Approximating the Frequency Response of Contractive Systems

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Abstract

We consider contractive systems whose trajectories evolve on a compact and convex state-space. It is well-known that if the time-varying vector field of the system is periodic then the system admits a unique globally asymptotically stable periodic solution. Obtaining explicit information on this periodic solution and its dependence on various parameters is important both theoretically and in numerous applications. We develop an approach for approximating such a periodic trajectory using the periodic trajectory of a simpler system (e.g. an LTI system). Our approximation includes an error bound that is based on the input-to-state stability property of contractive systems. We show that in some cases this error bound can be computed explicitly. We also use the bound to derive a new theoretical result, namely, that a contractive system with an additive periodic input behaves like a low pass filter. We demonstrate our results using several examples from systems biology.

1 Introduction

A dynamical system is called *contractive* if any two trajectories approach each other [1, 2]. This is a strong property with many important implications. For example, if the trajectories evolve on a compact and convex state-space Ω then the system admits an equilibrium point $e \in \Omega$, and since every trajectory converges to the trajectory emanating from e, e is globally asymptotically stable. Note that establishing this does not require an explicit description of e.

More generally, contractive systems with a periodic excitation *entrain*, that is, their trajectories converge to a periodic solution with the same period as the excitation. This property is very important in applications ranging from entrainment of biological systems to periodic excitations (e.g., the 24h solar day or the periodic cell-cycle division program) to the entrainment of synchronous generators to the frequency of the electric grid. However, the proof of the entrainment property of contractive systems is based on implicit arguments (see, e.g. [3]) and provides no explicit information on the periodic trajectory (except for its period).

Contraction theory has found numerous applications in systems and control theory, systems biology [4], and more (see e.g. the recent survey [2]). A particularly interesting line of research is based on combining contraction theory and graph theory in order to study various networks of multi-agent systems (see, e.g. [5, 6, 7, 8]).

As already noted by Desoer and Haneda [9], contractive systems satisfy a special case of the input-to-state stability (ISS) property (see the survey paper [10]). Desoer and Haneda used this to derive bounds on the error between trajectories of a continuous-time contractive system and its time-discretized model. This is important when computing solutions of contractive systems using numerical integration methods [11]. Sontag [12] has shown that contractive systems satisfy a "converging-input converging output" property. A recent paper [13] used the ISS property to derive a bound on the error between trajectories of a continuous-time contractive system and those of some "simpler" continuous-time system (e.g. an LTI system). This bound is particularly useful when the simpler model can be solved explicitly.

Here, we derive new bounds on the distance between the periodic trajectory of a contractive system and the periodic trajectory of a "simpler" system, e.g. an LTI system with a periodic forcing. We show

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several cases where the periodic trajectory of the simpler system is explicitly known and the bound is also explicit, so this provides considerable information on the unknown periodic trajectory of the contractive system. The explicit bounds also pave the way for new theoretical results. We demonstrate this by using one of the bounds to prove that any contractive system with an additive sinusoidal forcing behaves like a low-pass filter, that is, as the frequency of the sinusoidal signal goes to infinity the corresponding solution of the system converges to an equilibrium state. This generalizes the well-known behavior of asymptotically stable LTI systems.

The remainder of this paper is organized as follows. The next section reviews some properties of contractive systems and in particular their ISS property. For more details, including the historic development of contraction theory, see e.g. [14, 15]. The next three sections describe our main results. Section 3 develops a bound for the difference between the periodic trajectories of two systems: a contractive system and some simpler "approximating" system. We show using an example that in general this bound cannot be improved. Section 4 suggests two possible approximating systems for the case of a contractive system with a periodic forcing. Section 5 shows how the explicit bounds can be used to derive a new theoretical result on the frequency response of contractive systems. The final section concludes and describes possible directions for further research.

2 Preliminaries

Consider the time-varying dynamical system

$$\dot{x}(t) = f(t, x(t)), \tag{1}$$

with the state x evolving on a positively invariant convex set $\Omega \subseteq \mathbb{R}^n$. We assume that $f(t,x)$ is differentiable with respect to x, and that both $f(t,x)$ and its Jacobian $J(t,x) := \frac{\partial f}{\partial x}(t,x)$ are continuous in (t,x). Let $x(t,t_0,x_0)$ denote the solution of (1) at time $t \geq t_0$ for the initial condition $x(t_0) = x_0$. For the sake of simplicity, we assume from here on that $x(t,t_0,x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$.

The system (1) is said to be *contractive* on Ω with respect to a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $\eta > 0$ such that

$$|x(t,t_0,a) - x(t,t_0,b)| \leq e^{-(t-t_0)\eta}|a-b| \tag{2}$$

for all $t \geq t_0 \geq 0$ and all $a, b \in \Omega$. This means that any two trajectories approach one another at an exponential rate η. This implies in particular that the initial condition is "quickly forgotten".

Note that contraction can be defined in a more general way, for example with respect to a time- and space-varying norm [1] (see also [16]). We focus here on exponential contraction with respect to a *fixed* vector norm because there exist easy to check sufficient conditions, based on matrix measures, guaranteeing that (2) holds. A vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a *matrix measure* $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

$$\mu(A) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(||I + \varepsilon A|| - 1),$$

where $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by $|\cdot|$. For example, for the ℓ_1 vector norm, denoted $|\cdot|_1$, the induced matrix norm is the maximum absolute column sum of the matrix, and the induced matrix measure is

$$\mu_1(A) = \max\{c_1(A), \dots, c_n(A)\},$$

where

$$c_j(A) := A_{jj} + \sum_{\substack{1\leq i \leq n \\ i \neq j}} |A_{ij}|,$$

i.e., the sum of the entries in column j of A, with non-diagonal elements replaced by their absolute values. Matrix measures satisfy several useful properties (see, e.g. [17, 9]). We list here two properties that will be used later on:

$$\begin{aligned} \mu(A+B) &\leq \mu(A) + \mu(B), && \text{(subadditivity)}, \\ \mu(cA) &= c\mu(A) \text{ for all } c \geq 0, && \text{(homogeneity)}. \end{aligned}$$

If the Jacobian of f satisfies

$$\mu(J(t,x)) \le -\eta, \quad \text{for all } x \in \Omega \text{ and all } t \ge t_0 \ge 0, \tag{3}$$

then (2) holds (see [3] for a self-contained proof). This is in fact a particular case of using a Lyapunov-Finsler function to prove contraction [16]. We will focus on the case where $\eta > 0$, but some of our results hold when $\eta \le 0$ as well. In this case, (2) provides a bound on how quickly can trajectories of (1) separate from one another.

Often it is useful to work with scaled vector norms (see, e.g. [18, 19]). Let $|\cdot|_* : \mathbb{R}^n \to \mathbb{R}_+$ be some vector norm, and let $\mu_* : \mathbb{R}^{n\times n} \to \mathbb{R}$ denote its induced matrix measure. If $D \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $|\cdot|_{*,D} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,D} := |Dz|_*$, then the induced matrix measure is $\mu_{*,D}(A) = \mu_*(DAD^{-1})$. For example, the matrix measure induced by the Euclidean norm $|\cdot|_2$ is $\mu_2(A) = \max\{\lambda : \lambda \text{ is an eigenvalue of } (A + A')/2\}$, so

$$\begin{aligned} \mu_{2,D}(A) &= \mu_2(DAD^{-1}) \\ &= \max\{\lambda : \lambda \text{ is an eigenvalue of } (DAD^{-1} + (DAD^{-1})')/2\}. \end{aligned} \tag{4}$$

The next result describes an ISS property of contractive systems with an additive input.

Theorem 1 *[9] Consider the system*

$$\dot{x}(t) = f(t, x(t)) + u(t), \tag{5}$$

where $y \to f(t,y)$ is C^1 for all $t \ge t_0$. Fix some vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ and suppose that (3) holds for the induced matrix measure $\mu(\cdot)$. Then the solution of (5) with $x(t_0) = x_0$ satisfies

$$|x(t, t_0, x_0)| \le e^{-\eta(t-t_0)}|x_0| + \int_{t_0}^{t} e^{-\eta(t-s)}|u(s)|\, ds$$

for all $t \ge t_0$.

Ref. [13] has applied the ISS property to derive a bound on the error between trajectories of the contractive system (1) and those of a "simpler" dynamical system $\dot{y} = g(t, y(t))$. For such a system, pick $y_0 \in \Omega$, and let $\tau \ge t_0$ be such that the solution $y(t, t_0, y_0)$ belongs to Ω for all $t \in [t_0, \tau]$. Then the difference between the trajectories of the two systems $d(t) := x(t, t_0, x_0) - y(t, t_0, y_0)$ satisfies

$$|d(t)| \le e^{-\eta(t-t_0)}|x_0 - y_0| + \int_{t_0}^{t} e^{-\eta(t-s)}|f(s, y(s, t_0, y_0)) - g(s, y(s, t_0, y_0))|\, \mathrm{d}s \tag{6}$$

for all $t \in [t_0, \tau]$. The proof of this result is based on noting that

$$\begin{aligned} \dot{d}(t) &= f(t, x(t)) - f(t, y(t)) + f(t, y(t)) - g(t, y(t)) \\ &= M(t)d + u(t), \end{aligned}$$

where $M(t) := \int_0^1 J(t, sx(t) + (1-s)y(t))\, \mathrm{d}s$, and $u(t) := f(t, y(t)) - g(t, y(t))$. Since $y(t) \in \Omega$ for all $t \in [0, \tau]$ and Ω is convex, $sx(t) + (1-s)y(t) \in \Omega$ for all $t \in [0, \tau]$ and all $s \in [0, 1]$. Using (3) and the subadditivity of matrix measures, which, by continuity, extends to integrals yields $\mu(M(t)) \le -\eta$ for all $t \in [0, \tau]$. Summarizing, $\dot{d}(t) = M(t)d(t) + u(t)$ is a contractive system with an additive "disturbance" u and applying the ISS property of contractive systems yields (6).

Note that the integrand in (6) depends on the difference between the vector fields f and g *evaluated along the trajectory of the y system*. This is useful, for example, when the trajectory of the y system is explicitly known.

The applications studied in [13] were contractive systems with time-invariant vector fields approximated by time-invariant LTI systems. Here, we consider a different case, namely, when the vector field $f(t, x)$ is time-varying and T-periodic for some $T > 0$, that is,

$$f(t, z) = f(t + T, z)$$

for all $t \geq t_0$ and all $z \in \Omega$. It is well-known that in this case every trajectory of (1) converges to a unique periodic solution $\gamma(t)$ of (1) with period T (see [3] for a self-contained proof). This entrainment property is very important in applications (see, e.g. [20, 3]). However, the proof of entrainment is based on implicit arguments and provides no information on the properties of the period trajectory (except for its period). Our goal here is to develop a suitable bound for the difference between $\gamma(t)$ and the periodic solution $\kappa(t)$ of some simpler approximating y system, and to suggest suitable approximating systems. We also show that these explicit bounds can be used to derive new theoretical results on the response of contractive systems to a sinusoidal input. The next three sections present our main results.

3 Bounds on the difference between two periodic trajectories

In this section, we consider the T-periodic orbit of a T-periodic contractive system. Theorem 2 below is our main result in this section, and provides a bound on the distance of this periodic orbit to a T-periodic orbit of some approximating system.

Theorem 2 *Consider the system*

$$\dot{x} = f(t, x) \tag{7}$$

whose trajectories evolve on a compact and convex state-space $\Omega \subseteq \mathbb{R}^n$. Suppose that $f(t,x)$ is T-periodic and that $f(t,x)$ and $J(t,x)$ are continuous in (t,x). Let $|\cdot|$ be some vector norm on $\mathbb{R}^n$ and $\mu(\cdot)$ its induced matrix measure, and suppose that $\mu(J(t,x)) \leq -\eta < 0$ for all $t \geq 0$ and all $x \in \Omega$. Let $\gamma(t)$ be the unique periodic trajectory of (7) with period T. Consider another time-varying system

$$\dot{y} = g(t, y) \tag{8}$$

and suppose that $g(t,y)$ is also T-periodic and that $\kappa(t)$ is a T-periodic trajectory of (8) with $\kappa(t) \in \Omega$ for all $t \in [0,T]$. Define $c : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$c(\alpha) := \int_0^\alpha e^{-\eta(\alpha - s)} |f(s, \kappa(s)) - g(s, \kappa(s))| \, \mathrm{d}s. \tag{9}$$

Then the difference between the two periodic trajectories satisfies

$$|\gamma(\tau) - \kappa(\tau)| \leq \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} c(T) + c(\tau) \tag{10}$$

for all $\tau \in [0,T]$.

Note that the bound here depends on the difference between the vector fields f and g evaluated along the periodic trajectory $\kappa(s)$ of the "simpler" y system. This is useful for example when the y system is an asymptotically stable LTI system with a sinusoidal forcing term, as then $\kappa(t)$ is known explicitly.

Proof. Define the T-periodic function $u(t) := f(t, \kappa(t)) - g(t, \kappa(t))$. For any $t \geq 0$, Theorem 1 gives

$$|\gamma(t) - \kappa(t)| \leq e^{-\eta t} |\gamma(0) - \kappa(0)| + \int_0^t e^{-\eta(t-s)} |u(s)| \, \mathrm{d}s. \tag{11}$$

There exist $\tau \in [0, T)$ and a non-negative integer k such that $t = kT + \tau$. Observe that $\gamma(t) - \kappa(t) = \gamma(\tau) - \kappa(\tau)$. Write the integral on the right-hand side of (11) as

$$\begin{aligned}
\int_0^t e^{-\eta(t-s)} |u(s)| \, \mathrm{d}s &= e^{-\eta\tau} \int_0^{kT+\tau} e^{-\eta(kT-s)} |u(s)| \, \mathrm{d}s \\
&= e^{-\eta\tau} \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} e^{-\eta(kT-s)} |u(s)| \, \mathrm{d}s + \int_0^\tau e^{-\eta(\tau-p)} |u(p)| \, \mathrm{d}p.
\end{aligned}$$

Moreover,

$$\sum_{i=0}^{k-1}\int_{iT}^{(i+1)T} e^{-\eta(kT-s)}|u(s)|\,\mathrm{d}s = c(T)\frac{1-e^{-\eta k}}{1-e^{-\eta T}}$$

so that

$$|\gamma(\tau)-\kappa(\tau)| = |\gamma(t)-\kappa(t)| \leq e^{-\eta(kT+\tau)}|\gamma(0)-\kappa(0)| + e^{-\eta\tau}c(T)\frac{1-e^{-k\eta T}}{1-e^{-\eta T}} + c(\tau)$$

for all $\tau \in [0,T)$. Taking $k \to \infty$ completes the proof. ■

Note that the bound is actually based on taking the time $t \to \infty$. This is possible because we are considering the difference between two periodic trajectories.

The next example is important, as it shows that in general the bound (10) cannot be improved.

Example 1 Consider the scalar system

$$\dot{x} = f(t,x) := -x + 1 + \sin(2\pi t/T), \tag{12}$$

with $T > 0$. Note that $\Omega := [0,2]$ is an invariant set of this dynamics, and that f is T-periodic. The Jacobian of f is $J(x) = -1$, so for any vector norm the induced matrix measure satisfies $\mu(J(x)) = -1$. For any initial condition the solution of (12) converges to the T-periodic trajectory:

$$\gamma(t) = 1 + \frac{T^2\sin(2\pi t/T) - 2\pi T\cos(2\pi t/T)}{4\pi^2+T^2}. \tag{13}$$

Consider the approximating system $\dot{y} = -y$, which is (vacuously) T-periodic, and admits the T-periodic solution $\kappa(t) \equiv 0$, that belongs to Ω for all t. In this case, (9) yields

$$\begin{aligned} c(\alpha) &= \int_0^\alpha e^{-(\alpha-s)}|1+\sin(2\pi s/T)|\,\mathrm{d}s \\ &= 1 - e^{-\alpha} + \frac{2\pi T e^{-\alpha} - 2\pi T\cos(2\pi\alpha/T) + T^2\sin(2\pi\alpha/T)}{4\pi^2+T^2}. \end{aligned}$$

Thus for large values of α,

$$c(\alpha) \approx \frac{-2\pi T\cos(2\pi\alpha/T) + T^2\sin(2\pi\alpha/T)}{4\pi^2+T^2} + 1. \tag{14}$$

Now consider the case where $T \to \infty$ and $\tau = T - \varepsilon$, with $\varepsilon > 0$ and very small. Then (13) implies that the term on the left-hand side of (10) is

$$|\gamma(\tau)| \approx 1 + \sin(2\pi(T-\varepsilon)/T),$$

whereas (14) implies that the term on the right-hand side of (10) is

$$\begin{aligned} \frac{e^{-\eta\tau}}{1-e^{-\eta T}}c(T) + c(\tau) &\approx c(\tau) \\ &\approx 1 + \sin(2\pi(T-\varepsilon)/T). \end{aligned}$$

Thus, this example shows that in general the bound (10) cannot be improved. □

We now derive a simpler (and less tight) bound. By the definition of $c(\cdot)$,

$$c(\alpha) \leq \frac{1-e^{-\eta\alpha}}{\eta}\max_{t\in[0,\alpha]}|f(t,\kappa(t)) - g(t,\kappa(t))|.$$

for all $\alpha \geq 0$, and combining this with (10) yields the following result.

Corollary 1 *Under the hypotheses of Theorem 2,*

$$|\gamma(\tau) - \kappa(\tau)| \leq \frac{1}{\eta} \max_{t\in[0,T]} |f(t,\kappa(t)) - g(t,\kappa(t))|$$

for all $\tau \geq 0$.

This bound is useful in cases where one can establish a bound on the difference between the vector fields f and g along the periodic trajectory κ of the approximating system. Note that the bound here demonstrates a clear tradeoff: if g is "close" to f then the error $f - g$ will be small, yet κ may be an unknown complicated trajectory (as we assume that f is a nonlinear vector filed). On the other hand, if g is relatively simple (e.g., the vector field of an LTI system) then κ may be known explicitly yet that difference $|f - g|$ may be large.

To summarize, Theorem 2 and Corollary 1 provide a bound on the distance of the unique T-periodic trajectory of a contractive system and some T-periodic trajectory of an approximating system. The next step is to determine a suitable approximating system. We propose two natural approximating systems for the case where the periodic vector field arises via a periodic forcing function. The first approximating system considers the time-averaged periodic forcing function to arrive at an autonomous dynamical system with a unique equilibrium. The second approximating system results from a linearization of the dynamics, keeping the periodic excitation as is.

4 Approximating Systems

From hereon, we consider a special case of the contractive system (7) with the form

$$\dot{x}(t) = f(t, x(t)) = F(x(t), u(t))$$

where $u(t)$ is a given m-dimensional, T-periodic excitation.

4.1 Averaging the input

Our first result is based on using a "simpler" y system derived by averaging the excitation u over a period. The excitation in the y system is thus constant. We assume that the y system admits an equilibrium point $e \in \Omega$, and apply Theorem 2 to derive a bound on the distance between the periodic trajectory $\gamma(t)$ of the original x system and the point e.

Theorem 3 *Consider the system*

$$\dot{x} = F(x, u), \tag{15}$$

where u is an m-dimensional periodic excitation with period $T \geq 0$. Suppose that the trajectories of (15) evolve on a compact and convex state space $\Omega \subset \mathbb{R}^n$. Assume that for some vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ and induced matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$,

$$\mu\left(\frac{\partial F}{\partial x}(x, u(t))\right) \leq -\eta < 0$$

for all $t \geq 0$ and all $x \in \Omega$. Let $\gamma(t)$ be the unique, attracting, T-periodic orbit of (15) in Ω. Then, for any $z \in \Omega$,

$$|x(t, 0, z) - z| \leq \int_0^t e^{-\eta(t-s)} |F(z, u(s))| \,\mathrm{d}s \tag{16}$$

for all $t \geq 0$. In particular, for all $t \geq 0$,

$$|x(t, 0, z) - z| \leq (1 - e^{-\eta t})c/\eta,$$

where $c := \max_{t\in[0,T]} |F(z, u(t))|$. Moreover, for all $\tau \in [0, T]$,

$$\begin{aligned} |\gamma(\tau) - z| &\leq \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |F(z, u(s))| \,\mathrm{d}s + \int_0^\tau e^{-\eta(\tau-s)} |F(z, u(s))| \,\mathrm{d}s & (17) \\ &\leq c/\eta. & (18) \end{aligned}$$

Proof. Define the approximating system $\dot{y} = G(y) \equiv 0$ so that any $z \in \Omega$ is an equilibrium. Then (16) follows from the bound (6), and the bounds (17) and (18) follow from Theorem 2 and Corollary 1. ■

The following simple example demonstrates a case where the bounds in Theorem 3 are tight.

Example 2 Consider the scalar system $\dot{x} = F(x, u) := -ax + b$ with $a > 0$. Then $\gamma(t) \equiv b/a =: e$ is a periodic trajectory. This system is contracting with rate $\eta = a$. The bound (18) gives $|e-z| \leq |-az+b|/a = |e - z|$ for any $z \in \mathbb{R}$ so that this bound is tight. □

The next two examples demonstrate that a natural choice for z in Theorem 3 is the equilibrium point induced by the average of the periodic excitation.

Example 3 Our focus here is on nonlinear dynamical systems, but it is still useful to begin by considering the linear system

$$\dot{x} = Ax + Bu, \tag{19}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $B \in \mathbb{R}^{n\times m}$, and u is an m-dimensional T-periodic control. It is well-known that such a system is contractive [2, 21]. For the sake of completeness we repeat the argument here. We use the notation $Q > 0$ to denote that a matrix Q is symmetric and positive-definite. Since A is Hurwitz, there exist $\eta > 0$ and $Q > 0$ such that

$$QA + A'Q \leq -2\eta Q. \tag{20}$$

Let $P > 0$ be a matrix such that $P^2 = Q$. Then multiplying (20) by P^{-1} on the left and on the right yields

$$PAP^{-1} + P^{-1}A'P \leq -2\eta I. \tag{21}$$

This means that the Jacobian A of (19) satisfies $\mu_{2,P}(A) \leq -\eta$, where $\mu_{2,P}$ is the matrix measure induced by the scaled Euclidean norm $|z|_{2,P} := |Pz|_2$ (see (4)). Thus, (19) is contractive with respect to this scaled norm with contraction rate η, and every solution of (19) converges to the unique T-periodic solution $\gamma(t)$ of (19). Let $\bar{u} := \frac{1}{T}\int_0^T u(s)ds$ and choose $z = A^{-1}B\bar{u} =: e$, the equilibrium of the time-invariant system with input equal to $\bar{u}$. To apply the bound (18), note that

$$F(e, u(s)) = Ae + Bu(s) = B(u(s) - \bar{u}).$$

Thus, $|F(e, u(s))|_{2,P} = ((u(s) - \bar{u})'B'P'PB(u(s) - \bar{u}))^{1/2}$, and the bound (18) yields

$$|\gamma(\tau) - e|_{2,P} \leq \frac{1}{\eta} \max_{t\in[0,T]} ((u(t) - \bar{u})'B'P'PB(u(t) - \bar{u}))^{1/2} \tag{22}$$

for all $\tau \in [0, T]$.

Of course, for linear systems the periodic solution corresponding to sinusoidal excitations is known explicitly in terms of the system's frequency response. Nevertheless, (22) seems to be new and provides considerable intuition: the bound on the distance between $\gamma(t)$ and e decreases when: the contraction rate η increases; the input channel B becomes "more orthogonal" to the matrix P in (21); or $\max_{t\in[0,T]} |u(t) - \bar{u}|$ decreases, that is, the periodic excitation becomes more similar to its mean. □

The next example demonstrates an application of Theorem 3 for a nonlinear contractive system.

Example 4 The ribosome flow model (RFM) [22] is a nonlinear compartmental model describing the unidirectional flow of particles along a 1D chain of n sites using n non-linear first-order differential equations:

$$\begin{aligned}
\dot{x}_1 &= \lambda_0(1 - x_1) - \lambda_1 x_1(1 - x_2),\\
\dot{x}_2 &= \lambda_1 x_1(1 - x_2) - \lambda_2 x_2(1 - x_3),\\
\dot{x}_3 &= \lambda_2 x_2(1 - x_3) - \lambda_3 x_3(1 - x_4),\\
&\vdots\\
\dot{x}_{n-1} &= \lambda_{n-2} x_{n-2}(1 - x_{n-1}) - \lambda_{n-1} x_{n-1}(1 - x_n),\\
\dot{x}_n &= \lambda_{n-1} x_{n-1}(1 - x_n) - \lambda_n x_n.
\end{aligned} \tag{23}$$

Here $x_i(t) \in [0,1]$ represents the level of occupancy of site i at time t, normalized such that $x_i(t) = 1$ [$x_i(t) = 0$] means that site i is completely full [empty]. The state-space is thus $[0,1]^n$, and this is an invariant set of (23) (see [20]). The transition rate $\lambda_i > 0$ controls the flow from site i to site $i+1$, with λ_0 [λ_n] called the initiation [exit] rate. To understand these equations, note that they may be written as $\dot{x}_i = g_{i-1}(x) - g_i(x)$, where $g_k(x)$ is the flow from site k to site $k+1$ at time t. This flow increases with x_k and decreases with x_{k+1}. In other words, the flow satisfies a "soft" excluded volume principle: as site $k+1$ becomes fuller the flow from site k to site $k+1$ decreases. This models the fact that the particles have volume and thus cannot overtake one another. The rate at which particles leave the chain, that is, $R(t) := \lambda_n x_n$ is called the *production rate*.

The RFM with $n > 2$ is actually not contractive in the sense defined above on $[0,1]^n$, as there exists $p \in [0,1]^n$ such that $J(p)$ is singular, but it is "weakly" contractive in a well-defined sense; see [23, 24].

Recently, the RFM has been used to model and analyze the flow of ribosomes (the particles) along groups of codons (the sites) along the mRNA molecule during translation (see, e.g. [25, 26, 27, 20, 28, 29, 30, 31, 32]). In this case, every ribosome that leaves the chain releases the produced protein, so $R(t)$ is the protein production rate at time t. The values of the transition rates depend on various biophysical properties, e.g. the abundance of tRNA molecules that carry the corresponding amino-acids.

Consider the RFM with $n = 2$ and a *time-varying* initiation rate $u_0(t)$, that is,

$$
\begin{aligned}
\dot{x}_1 &= (1 - x_1)u_0 - \lambda_1 x_1(1 - x_2), \\
\dot{x}_2 &= \lambda_1 x_1(1 - x_2) - \lambda_2 x_2,
\end{aligned} \tag{24}
$$

where λ_1, λ_2 are positive constants. Suppose that $u_0(t) = \lambda_0 + \sin(2\pi t/T)$, with $\lambda_0 > 1$, $T > 0$, i.e. the initiation rate is a strictly positive periodic function with (minimal) period T. The state space here is $\Omega := [0,1]^2$. The Jacobian of (24) is

$$
J(t,x) = \begin{bmatrix} -u_0(t) - \lambda_1(1 - x_2) & \lambda_1 x_1 \\ \lambda_1(1 - x_2) & -\lambda_1 x_1 - \lambda_2 \end{bmatrix}.
$$

The off-diagonal terms are non-negative for any $x \in [0,1]^2$, so $\mu_1(J(t,x)) = \max\{-u_0(t), -\lambda_2\}$ for all $t \geq 0$ and all $x \in [0,1]^2$. Thus, the system is contractive with respect to the ℓ_1 norm with contraction rate $\eta := \min\{\lambda_0 - 1, \lambda_2\} > 0$. This means that it admits a unique periodic solution $\gamma \in [0,1]^2$, with period T, and that every solution converges to γ. Entrainment in mRNA translation is important as biological organisms are often exposed to periodic excitations, for example the periodic cell-cycle division process. Proper biological functioning requires entrainment to such excitations [20].

Let $\bar{u}_0 = \frac{1}{T}\int_0^T u_0(s)\,\mathrm{d}s = \lambda_0$ and consider the system

$$
\begin{aligned}
\dot{y}_1 &= \lambda_0(1 - y_1) - \lambda_1 y_1(1 - y_2), \\
\dot{y}_2 &= \lambda_1 y_1(1 - y_2) - \lambda_2 y_2.
\end{aligned} \tag{25}
$$

This system admits an equilibrium point

$$
e = \begin{bmatrix} \frac{\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2 + \sqrt{d}}{2\lambda_0\lambda_1} & \frac{\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2 - \sqrt{d}}{2\lambda_1\lambda_2} \end{bmatrix}' \in (0,1)^2, \tag{26}
$$

where $d := 4\lambda_0^2\lambda_1\lambda_2 + (\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2)^2$.

Here,

$$
F(e, u(s)) = \begin{bmatrix} (\lambda_0 + \sin(2\pi s/T))(1 - e_1) - \lambda_1 e_1(1 - e_2) \\ \lambda_1 e_1(1 - e_2) - \lambda_2 e_2 \end{bmatrix},
$$

and since e is an equilibrium point of (25), $F(e, u(s)) = \begin{bmatrix}(1 - e_1)\sin(2\pi s/T) & 0\end{bmatrix}'$. Thus, the bound (16) yields

$$
|x(t, 0, e) - e|_1 \leq (1 - e_1) \int_0^t e^{-\eta(t-s)} |\sin(2\pi s/T)| \,\mathrm{d}s. \tag{27}
$$

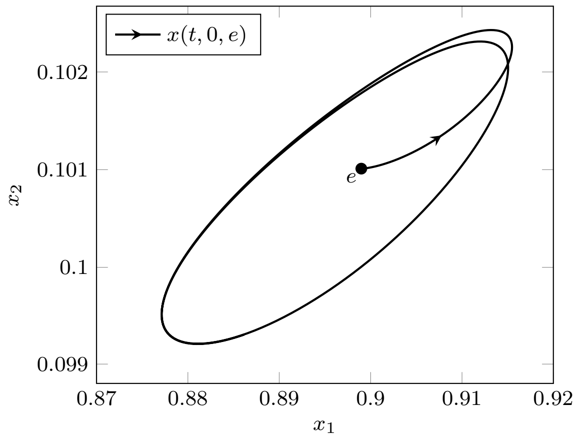

Figure 1: RFM in Example 4: Averaging the periodic excitation leads to an autonomous approximating system with a unique equilibrium e. For the periodic excitation the trajectory $x(t, 0, e)$ converges to the unique periodic solution $\gamma(t)$ of the RFM, and Theorem 1 provides a bound on the distance between $x(t, 0, e)$ and e for all $t \geq 0$.

Likewise, (17) implies

$$|\gamma(\tau) - e|_1 \leq (1 - e_1)\frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |\sin(2\pi s/T)| \, \mathrm{d}s + (1 - e_1) \int_0^\tau e^{-\eta(\tau-s)} |\sin(2\pi s/T)| \, \mathrm{d}s \quad (28)$$

for all τ. Furthermore,

$$|F(e, u(t))|_1 = (1 - e_1)|\sin(2\pi t/T)| \leq 1 - e_1$$

so (17) implies the simpler yet more conservative bound

$$|\gamma(\tau) - e|_1 \leq (1 - e_1)/\eta \quad (29)$$

for all τ.

Note that all the bounds above can be computed *explicitly*. For example, a tedious yet straightforward calculation of the integrals in (28) yields

$$|\gamma(\tau) - e|_1 \leq \frac{2\pi T(1 - e_1)\coth(\eta T/4)}{e^{\eta\tau}(4\pi^2 + \eta^2 T^2)} + \frac{T(1 - e_1)}{4\pi^2 + \eta^2 T^2} r(\tau),$$

where

$$r(\tau) := \begin{cases} 2\pi e^{-\eta\tau} + \eta T \sin(2\pi\tau/T) - 2\pi\cos(2\pi\tau/T) & \text{if } 0 \leq \tau < T/2, \\ 2\pi e^{-\eta\tau}(1 + 2e^{\eta T/2}) - \eta T \sin(2\pi\tau/T) + 2\pi\cos(2\pi\tau/T) & \text{if } T/2 \leq \tau < T. \end{cases}$$

Summarizing, in this example, we have an analytical expression both for e and for the error bounds. Taken together, this provides considerable explicit information on the periodic trajectory γ.

For the case $\lambda_0 = 4$, $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$, Fig. 1 depicts the periodic trajectory of (24) and the equilibrium point e of (25), and Fig. 2 depicts the error $|x(t, 0, e) - e|_1$ and the bound (27). In this case, (26) yields $e \approx \begin{bmatrix} 0.8990 & 0.1010 \end{bmatrix}'$ (all numerical values in this note are to four digit accuracy). Fig. 3 illustrates the other bounds on the periodic trajectory. It may be seen that these bounds indeed provide a reasonable approximation for the ℓ_1 distance between the unknown periodic trajectory and the point e. □

Thm. 3 is based on averaging the excitation over a period, thus obtaining a constant input. Such an approximation is not always suitable. For example, when $u(t) = \sin(2\pi t/T)$ then $\bar{u} := \frac{1}{T}\int_0^T u(t)dt = 0$ for all T. This may obscure the effect of the *frequency* of the excitation in the derived bounds. The approach in the next subsection tries to overcome this using a different approximating system, namely, an LTI system that is excited by the original periodic input.

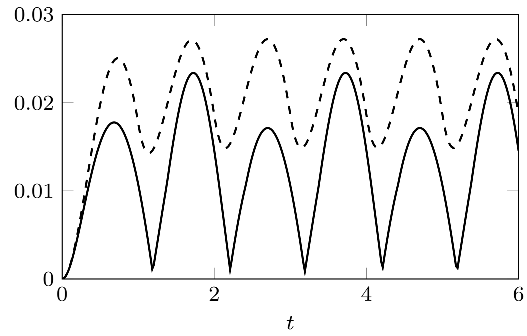

Figure 2: The error $|x(t,0,e)-e|_1$ (solid line) and the bound provided by (27) (dashed line) as a function of time for Example 4.

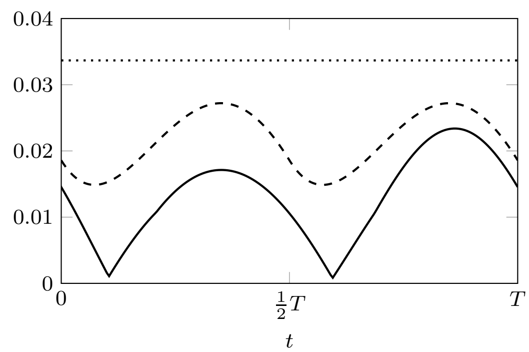

Figure 3: RFM in Example 4. The error $|\gamma(t)-e|_1$ (solid line) and the bounds in Theorem 3: the bound (28) (dashed line) and the bound (29) (dotted line). These bounds can be obtained analytically for this example.

4.2 An LTI approximation

Theorem 4 *Consider the system*

$$\dot{x} = F(x,u), \tag{30}$$

where u is an m-dimensional periodic excitation with period $T>0$. Suppose that the trajectories of (30) evolve on a compact and convex state space $\Omega \subset \mathbb{R}^n$. Assume that for some vector norm $|\cdot|:\mathbb{R}^n \to \mathbb{R}_+$ and the induced matrix measure $\mu:\mathbb{R}^{n\times n}\to\mathbb{R}$,

$$\mu\left(\frac{\partial F}{\partial x}(x,u(t))\right) \le -\eta < 0$$

for all $t \ge 0$, all $x \in \Omega$. Let $\gamma(t)$ be the unique, attracting, T-periodic orbit of (30) in Ω.

Suppose also that for the unforced dynamics, i.e. $\dot{x} = F(x,0)$, there exists a locally stable equilibrium point $e \in \Omega$, and without loss of generality, that $e = 0$. Let

$$A := \frac{\partial F}{\partial x}(0,0), \qquad B := \frac{\partial F}{\partial u}(0,0),$$

and consider the LTI approximating system

$$\dot{y} = Ay + Bu := G(y,u). \tag{31}$$

Pick $x_0, y_0 \in \Omega$ and let $\tau \ge 0$ be such that $y(t) \in \Omega$ for all $t \in [0,\tau]$ where $y(t)$ is the solution of (31) with

$y(0) = y_0$. *Then*

$$|x(t) - y(t)| \le e^{-\eta t}|x_0 - y_0| + \int_0^t e^{-\eta(t-s)}|F(y(s), u(s)) - G(y(s), u(s))|\,\mathrm{d}s$$

for all $t \in [0, \tau]$. *Moreover, let* $\kappa(t)$ *be the unique* T*-periodic trajectory of* (31) *and assume that* $\kappa(t) \in \Omega$ *for all* t. *Then, for all* $\tau \in [0, T]$,

$$\begin{aligned}|\gamma(\tau) - \kappa(\tau)| &\le \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)}|F(\kappa(s), u(s)) - G(\kappa(s), u(s))|\,\mathrm{d}s \\ &\quad + \int_0^\tau e^{-\eta(\tau-s)}|F(\kappa(s), u(s)) - G(\kappa(s), u(s))|\,\mathrm{d}s \\ &\le \frac{1}{\eta}\max_{t\in[0,T]} |F(\kappa(t), u(t)) - G(\kappa(t), u(t))|.\end{aligned} \tag{32}$$

We emphasize again that the advantage of the bounds here is that the integrand depends on the difference between the vector fields F and G evaluated along the solution κ of the LTI system (31). Note that our assumptions imply that A is Hurwitz and thus, for any initial condition, $y(t)$ converges to the periodic trajectory $\kappa(t)$. In some cases, this solution can be written explicitly, and the integral can be computed explicitly. For example, if $u(t)$ is a complex exponential, then $\kappa(t)$ is also a complex exponential and can be easily computed using a Fourier transform. Then a bound on $|F(\kappa(t), u(t)) - G(\kappa(t), u(t))|$, $t \in [0, T]$, may be straightforward to establish. This leads to the following corollary of Theorem 4. For the sake of simplicity, we state this for the case of a scalar control.

Corollary 2 *Consider the system* (30) *where* $u(t) = \sum_{i=1}^p a_i \cos(\omega_i t)$ *with* $a_i \in \mathbb{R}$ *and every* ω_i *has the form* $\omega_i = 2\pi k_i/T$, *with* k_i *a non-negative integer. Suppose that the trajectories of* (30) *evolve on a compact and convex state space* $\Omega \subset \mathbb{R}^n$. *Assume that for some vector norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *the induced matrix measure* $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ *satisfies*

$$\frac{\partial F}{\partial x}\left(x, \sum_{i=1}^p a_i \cos(\omega_i t)\right) \le -\eta < 0$$

for all $t \ge 0$ *and all* $x \in \Omega$. *Let* $\gamma(t)$ *be the unique, attracting,* T*-periodic orbit of* (30) *in* Ω.

Suppose also that the unforced dynamics, i.e. $\dot{x} = F(x, 0)$ *admits a locally stable equilibrium point* $e \in \Omega$, *and without loss of generality, that* $e = 0$. *Let*

$$A := \frac{\partial F}{\partial x}(0, 0), \qquad B := \frac{\partial F}{\partial u}(0, 0),$$

and consider the approximating system

$$\dot{y} = Ay + Bu := G(y, u). \tag{33}$$

Let $\hat{g}(s) := (sI - A)^{-1}b$ *and let* $\kappa(t)$ *be the unique* T*-periodic trajectory of* (33), *that is,*

$$\kappa_r(t) = \sum_{i=1}^p a_i|\hat{g}_r(j\omega_i)| \cos(\omega_i t + \angle\hat{g}_r(j\omega_i)), \quad r = 1, \ldots, n,$$

and assume that $\kappa(t) \in \Omega$ *for all* $t \in [0, T]$. *Then for all* $\tau \in [0, T]$,

$$|\gamma(\tau) - \kappa(\tau)| \le \frac{1}{\eta}\max_{t\in[0,T]} \left| H\left(\kappa(t), \sum_{i=1}^p a_i \cos(\omega_i t)\right)\right| \tag{34}$$

where $H(z, v) := F(z, v) - G(z, v)$.

Remark 1 *Our focus here is on cases where $\kappa(t)$ is explicitly known. However, the derived bounds are useful even when this is not the case. For example, suppose that $\kappa(t)$ is not known, yet the bounds*

$$C_r := \max_{t\in[0,T]} |\kappa_r(t)|, \quad r = 1, \dots, n,$$
$$C_v := \max_{t\in[0,T]} |u(t)|,$$

are known. Then (34) implies that

$$\begin{aligned} |\gamma(\tau) - \kappa(\tau)| \leq & \quad \frac{1}{\eta} \max_{z,v} \quad |H(z,v)| \\ & \text{subject to } |z_r| \leq C_r, \quad r = 1, \dots, n \\ & \qquad |v| \leq C_v. \end{aligned}$$

This nonlinear optimization program is useful because the feasible set is a box constraint and thus is a convex set.

Example 5 We again consider the RFM with $n = 2$ and the periodic initiation rate $u_0(t) := \lambda_0 + u(t)$, with $\lambda_0 > 1$ and $u(t) = \sin(2\pi t/T)$. Again, let e be the unique equilibrium of the system when the initiation rate is λ_0 (see (26)). Let $\delta x := x - e$. Then the linearized system is $\dot{\delta x} = A\delta x + bu$, where

$$A = \begin{bmatrix} -\lambda_0 - \lambda_1(1-e_2) & \lambda_1 e_1 \\ \lambda_1(1-e_2) & -\lambda_1 e_1 - \lambda_2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 - e_1 \\ 0 \end{bmatrix}.$$

Note that $\mu_1(A) = \max\{-\lambda_0, -\lambda_2\} < 0$, so, in particular, A is Hurwitz. Thus, the approximating system is

$$\dot{y} = A(y - e) + bu =: G(y, u), \quad u(t) = \sin(2\pi t/T). \tag{35}$$

The difference between the vector fields evaluated along a solution of the y system is

$$F(y, \sin(2\pi t/T)) - G(y, \sin(2\pi t/T)) = \begin{bmatrix} \lambda_1(y_1 - e_1)(y_2 - e_2) - (y_1 - e_1)\sin(2\pi t/T) \\ -\lambda_1(y_1 - e_1)(y_2 - e_2) \end{bmatrix}.$$

Let $\hat{g}(s) := \begin{bmatrix} \hat{g}_1(s) \\ \hat{g}_2(s) \end{bmatrix} = (sI - A)^{-1}b$, and let $\kappa(t) : \mathbb{R} \to \mathbb{R}^2$ be the unique periodic trajectory of (35) defined for all $-\infty < t < \infty$. Then

$$\kappa(t) - e = \begin{bmatrix} |\hat{g}_1(j\omega)| \sin(\omega t + \angle \hat{g}_1(j\omega)) \\ |\hat{g}_2(j\omega)| \sin(\omega t + \angle \hat{g}_2(j\omega)) \end{bmatrix},$$

with $\omega := 2\pi/T$. By Remark 1,

$$\begin{aligned} |\gamma(t) - \kappa(t)|_1 \leq & \quad \max_{z_1,z_2,v} \frac{1}{\eta}(|\lambda_1 z_1 z_2 - z_1 v| + |\lambda_1 z_1 z_2|) \\ & \text{subject to } |z_1| \leq |\hat{g}_1(j\omega)| \\ & \qquad |z_2| \leq |\hat{g}_2(j\omega)| \\ & \qquad |v| \leq 1 \\ & = \frac{1}{\eta}\left(2\lambda_1 |\hat{g}_1(j\omega)||\hat{g}_2(j\omega)| + |\hat{g}_1(j\omega)|\right) \end{aligned} \tag{36}$$

where $\eta := \min\{\lambda_0 - 1, \lambda_2\}$ as before. Note that the bound here depends on the frequency of the periodic excitation. The more exact bound in (32) can be computed numerically.

For the parameters $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$, Figure 4 shows the equilibrium point when $\lambda_0 = 4$, the periodic trajectory for the case when the initiation rate is $u_0(t) = 4 + \sin(2\pi t/T)$, and the periodic trajectory of the linearized system. Figure 5 illustrates the bounds from Theorem 4. It may be observed that these bounds provide a reasonable estimate of the error. □

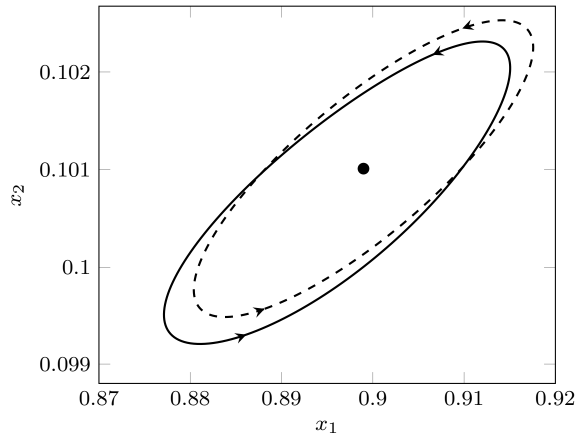

Figure 4: RFM in Example 5. The equilibrium e for $\lambda_0 = 4$ is marked by a dot. The periodic trajectory $\gamma(t)$ of the RFM (solid line) and the periodic trajectory $\kappa(t)$ of the linearized system (dashed line) when $u_0(t) = 4 + \sin(2\pi t/T)$.

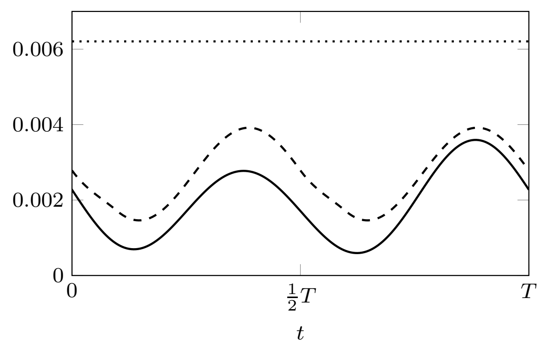

Figure 5: RFM in Example 5. The error $|\gamma(t) - \kappa(t)|_1$ (solid line) and the bounds (32) (dashed lines) and (36) (dotted line).

The bound (36) has some interesting implications. For example, if $\hat{g}_1(j\omega) = 0$ for some ω then (36) implies that $\gamma(t) \equiv \kappa(t)$ for a sinusoidal excitation with frequency ω. Similarly, if $\lim_{\omega\to\infty} \hat{g}_1(j\omega) = 0$ then (36) implies that for a high frequency sinusoidal forcing term, γ will approach κ. Note that the conclusions on γ here are based on *properties of the LTI system*. In the next section, we use this idea to derive a theoretical result on the response of contractive systems to a sinusoidal input.

5 Contractive Systems as Low-Pass Filters

We consider a contractive systems with an additive input and show that for a high-frequency sinusoidal input, the periodic trajectory of the contractive system is very similar to that of a suitable LTI system. For the sake of simplicity, we state this for the case of a scalar control.

Theorem 5 *Consider the system*

$$\dot{x} = f(x) + bu \tag{37}$$

where

$$u(t) = a\cos(\omega t + \phi).$$

Suppose that the trajectories of (37) *evolve on a compact and convex state space* $\Omega \subset \mathbb{R}^n$*. Assume that for some vector norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$*, and induced matrix measure* $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$*,*

$$\mu\left(\frac{\partial f}{\partial x}(x)\right) \le -\eta < 0$$

for all $x \in \Omega$*. Denote* $T := 2\pi/\omega$*, and let* $\gamma(t)$ *be the unique, attracting,* T*-periodic orbit of* (37) *in* Ω*.*

Suppose also that for the unforced dynamics, i.e. $\dot{x} = f(x)$ *there exists a locally stable equilibrium point* $e \in \Omega$*, and without loss of generality, that* $e = 0$*. Let* $A := \frac{\partial f}{\partial x}(0)$*, and consider the approximating system*

$$\dot{y} = Ay + bu := G(y, u). \tag{38}$$

Let $\hat{g}(s) := (sI - A)^{-1}b$ *and let* $\kappa(t)$ *be the unique* T*-periodic trajectory of* (38)*, that is,*

$$\kappa_r(t) = a|\hat{g}_r(j\omega)|\cos(\omega t + \phi + \angle\hat{g}_r(j\omega)), \quad r = 1, \dots, n. \tag{39}$$

Then

$$\max_{t\in[0,T]} |\gamma(t) - \kappa(t)| = o(1/\omega). \tag{40}$$

Proof. It follows from Corollary 2 that for any τ,

$$\begin{aligned}|\gamma(\tau) - \kappa(\tau)| &\le \frac{1}{\eta}\max_{t\in[0,T]} |f(\kappa(t)) - A\kappa(t)| \\ &\le \frac{1}{\eta}\max_{t\in[0,T]} o(|\kappa(t)|).\end{aligned} \tag{41}$$

Since $\hat{g}(s) = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)}b$, where adj denotes the adjugate, (39) implies that $|\kappa_r(t)| = O(1/\omega)$ for all r and all $t \in [0, T]$. Combining this with (41) completes the proof. ∎

The next two examples demonstrate Theorem 5.

Example 6 We consider a basic model for an externally driven transcriptional module that is ubiquitous in both biology and synthetic biology (see, e.g., [33, 3]):

$$\begin{aligned}\dot{x}_1 =& u - \delta x_1 + k_1 x_2 - k_2(e_T - x_2)x_1, \\ \dot{x}_2 =& -k_1 x_2 + k_2(e_T - x_2)x_1,\end{aligned} \tag{42}$$

where δ, k_1, k_2, e_T are strictly positive parameters. Here $x_1(t)$ is the concentration at time t of a transcriptional factor X that regulates a downstream transcriptional module by binding to a promoter with concentration $e(t)$ yielding a protein-promoter complex Y with concentration $x_2(t)$. The binding reaction is reversible with binding and dissociation rates k_2 and k_1, respectively. The linear degradation rate of X is δ, and as the promoter is not subject to decay, its total concentration, e_T, is conserved, so $e(t) = e_T - x_2(t)$ for all $t \ge 0$. The input $u(t)$ might represent for example the concentration of an enzyme or of a second messenger that activates X, so we assume that $u(t) \ge 0$ for all $t \ge 0$.

Trajectories of (42) evolve on $[0, \infty) \times [0, e_T]$. For an input satisfying $0 \le u(t) \le c$ for all $t \ge 0$, the set $\Omega := [0, (c + k_1 e_T)/\delta] \times [0, e_T]$ is a convex and compact invariant set.

Ref. [3] has shown that (42) is contractive with respect to a certain weighted L_1 norm. Indeed, the Jacobian of (42) is

$$J(x) = \begin{bmatrix} -\delta - k_2(e_T - x_2) & k_1 + k_2 x_1 \\ k_2(e_T - x_2) & -k_1 - k_2 x_1 \end{bmatrix},$$

so for $D := \operatorname{diag}(d, 1)$, with $d > 0$,

$$DJ(x)D^{-1} = \begin{bmatrix} -\delta - k_2(e_T - x_2) & (k_1 + k_2 x_1)d \\ k_2(e_T - x_2)/d & -k_1 - k_2 x_1 \end{bmatrix}. \tag{43}$$

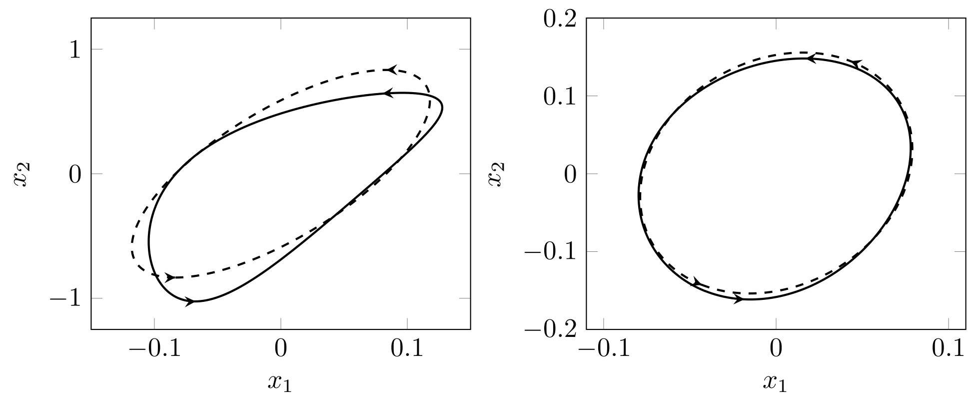

Figure 6: Trajectories γ (solid line) and κ (dashed line) for the system in Example 6 for $\omega = 1$ (top) and $\omega = 5$ (bottom). Note the different scales in the figures.

The off-diagonal terms here are non-negative, and this means that for any $d \in (\frac{k_2 e_T}{k_2 e_T + \delta}, 1)$,

$$\mu_{1,D}(J(x)) \leq -\eta, \text{ for all } \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \Omega,$$

where $\eta := \min\{k_1(1-d), \delta + k_2 e_T(1-d^{-1})\} > 0$. Thus, (42) is contractive with respect to the scaled norm $|\cdot|_{1,D}$ with contraction rate η.

Linearizing (42) yields $\dot{y} = G(y,u) = Ay + bu$, with

$$A := \begin{bmatrix} -\delta - k_2 e_T & k_1 \\ k_2 e_T & -k_1 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$\hat{g}(s) = (sI - A)^{-1} b = \frac{1}{s^2 + (\delta + k_1 + k_2 e_T)s + \delta k_1} \begin{bmatrix} s + k_1 \\ k_1 \end{bmatrix}.$$

Since $f(y) - Ay = k_2 y_1 y_2 \begin{bmatrix} 1 & -1 \end{bmatrix}'$, the bound (41) yields

$$\begin{aligned} |D(\gamma(\tau) - \kappa(\tau))|_1 &\leq \frac{k_2}{\eta} \max_{t \in [0,T]} |\kappa_1(t)\kappa_2(t) \begin{bmatrix} 1 & -1 \end{bmatrix}'|_{1,D} \\ &\leq \frac{k_2(d+1)}{\eta} \max_{t \in [0,T]} |\kappa_1(t)\kappa_2(t)|. \end{aligned} \tag{44}$$

Note that for any input in the form

$$u(t) = \sum_{i=1}^{p} a_i \cos(\omega_i t),$$

the periodic trajectory $\kappa(t)$ is explicitly known and thus the bound (44) is explicit. Figure 6 depicts the trajectories of both the contractive system (42) and of the LTI system for the parameters $k_1 = 1$, $k_2 = 5$, $\delta = 1$, $e_T = 2$, and the excitation $u(t) = \cos(\omega t)$ for two different values of ω.[1] It may be seen that for a larger value of ω the difference between γ and κ decreases, as anticipated by (40). □

The next example demonstrates the result in Theorem 5 using a nonlinear system for which the frequency response has been computed explicitly in [34].

[1]This control is not positive for all times, yet for the initial conditions in the simulations the trajectory remains in a convex and compact region in which the off-diagonal terms in (43) are non-negative and contraction holds.

Example 7 Consider the system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2,\\ \dot{x}_2 &= -x_2 + u,\end{aligned} \tag{45}$$

where the excitation is $u(t) = a\sin(\omega t)$, with $a, \omega > 0$. It is clear that $\Omega_2 := [-a, a]$ is an invariant set of x_2. The Jacobian of (45) is $J(x) = \begin{bmatrix} -1 & 2x_2 \\ 0 & -1 \end{bmatrix}$. For any $c > 0$ and $D := \operatorname{diag}(1, c)$, we have $DJ(x)D^{-1} = \begin{bmatrix} -1 & 2x_2/c \\ 0 & -1 \end{bmatrix}$, so $\mu_1(DJ(x)D^{-1}) \leq -1 + \frac{2a}{c}$ for all $x_2 \in \Omega_2$. If $c > 2a$, then this systems is contractive with respect to the scaled norm $|z|_{1,D} := |Dz|_1$ with contraction rate

$$\eta = 1 - \frac{2a}{c}. \tag{46}$$

Note that, by taking c arbitrarily large, we may obtain a contraction rate arbitrarily close to 1. The periodic trajectory $\gamma(t)$ can be computed explicitly as follows. First, it is clear that

$$\gamma_2(t) = \frac{a}{\sqrt{1+\omega^2}} \sin(\omega t - \tan^{-1}(\omega)),$$

and substituting this in the first equation of (45) yields

$$\gamma_1(t) = M\left[1 + 5\omega^2 + 4\omega^4 + (5\omega^2 - 1)\cos(2\omega t) + 2\omega(\omega^2 - 2)\sin(2\omega t)\right], \tag{47}$$

where $M := \frac{a^2}{2(1+\omega^2)^2(1+4\omega^2)}$.

Note that the unforced dynamics admits an equilibrium $e = 0$. The approximating system is $\dot{y} = G(y, u) = -y + bu$, with $b := \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\hat{g}(s) = (s+1)^{-1}b$. Thus, $\kappa(t) = \begin{bmatrix} 0 & \gamma_2(t) \end{bmatrix}'$, so $\gamma(t) - \kappa(t) = \begin{bmatrix} \gamma_1(t) & 0 \end{bmatrix}'$, and $|\gamma(t) - \kappa(t)|_{1,D} = |\gamma_1(t)|$. To apply Corollary 2 note that $H(z, v) := F(z, v) - G(z, v) = \begin{bmatrix} z_2^2 & 0 \end{bmatrix}'$, so applying the bound (34) gives

$$\begin{aligned}\max_{t\in[0,T]} |\gamma(t) - \kappa(t)|_{1,D} &= \max_{t\in[0,T]} |\gamma_1(t)| \\ &\leq \eta^{-1} \max_{t\in[0,T]} |\kappa_2^2(t)| \\ &= \eta^{-1} \frac{a^2}{1+\omega^2}\end{aligned} \tag{48}$$

where η is given in (46) with $c > 2a$, and $T = 2\pi/\omega$. Taking $c \to \infty$ gives the explicit bound $\max_{t\in[0,T]} |\gamma(t) - \kappa(t)|_{1,D} \leq a^2/(1+\omega^2)$. In fact, it follows from (47), after some calculation, that

$$\max_{t\in[0,T]} |\gamma_1(t)| = \frac{a^2(1 + \sqrt{4\omega^2+1})}{2(1+\omega^2)\sqrt{4\omega^2+1}}. \tag{49}$$

Fig. 7 depicts the exact difference $\max_{t\in[0,T]} |\gamma(t) - \kappa(t)|_{1,D}$ given in (49) and the bound $a^2/(1+\omega^2)$ implied by (48), as a function of ω for $a = 1$. Theorem 5 guarantees $\max_{t\in[0,T]} |\gamma(t) - \kappa(t)| = o(1/\omega)$, as seen in the figure. □

6 Discussion

Contractive systems entrain to periodic excitations. Analyzing the corresponding periodic solution of the contractive system and its dependence on various parameters is an important theoretical question with many potential applications. We developed approximation schemes for this periodic solution using LTI systems and, using the ISS property of contractive systems, provided bounds on the approximation error. An important advantage of these bounds is that in some cases they can be computed explicitly. This also led to a new theoretical result on the behavior of contractive systems for a high frequency excitation.

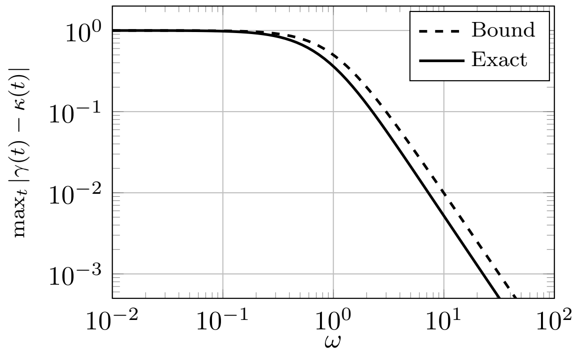

Figure 7: Maximum distance between the exact periodic trajectory $\gamma(t)$ and the approximate periodic trajectory $\kappa(t)$ as a function of the excitation frequency ω for Example 7 with $a = 1$. The solid line is the exact difference, and the dashed line is the bound on the difference determined by Corollary 2.

More generally, it is well-known that contractive systems whose solutions evolve on a compact state-space have a well-defined frequency response [34, 35]. For the contractive system $\dot{x} = F(x, u)$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, this means that there exists a continuous function $\alpha : \mathbb{R}^3 \to \mathbb{R}^n$ such that the following property holds. For the sinusoidal input $u(t) = a\sin(\omega t)$, with frequency $\omega := 2\pi/T$ and amplitude $a \geq 0$, the solution of the contractive system converges to a periodic solution $\gamma_{a\omega}$ satisfying

$$\gamma_{a\omega}(t) = \alpha(a\sin(\omega t), a\cos(\omega t), \omega)$$

(see [34, Theorem 3]). The function $\alpha(v_1, v_2, w)$ is called the *state frequency response*. For the special case of a linear system, i.e. $F(x, u) = Ax + bu$ the state frequency response is known explicitly:

$$\alpha(v_1, v_2, \omega) = \Pi(\omega) \begin{bmatrix} v_1 & v_2 \end{bmatrix}'$$

with $\Pi(\omega) := \begin{bmatrix} \text{Re}(\hat{g}(j\omega)) & \text{Im}(\hat{g}(j\omega)) \end{bmatrix} \in \mathbb{R}^{n\times 2}$, and $\hat{g}(s) := (sI - A)^{-1}b$. That is, for linear systems, the state frequency response recovers the standard notion of frequency response.

Of course, for nonlinear systems it is typically not possible to compute the frequency response analytically. Our results may be interpreted in this context as follows. Considering Theorem 3, we have that $\bar{u} = \frac{1}{T}\int_0^T u(t)dt = 0$ for any a and ω and that $\dot{y} = F(y, 0)$ admits an equilibrium point e. Thus, $e = \alpha(0, 0, \omega)$ (where e is in fact independent of ω), and (17) may be interpreted as providing bounds on

$$|\alpha(a\sin(\omega t), a\cos(\omega t), \omega) - e|.$$

On the other-hand, the results in Theorem 4 may be interpreted as bounds on the difference

$$|\alpha(a\sin(\omega t), a\cos(\omega t), \omega) - \bar{\alpha}(a\sin(\omega t), a\cos(\omega t), \omega)|,$$

where $\bar{\alpha}$ is the state frequency response of the linearized system $\dot{y} = Ay + Bu$, with $A = \frac{\partial F}{\partial x}(0, 0)$ and $B = \frac{\partial F}{\partial u}(0, 0)$.

An interesting topic for further research is deriving more theoretical results using the explicit bounds described here. Other possible topics include the design of an excitation signal that yields a pre-specified periodic trajectory for a contractive system. This issue is important for example in synthetic biology, where an important goal is to design programmable biochemical oscillators (see e.g., [36, 37, 38, 39]). Another possible research topic is the extension of the results presented here to more general classes of dynamical systems (see, e.g., [40] for a special class of infinite dimensional systems that admit a frequency response).

Acknowledgments

We are grateful to Eduardo D. Sontag for reading an earlier version of this paper and providing us with many useful comments.

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