University of Manchester (Manchester, United Kingdom).

Currently he is a university reader in the Faculty of Mathematics and Natural Sciences at University of Groningen, Netherlands. His research interests are on the analysis of nonlinear systems, systems with hysteresis, mechatronics, systems biology and micro-nanotechnology. He is a senior member of IEEE, a subject editor of International Journal of Robust and Nonlinear Control, an associate editor of European Journal of Control, a member of the Conference Editorial Board of the IEEE Control Systems Society and chair of the IEEE CSS TC on Systems Biology.