Observer design for piecewise smooth and switched systems via contraction theory [★]

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Abstract: The aim of this paper is to present the application of an approach to study contraction theory recently developed for piecewise smooth and switched systems. The approach that can be used to analyze incremental stability properties of so-called Filippov systems (or variable structure systems) is based on the use of regularization, a procedure to make the vector field of interest differentiable before analyzing its properties. We show that by using this extension of contraction theory to nondifferentiable vector fields it is possible to design observers for a large class of piecewise smooth systems using not only Euclidean norms, as also done in previous literature, but also non-Euclidean norms. This allows greater flexibility in the design and encompasses the case of both piecewise-linear and piecewise-smooth (nonlinear) systems. The theoretical methodology is illustrated via a set of representative examples.

Keywords: contraction theory, observer design, incremental stability, discontinuous control, regularization

1. INTRODUCTION

The problem of designing state observers for nondifferentiable systems is the subject of current research. For example, the design of observers for Lipschitz continuous nonlinear systems was investigated in (Rajamani, 1998; Zemouche and Boutayeb, 2013), while in (Arcak and Kokotovic, 2001; Brogliato and Heemels, 2009) design approaches based on passivity theory were proposed for Lur'e-type systems. Also, in (Juloski et al., 2007; Doris et al., 2008) sufficient conditions were presented to ensure stability of the estimation error for state observers of bimodal piecewise linear (PWL) systems (both continuous or discontinuous on the switching surface). The analysis was conducted analyzing the quadri-modal estimation error dynamics based on quadratic Lyapunov functions and LMI. Related results were presented in (van de Wouw and Pavlov, 2008) for the case of piecewise affine (PWA) systems. Therein, using theoretical results developed in (Pavlov et al., 2007), sufficient conditions guaranteeing exponential stability of the estimation error were given in terms of a set of appropriate LMIs. More recently, the state estimation problem was investigated in (Heemels et al., 2011) for linear complementarity systems and in (Forni et al., 2013) for hybrid systems with impacts.

Contraction theory (Lohmiller and Slotine, 1998; Russo et al., 2010; Jouffroy, 2005; Forni and Sepulchre, 2014; Aminzare and Sontag, 2014) is a powerful analysis tool providing sufficient conditions for incremental stability (Angeli, 2002) of a dynamical system. Namely, if the system vector field is contracting in a set of interest, any two of its trajectories will converge towards each other in that set, a property that can be effectively exploited to design state observers and solve tracking control problems as discussed, for instance, in (Lohmiller and Slotine, 1998; van de Wouw and Pavlov, 2008; Bonnabel et al., 2011; Dinh et al., 2013; Manchester and Slotine, 2014b,a; di Bernardo and Fiore, 2016). More specifically, incremental exponential stability over a given forward invariant set is guaranteed if some matrix measure μ of the system Jacobian matrix is uniformly negative in that set for all time.

The original results on contraction analysis were presented for continuously differentiable vector fields limiting their application to observer design for this class of dynamical systems. Recently, extensions have been presented in literature for applying contraction and convergence analysis to different classes of nondifferentiable and discontinuous vector fields (Lohmiller and Slotine, 2000; Pavlov et al., 2007; di Bernardo et al., 2014; Lu and di Bernardo, 2016; di Bernardo and Liuzza, 2013; di Bernardo and Fiore, 2014; Fiore et al., 2016).

In this paper we propose a methodology to design state observers for nondifferentiable bimodal vector fields, which stems from the results presented in (Fiore et al., 2016) on extending contraction analysis to Filippov systems. Specifically, we derive conditions on the observer dynamics for the estimation error to converge exponentially to zero. These conditions, when particularized to the case of PWA systems, generalize those presented in (van de Wouw and Pavlov, 2008) to the case of non-Euclidean norms.

In what follows, after reviewing some key results on contraction analysis of switched systems, we present our

procedure for state observer design complementing the theoretical derivations with some illustrative examples.

2. CONTRACTION ANALYSIS OF SWITCHED SYSTEMS

2.1 Incremental Stability and Contraction Theory

Let $U \subseteq \mathbb{R}^n$ be an open set. Consider the system of ordinary differential equations

$$\dot{x} = f(t, x), \qquad (1)$$

where f is a continuously differentiable vector field defined for $t \in [0, \infty)$ and $x \in U$, that is $f \in C^1(\mathbb{R}^+ \times U, \mathbb{R}^n)$.

We denote by $\psi(t, t_0, x_0)$ the value of the solution $x(t)$ at time t of the differential equation (1) with initial value $x(t_0) = x_0$. We say that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is *forward invariant* for system (1) if $x_0 \in \mathcal{C}$ implies $\psi(t, t_0, x_0) \in \mathcal{C}$ for all $t \geq t_0$.

Definition 1. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a forward invariant set and $|\cdot|$ some norm on $\mathbb{R}^n$. The system (1) is said to be *incrementally exponentially stable* (IES) in $\mathcal{C}$ if there exist constants $K \geq 1$ and $\lambda > 0$ such that

$$|x(t) - y(t)| \leq K e^{-\lambda(t - t_0)} |x_0 - y_0|, \qquad (2)$$

$\forall t \geq t_0$, $\forall x_0, y_0 \in \mathcal{C}$, where $x(t) = \psi(t, t_0, x_0)$ and $y(t) = \psi(t, t_0, y_0)$ are its two solutions.

Results in contraction theory can be applied to a quite general class of subsets $\mathcal{C} \subseteq \mathbb{R}^n$, known as K-reachable subsets (Russo et al., 2010). See Appendix A for a definition.

Definition 2. The continuously differentiable vector field (1) is said to be *contracting* on a K-reachable set $\mathcal{C} \subseteq U$ if there exists some norm in $\mathcal{C}$, with associated matrix measure μ (see Appendix A), such that, for some constant $c > 0$ (the *contraction rate*),

$$\mu \left(\frac{\partial f}{\partial x}(t, x) \right) \leq -c, \quad \forall x \in \mathcal{C}, \ \forall t \geq t_0. \qquad (3)$$

A contracting dynamical system forgets initial conditions or temporary state perturbations exponentially fast, implying convergence of system trajectories towards each other and consequently towards a steady-state solution which is determined only by the input (the *entrainment* property, e.g. (Russo et al., 2010; Lohmiller and Slotine, 1998)), as stated in next theorem.

Theorem 3. Suppose that $\mathcal{C}$ is a K-reachable forward-invariant subset of U and that the vector field (1) is contracting with contraction rate c therein. Then, for every two solutions $x(t) = \psi(t, t_0, x_0)$ and $y(t) = \psi(t, t_0, y_0)$ with $x_0, y_0 \in \mathcal{C}$ we have that (2) holds with $\lambda = c$.

In this paper we analyze contraction properties of dynamical systems based on norms and matrix measures (Lohmiller and Slotine, 1998; Russo et al., 2010). Other more general definitions exist in the literature, for example results based on Riemannian metrics (Lohmiller and Slotine, 1998) and Finsler-Lyapunov functions (Forni and Sepulchre, 2014). The relations between these three definitions and the definition of convergence (Pavlov et al., 2004) were investigated in (Forni and Sepulchre, 2014).

2.2 Switched systems

Switched (or bimodal) Filippov systems are dynamical systems $\dot{x} = f(x)$ where $f(x)$ is a piecewise continuous vector field having a codimension-one submanifold Σ as its discontinuity set. This class of dynamical systems were thoroughly investigated in the classical work by Filippov (Filippov, 1988) and Utkin (Utkin, 1992).

The submanifold Σ is called the *switching manifold* and is defined as the zero set of a smooth function $H : U \subseteq \mathbb{R}^n \to \mathbb{R}$, that is

$$\Sigma := \{x \in U : H(x) = 0\}, \qquad (4)$$

where $0 \in \mathbb{R}$ is a regular value of H, i.e. $\nabla H(x) \neq 0$, $\forall x \in \Sigma$. Σ divides U in two disjoint regions, $\mathcal{S}^+ := \{x \in U : H(x) > 0\}$ and $\mathcal{S}^- := \{x \in U : H(x) < 0\}$ (see Fig. 1).

Hence, a bimodal Filippov system can be defined as

$$\dot{x} = \begin{cases} F^+(x), & \text{if } x \in \mathcal{S}^+ \\ F^-(x), & \text{if } x \in \mathcal{S}^- \end{cases}, \qquad (5)$$

where $F^+, F^- \in C^1(U, \mathbb{R}^n)$. When the normal components of the vector fields either side of Σ point in the *same* direction, the gradient of a trajectory is discontinuous, leading to Carathéodory solutions (Filippov, 1988). In this case, the dynamics is described as *crossing* or *sewing*. But when the vector fields on either side of Σ both point towards it, the solutions are constrained to evolve along Σ and some additional dynamics needs to be given when such *sliding* behavior occurs. To define this sliding vector field Filippov convexification approach is widely adopted (Filippov, 1988).

Remark 1. In the following we assume that solutions of system (5) are defined in the sense of Filippov (Filippov, 1988) and they have the property of *right-uniqueness* in U (Filippov, 1988, pag. 106), i.e. for each point $x_0 \in U$ there exists $t_1 > t_0$ such that any two solutions satisfying $x(t_0) = x_0$ coincide on the interval $[t_0, t_1]$. Therefore, the escaping region is excluded from our analysis.

Definition 2 was previously presented as a sufficient condition for a dynamical system to be incrementally exponentially stable, but condition (3) can not be directly applied to system (5) because its vector field is not continuously differentiable. Therefore an extension of contraction analysis to PWS systems is not straightforward. In recent work reported in (Fiore et al., 2016), sufficient conditions were derived for convergence of any two trajectories of a Filippov system towards each other. Instead of directly analyzing the Filippov vector field on Σ, the approach in (Fiore et al., 2016) is based on looking at its regularized version, say $f_\varepsilon(x)$, given as

$$f_\varepsilon(x) = \frac{1 + \varphi_\varepsilon(H(x))}{2} F^+(x) + \frac{1 - \varphi_\varepsilon(H(x))}{2} F^-(x),$$

where $\varphi_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ is the so-called transition function. In this new system the switching manifold Σ is replaced by a boundary layer $\mathcal{S}_\varepsilon$ (Fig. 1) of width 2ε, defined as

$$\mathcal{S}_\varepsilon := \{x \in U : -\varepsilon < H(x) < \varepsilon\}, \qquad (6)$$

and, more importantly, f_ε is continuously differentiable in U, so that condition (3) can be applied to it. Finally, contraction properties of Filippov systems (5) are recovered taking the limit for $\varepsilon \to 0$ and considering the following Lemma.
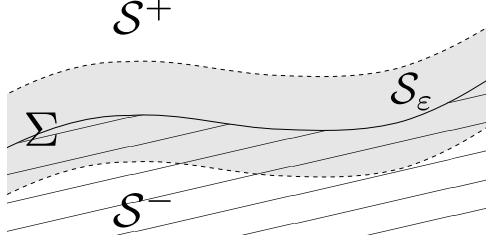
Fig. 1. Regions of state space: the switching manifold $\Sigma := \{x \in U : H(x) = 0\}$ from (4), $\mathcal{S}^+ := \{x \in U : H(x) > 0\}$, $\mathcal{S}^- := \{x \in U : H(x) < 0\}$ (hatched zone) and $\mathcal{S}_\varepsilon := \{x \in U : -\varepsilon < H(x) < \varepsilon\}$ (grey zone) from (6).

Lemma 4. Denoting by $x_\varepsilon(t)$ a solution to the regularized system and by $x(t)$ a solution to the discontinuous system with the same initial conditions x_0, then $|x_\varepsilon(t) - x(t)| = O(\varepsilon)$, uniformly for all $t \geq t_0$ and for all $x_0 \in U$.

For further details see (Sotomayor and Teixeira, 1996; Utkin, 1992; Fiore et al., 2016).

The resulting sufficient conditions for a bimodal Filippov system to be incrementally exponentially stable in a certain set are stated in the following theorem from (Fiore et al., 2016).

Theorem 5. The bimodal switched system (5) is incrementally exponentially stable in a K-reachable set $\mathcal{C} \subseteq U$ with convergence rate $c := \min\{c_1, c_2\}$ if there exists some norm in $\mathcal{C}$, with associated matrix measure μ, such that, for some positive constants c_1, c_2,

$$\mu\left(\frac{\partial F^+}{\partial x}(x)\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+,$$

$$\mu\left(\frac{\partial F^-}{\partial x}(x)\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^-,$$

$$\mu\left(\left[F^+(x) - F^-(x)\right]\nabla H(x)\right) = 0, \quad \forall x \in \Sigma.$$

In the above relations $\bar{\mathcal{S}}^+$ and $\bar{\mathcal{S}}^-$ represent the closures of the sets $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively. The interested reader can refer to (Fiore et al., 2016) for a complete proof and further details.

3. STATE OBSERVER DESIGN

3.1 Problem formulation

Consider the bimodal switched system

$$\dot{x} = \begin{cases} f^+(x) + u(t), & H(x) > 0 \\ f^-(x) + u(t), & H(x) < 0 \end{cases}, \tag{7}$$

$$y = g(x), \tag{8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^n$ are the state, output and the input of the system, respectively, and f^+, f^-, g are continuously differentiable vector fields.

As an observer for the system (7)-(8), we propose a bimodal switched observer with the Luemberger-type structure

$$\dot{\hat{x}} = \begin{cases} f^+(\hat{x}) + L^+(y - \hat{y}) + u(t), & H(\hat{x}) > 0 \\ f^-(\hat{x}) + L^-(y - \hat{y}) + u(t), & H(\hat{x}) < 0 \end{cases}, \tag{9}$$

$$\hat{y} = g(\hat{x}), \tag{10}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state and $L^+, L^- \in \mathbb{R}^{n \times p}$ are observer gain matrices to be selected appropriately.

We are interested to derive conditions on the observer gain matrices L^+ and L^- that guarantee exponential convergence to 0 of the estimation error $e(t) := x(t) - \hat{x}(t)$ for all $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying (7)-(8) for any given continuous function $u(t) : \mathbb{R}^+ \to \mathbb{R}^n$. Note that in what follows it is not required for system (7)-(8) to be contracting, i.e. Theorem 5 must not necessarily hold for this system. Contraction will instead be used to analyze convergence of the system describing the dynamics of the estimation error.

3.2 Main results

Theorem 6. The state estimation error $e(t)$ converges exponentially to zero with convergence rate $c := \min\{c_1, c_2\}$, that is

$$|e(t)| \leq K\, e^{-c(t-t_0)}\, |x(t_0)|, \quad \forall t \geq t_0, \tag{11}$$

if there exists some norm, with associated matrix measure μ, such that, for some positive constants c_1, c_2,

$$\mu\left(\frac{\partial f^+}{\partial x}(\hat{x}) - L^+\frac{\partial g}{\partial x}(\hat{x})\right) \leq -c_1, \forall \hat{x} : H(\hat{x}) > 0, \tag{12}$$

$$\mu\left(\frac{\partial f^-}{\partial x}(\hat{x}) - L^-\frac{\partial g}{\partial x}(\hat{x})\right) \leq -c_2, \forall \hat{x} : H(\hat{x}) < 0, \tag{13}$$

$$\mu\left(\left[\Delta f(\hat{x}) + \Delta L(y - \hat{y})\right]\nabla H(\hat{x})\right) = 0, \forall \hat{x} : H(\hat{x}) = 0, \tag{14}$$

where $\Delta f(\hat{x}) = f^+(\hat{x}) - f^-(\hat{x})$ and $\Delta L = L^+ - L^-$.

Proof. Conditions (12)-(14) come from the application of Theorem 5 to the dynamics of state observer (9)-(10) by rewriting them as

$$\dot{\hat{x}} = \begin{cases} F^+(\hat{x}) + \eta^+(t), & H(\hat{x}) > 0 \\ F^-(\hat{x}) + \eta^-(t), & H(\hat{x}) < 0 \end{cases}, \tag{15}$$

where $F^\pm(\hat{x}) = f^\pm(\hat{x}) - L^\pm g(\hat{x})$ depends only on $\hat{x}$, and $\eta^\pm(t) = L^\pm g(x(t)) + u(t)$ is a function of t.
Hence, if such conditions are satisfied, then the state observer is contracting; this in turn implies that

$$|\hat{x}_1(t) - \hat{x}_2(t)| \leq K\, e^{-c(t-t_0)}|\hat{x}_1(t_0) - \hat{x}_2(t_0)|, \forall t \geq t_0.$$

Now, noticing that a solution $x(t)$ of system (7) is a particular solution of the observer (9), i.e. $\hat{x}_2(t) = x(t)$ (because (7) and (9) have the same structure, except for the correction term $g(x) - g(\hat{x})$, which is null if $\hat{x} = x$), we can write

$$|e(t)| = |x(t) - \hat{x}(t)| \leq K\, e^{-c(t-t_0)}|x(t_0)|,$$

for all $t \geq t_0$, where $\hat{x}(t_0) = 0$ as usual in observer design. Hence, the exponential convergence to zero of the estimation error is proved.

Remark 2. Alternatively, the theorem can be proved considering the regularized dynamics of both system (7) and observer (9). Denoting by $x_\varepsilon(t)$ a solution to the regularized switched system (7), and by $\hat{x}_\varepsilon(t)$ a solution to the regularized observer (9), we have

$$|e(t)| = |x(t) - \hat{x}(t)|$$
$$\leq |x(t) - x_\varepsilon(t)| + |x_\varepsilon(t) - \hat{x}_\varepsilon(t)| + |\hat{x}_\varepsilon(t) - \hat{x}(t)|.$$

The first and the third terms are the error between a solution to the discontinuous system and a solution to its regularized counterpart; hence, from Lemma 4 we know that

$$|x(t) - x_\varepsilon(t)| = O(\varepsilon),$$
$$|\hat{x}(t) - \hat{x}_\varepsilon(t)| = O(\varepsilon).$$

Furthermore, similarly to what done in (Fiore et al., 2016), it can be shown that conditions (12)-(14) imply incremental stability of the trajectories of the regularized observer, thus

$$|\hat{x}_{\varepsilon,1}(t) - \hat{x}_{\varepsilon,2}(t)| \leq K\, e^{-c(t-t_0)} |\hat{x}_{\varepsilon,1}(t_0) - \hat{x}_{\varepsilon,2}(t_0)|, \forall t \geq t_0.$$

The theorem is finally proved by taking the limit for $\varepsilon \to 0^+$ and following the last step as in the above proof.

Remark 3. In the case that one of the two modes, f^+ or f^-, of the observed system (7) is already contracting, the respective observer gain in (9) can be set to zero to simplify the design problem. The drawback is a convergence rate of the estimation error that depends on the open loop contraction rate of the respective mode.

Remark 4. In presence of bounded disturbances or uncertainties on the models, contraction properties of the vector fields guarantee boundedness of the estimation error (a more detailed analysis is not the aim of the current paper; the interested reader can refer to (Lohmiller and Slotine, 1998)).

4. EXAMPLES

Here we present examples to illustrate how state observers for different classes of piecewise smooth systems may be designed using Theorem 6. All simulations presented in this section have been computed using the numerical solver in (Piiroinen and Kuznetsov, 2008).

4.1 Example 1

Consider a nonlinear bimodal switched system as in (7)-(8) with

$$f^+(x) = \begin{bmatrix} -9x_1 - 3x_1^2 - 18 \\ -4x_2 \end{bmatrix}, \quad (16)$$

$$f^-(x) = \begin{bmatrix} -9x_1 + 3x_1^2 + 18 \\ -4x_2 \end{bmatrix}, \quad (17)$$

and $H(x) = x_1$, $y = g(x) = x_1^2$.

According to Theorem 6, a state observer as in (9)-(10) with $L^+ = \begin{bmatrix} \ell_1^+ & \ell_2^+ \end{bmatrix}^{\mathrm{T}}$ and $L^- = \begin{bmatrix} \ell_1^- & \ell_2^- \end{bmatrix}^{\mathrm{T}}$ for this system has the property that its estimation error converges exponentially to zero if there exist choices of the gain matrices L^+ and L^- so that all three conditions (12)-(14) are satisfied.

To find L^+ and L^-, it is first necessary to select a specific matrix measure; here we use the measure μ_1, associated to the so-called ℓ^1-norm, as defined in Appendix A. Therefore, conditions (12) and (13) translate respectively to

$$\mu_1\left(\begin{bmatrix} -9 - 6\hat{x}_1 - 2\ell_1^+\hat{x}_1 & 0 \\ -2\ell_2^+\hat{x}_1 & -4 \end{bmatrix}\right) < 0, \quad \text{with } \hat{x}_1 > 0, \quad (18)$$

$$\mu_1\left(\begin{bmatrix} -9 + 6\hat{x}_1 - 2\ell_1^-\hat{x}_1 & 0 \\ -2\ell_2^-\hat{x}_1 & -4 \end{bmatrix}\right) < 0, \quad \text{with } \hat{x}_1 < 0. \quad (19)$$
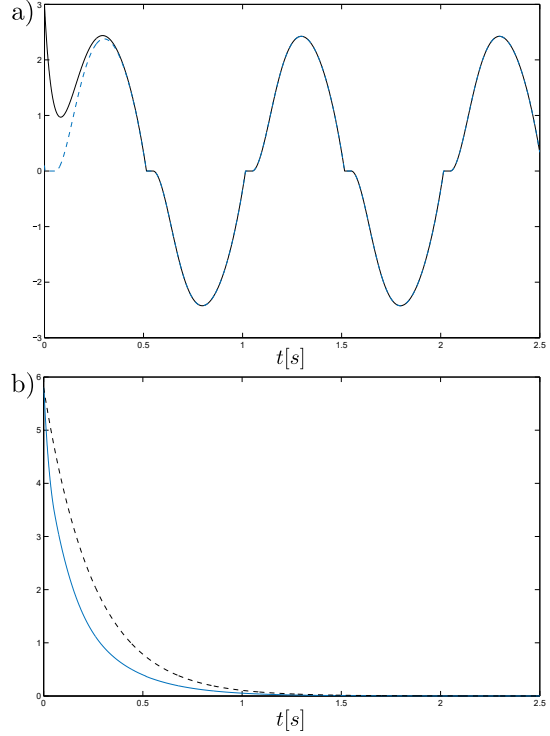

a)
b)

Fig. 2. Panel a: Time evolution of the states $x_1(t)$ (solid line) and $\hat{x}_1(t)$ (dashed line) of Example 1. Initial conditions are respectively $x_0 = \begin{bmatrix} 3 & 3 \end{bmatrix}^{\mathrm{T}}$, $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_1$. The dashed line represents the analytical estimate (11) with $c = 4$ and $K = 1$. Parameters: $L^+ = \begin{bmatrix} -2 & 0 \end{bmatrix}^{\mathrm{T}}$ and $L^- = \begin{bmatrix} 2 & 0 \end{bmatrix}^{\mathrm{T}}$.

Selecting for simplicity $\ell_2^+ = \ell_2^- = 0$, the above inequalities are satisfied if

$$\max\{-9 - 6\hat{x}_1 - 2\ell_1^+\hat{x}_1; -4\} < 0, \quad \text{with } \hat{x}_1 > 0,$$
$$\max\{-9 + 6\hat{x}_1 - 2\ell_1^-\hat{x}_1; -4\} < 0, \quad \text{with } \hat{x}_1 < 0.$$

This is true if $\ell_1^+ > -3$ and $\ell_1^- < 3$.

Next, from the the third condition (14), we have

$$\mu_1\left(\begin{bmatrix} -6\hat{x}_1^2 - 36 + (\ell_1^+ - \ell_1^-)(x_1^2 - \hat{x}_1^2) \\ 0 \end{bmatrix} [1\ 0]\right) = 0, \quad (20)$$

with $\hat{x}_1 = 0$, which is verified if

$$\max\{-36 + (\ell_1^+ - \ell_1^-)x_1^2; 0\} = 0,$$

i.e. if

$$-36 + (\ell_1^+ - \ell_1^-)x_1^2 < 0,$$

which holds for all x_1 if $\ell_1^+ < \ell_1^-$. Therefore, to satisfy all three conditions of Theorem 6 it is possible for example to select $L^+ = \begin{bmatrix} -2 & 0 \end{bmatrix}^{\mathrm{T}}$ and $L^- = \begin{bmatrix} 2 & 0 \end{bmatrix}^{\mathrm{T}}$. The resulting state observer is contracting and its estimation error satisfies (11) with convergence rate $c = 4$. In Fig. 2(a) we show numerical simulations of the evolution of the states x_1 and $\hat{x}_1$ when an input $u(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}} \sin(2\pi t)$ of period $T = 1$ is applied to the system. In Fig. 2(b) the evolution of the ℓ^1-norm of the state estimation error $e(t)$ is reported, confirming the analytical estimate (11).

4.2 Example 2

Consider a piecewise affine (PWA) system of the form

$$\dot{x} = \begin{cases} A_1 x + b_1 + Bu, & \text{if } h^{\mathrm{T}} x > 0 \\ A_2 x + b_2 + Bu, & \text{if } h^{\mathrm{T}} x < 0 \end{cases}, \quad (21)$$

$$y = c^{\mathrm{T}} x, \quad (22)$$

where

$$A_1 = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ -3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

and $B = [0 \ 1]^{\mathrm{T}}$, $h = [0 \ 1]^{\mathrm{T}}$, $c = [1 \ 1]^{\mathrm{T}}$.
A state observer as in (9)-(10) for this system has the structure

$$\dot{\hat{x}} = \begin{cases} A_1 \hat{x} + b_1 + L^+ (y - \hat{y}) + Bu, & \text{if } h^{\mathrm{T}} \hat{x} > 0 \\ A_2 \hat{x} + b_2 + L^- (y - \hat{y}) + Bu, & \text{if } h^{\mathrm{T}} \hat{x} < 0 \end{cases}, \quad (23)$$

$$\hat{y} = c^{\mathrm{T}} \hat{x}, \quad (24)$$

where, for the sake of simplicity, we choose $L^+ = L^- = L$. Again we decide to proceed using the matrix measure induced by the ℓ^1-norm. In this case, conditions (12) and (13) yield respectively

$$\mu_1 \left(A_1 - L c^{\mathrm{T}} \right) = \mu_1 \left(\begin{bmatrix} -1 - \ell_1 & -\ell_1 \\ 2 - \ell_2 & -2 - \ell_2 \end{bmatrix} \right) \\ = \max\{-1 - \ell_1 + |2 - \ell_2|; -2 - \ell_2 + |\ell_1|\} \quad (25)$$

and

$$\mu_1 \left(A_2 - L c^{\mathrm{T}} \right) = \mu_1 \left(\begin{bmatrix} -1 - \ell_1 & -\ell_1 \\ 2 - \ell_2 & -3 - \ell_2 \end{bmatrix} \right) \\ = \max\{-1 - \ell_1 + |2 - \ell_2|; -3 - \ell_2 + |\ell_1|\}. \quad (26)$$

It is easy to verify that choosing $\ell_1 = \ell_2 = 1$ both measures are equal to -1. Condition (14) translates into

$$\mu_1 \left(\begin{bmatrix} 0 & -3 \\ 0 & \hat{x}_2 - 7 \end{bmatrix} \right) = 0, \quad \text{with } \hat{x}_2 = 0, \quad (27)$$

which is identically verified, independently of L.

Hence, the designed observer (23) is contracting and the estimation error converges exponentially to zero with rate $c = 1$. In Fig. 3(a) we show numerical simulations of the evolution of the states x_2 and $\hat{x}_2$ when an input $u(t) = 4 \sin(2\pi t)$ of period $T = 1$ is applied to the system. In Fig. 3(b) the evolution is reported of the ℓ^1-norm of the state estimation error $e(t)$.

Note that faster convergence can be obtained by choosing higher values of ℓ_1 and ℓ_2 fulfilling conditions (25)-(26). For example choosing $L = [1.5 \ 2]^T$ we obtain a convergence rate $c = 2.5$, as shown in Fig. 3(c).

4.3 Example 3

Consider now a harmonic oscillator affected by Coulomb friction, described by the equations

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\omega_{\mathrm{n}} x_1 - \dfrac{\omega_{\mathrm{n}}}{Q} x_2 - \dfrac{F_{\mathrm{f}}}{m} \mathrm{sgn}(x_2) + \dfrac{F_{\mathrm{d}}}{m} \sin(\omega_{\mathrm{d}} t), \end{cases} \quad (28)$$

$$y = x_1, \quad (29)$$

where $x_1 \in \mathbb{R}$ is the position of the oscillator, $x_2 \in \mathbb{R}$ is its velocity, ω_{n} is its natural frequency, Q is said Q factor and is inversely proportional to the damping, m is the mass of
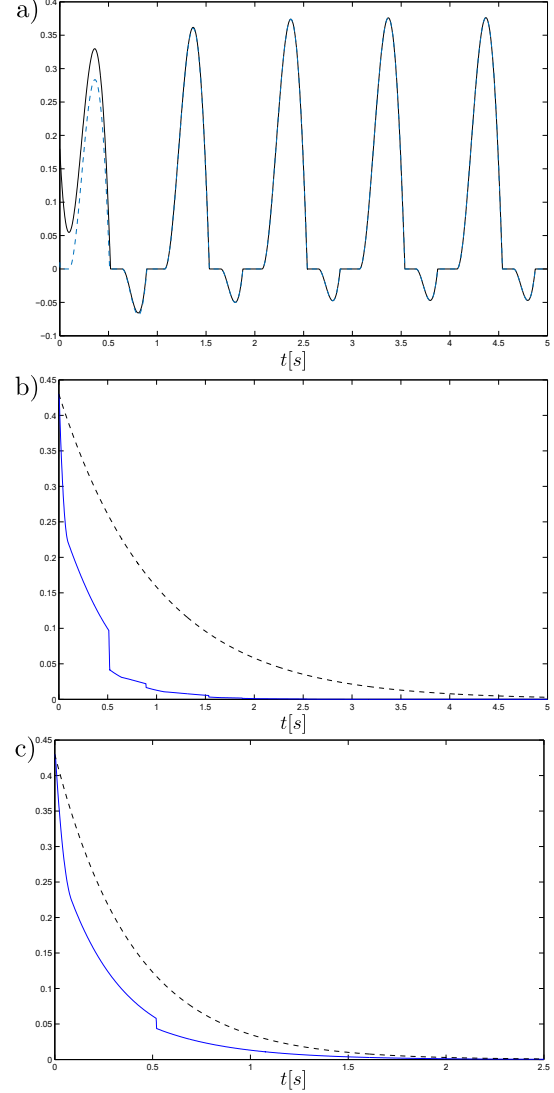

Fig. 3. Panel a: Time evolution of the states $x_2(t)$ (solid line) and $\hat{x}_2(t)$ (dashed line) of Example 2. Initial conditions are respectively $x_0 = [0.3 \ 0.3]^{\mathrm{T}}$, $\hat{x}_0 = [0 \ 0]^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_1$. The dashed line represents the analytical estimate (11) with $c = 1$ and $K = 1$. Parameters: $L^+ = L^- = [1 \ 1]^{\mathrm{T}}$. Panel c: Norm of the estimation error using observer gain $L = [1.5 \ 2]^{\mathrm{T}}$.

the oscillator, F_{d} is the amplitude of the driving force, ω_{d} is the driving frequency and F_{f} is the amplitude of the dry friction force which is modeled through the sign function as in (Csernák and Stépán, 2006). The proposed observer for system (28)-(29) has the form

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \ell_1 (x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = -\omega_{\mathrm{n}} \hat{x}_1 - \dfrac{\omega_{\mathrm{n}}}{Q} \hat{x}_2 - \dfrac{F_{\mathrm{f}}}{m} \mathrm{sgn}(\hat{x}_2) \\ \qquad + \ell_2 (x_1 - \hat{x}_1) + \dfrac{F_{\mathrm{d}}}{m} \sin(\omega_{\mathrm{d}} t) \end{cases}, \quad (30)$$

$$\hat{y} = \hat{x}_1. \quad (31)$$

Note that system (28) may be viewed as a PWA system (21) where

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -\omega_{\mathrm{d}} & -\omega_{\mathrm{d}}/Q \end{bmatrix},$$

$B = [0 \;\; 1/m]^{\mathrm{T}}$, $b_1 = [0 \;\; -F_{\mathrm{f}}/m]^{\mathrm{T}}$, $b_2 = [0 \;\; F_{\mathrm{f}}/m]^{\mathrm{T}}$, $h = [0 \;\; 1]^{\mathrm{T}}$, and excited by an input $u(t) = F_{\mathrm{d}} \sin(\omega_{\mathrm{d}} t)$.

Using the measure μ_∞, induced by the uniform norm, as defined in Appendix A, conditions (12) and (13) of Theorem 6, combined, translate to

$$\mu_\infty \left(\begin{bmatrix} -\ell_1 & 1 \\ -\omega_{\mathrm{n}} - \ell_2 & -\omega_{\mathrm{n}}/Q \end{bmatrix} \right) < 0, \quad \text{with } \hat{x}_2 \neq 0, \quad (32)$$

which in turn is equivalent to

$$\max \{-\ell_1 + 1; \; -\omega_{\mathrm{n}}/Q + |-\omega_{\mathrm{n}} - \ell_2|\} < 0, \quad \text{with } \hat{x}_2 \neq 0. \quad (33)$$

Therefore ℓ_1 and ℓ_2 must be chosen so that

$$\ell_1 > 1, \quad (34)$$

$$-\omega_{\mathrm{n}} \left(1 + \frac{1}{Q}\right) < \ell_2 < -\omega_{\mathrm{n}} \left(1 - \frac{1}{Q}\right). \quad (35)$$

Furthermore, condition (14) is verified if

$$\mu_\infty \left(\begin{bmatrix} 0 \\ -2F_{\mathrm{f}}/m \end{bmatrix} [0 \;\; 1] \right) = 0, \quad \text{with } \hat{x}_2 = 0, \quad (36)$$

i.e. if

$$\max \{0; \; -F_{\mathrm{f}}/m\} = 0, \quad \text{with } \hat{x}_2 = 0, \quad (37)$$

which always holds because $F_{\mathrm{f}}, m > 0$.

Numerical simulations reported in Fig. 4(a)-(b) confirm the theoretical predictions, showing that the estimation error converges to zero as expected.

5. CONCLUSIONS

We presented new conditions for the design of state observers for a large class of nonlinear switched systems including those exhibiting sliding motion. The design methodology is based on the analysis of incremental exponential stability based on the extension of contraction theory to switched bimodal Filippov systems derived in (Fiore et al., 2016). The conditions were formulated in terms of matrix measures of the Jacobians of the observer dynamics and of an additional condition on the vector fields on the discontinuity set. The theoretical results were illustrated through simple but representative examples demonstrating the effectiveness of the proposed methodology. Future work will be aimed at extending the approach to a wider class of switched systems and reformulating the design procedure as a convex optimization problem to compute numerically both metrics and observer gains.
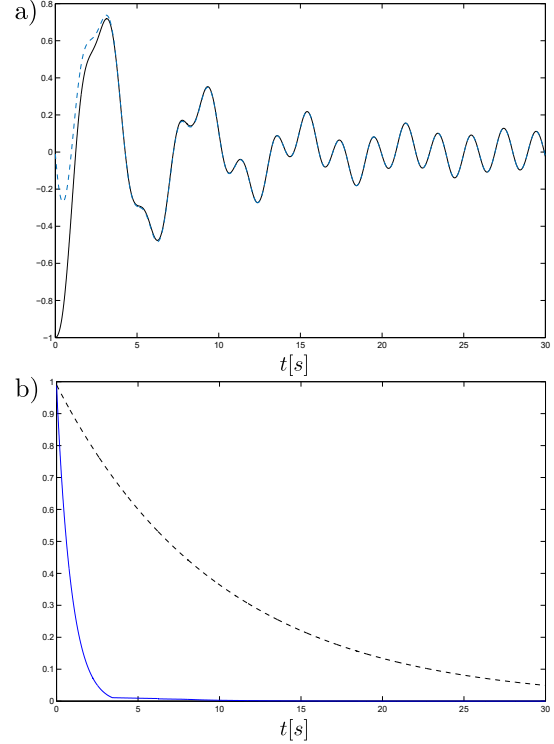

Fig. 4. Panel a: time evolution of the states $x_1(t)$ (solid line) and $\hat{x}_1(t)$ (dashed line) of Example 3. Initial conditions are respectively $x_0 = [-1 \;\; 0]^{\mathrm{T}}$, $\hat{x}_0 = [0 \;\; 0]^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_\infty$. The dashed line represents the analytical estimate (11) with $c = 0.1$ and $K = 1$. Parameters: $\omega_{\mathrm{n}} = 1$ rad/s, $Q = 10$, $m = 1$ kg, $F_{\mathrm{d}} = 1$ N, $\omega_{\mathrm{d}} = \pi$ rad/s, $F_{\mathrm{f}} = 0.1$ N, $\ell_1 = 1.1$, $\ell_2 = -1$.

REFERENCES

Aminzare, Z. and Sontag, E.D. (2014). Contraction methods for nonlinear systems: A brief introduction and some open problems. In *Proc. of Conference on Decision and Control*, 3835–3847. IEEE.

Angeli, D. (2002). A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3), 410–421.

Arcak, M. and Kokotovic, P. (2001). Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, 46(7), 1146–1150.

Bonnabel, S., Astolfi, A., and Sepulchre, R. (2011). Contraction and observer design on cones. In *Proc. of Conference on Decidion and Control*, 7147–7151. IEEE.

Brogliato, B. and Heemels, W.P.M.H. (2009). Observer design for Lur'e systems with multivalued mappings: a passivity approach. *IEEE Transactions on Automatic Control*, 54(8), 1996–2001.

Csernák, G. and Stépán, G. (2006). On the periodic response of a harmonically excited dry friction oscillator. *Journal of Sound and Vibration*, 295(3), 649–658.

Dahlquist, G. (1958). Stability and error bounds in the numerical integration of ordinary differential equations. *Trans. Royal Inst. of Technology*, (130).

Desoer, C. and Haneda, H. (1972). The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5), 480–486.

di Bernardo, M. and Fiore, D. (2014). Incremental stability of bimodal Filippov systems in R^n. In *Proc. of Conference on Decision and Control*, 4679–4684. IEEE.

di Bernardo, M. and Fiore, D. (2016). Switching control for incremental stabilization of nonlinear systems via contraction theory. In *Proc. of European Control Conference*.

di Bernardo, M. and Liuzza, D. (2013). Incremental stability of planar Filippov systems. In *Proc. of European Control Conference*, 3706–3711.

di Bernardo, M., Liuzza, D., and Russo, G. (2014). Contraction analysis for a class of nondifferentiable systems with applications to stability and network synchronization. *SIAM Journal on Control and Optimization*, 52(5), 3203–3227.

Dinh, T.N., Bonnabel, S., and Sepulchre, R. (2013). Contraction-based design of positive observers. In *Proc. of Conference on Decision and Control*, 6574–6579. IEEE.

Doris, A., Juloski, A.L., Mihajlovic, N., Heemels, W., van de Wouw, N., and Nijmeijer, H. (2008). Observer designs for experimental non-smooth and discontinuous systems. *IEEE Transactions on Control Systems Technology*, 16(6), 1323–1332.

Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides*. Kluwer.

Fiore, D., Hogan, S.J., and di Bernardo, M. (2016). Contraction analysis of switched systems via regularization. *Automatica*, 73, 279–288.

Forni, F. and Sepulchre, R. (2014). A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3), 614–628.

Forni, F., Teel, A.R., and Zaccarian, L. (2013). Follow the bouncing ball: Global results on tracking and state estimation with impacts. *IEEE Transactions on Automatic Control*, 58(6), 1470–1485.

Heemels, W.P.M.H., Camlibel, M.K., Schumacher, J.M., and Brogliato, B. (2011). Observer-based control of linear complementarity systems. *International Journal of Robust and Nonlinear Control*, 21(10), 1193–1218.

Jouffroy, J. (2005). Some ancestors of contraction analysis. In *Proc. of Conference on Decision and Control and European Control Conference*, 5450–5455. IEEE.

Juloski, A.L., Heemels, W.P.M.H., and Weiland, S. (2007). Observer design for a class of piecewise linear systems. *International Journal of Robust and Nonlinear Control*, 17(15), 1387–1404.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6), 683–696.

Lohmiller, W. and Slotine, J.J.E. (2000). Nonlinear process control using contraction theory. *AIChE Journal*, 46(3), 588–596.

Lu, W. and di Bernardo, M. (2016). Contraction and incremental stability of switched Carathéodory systems using multiple norms. *Automatica*, 70, 1 – 8.

Manchester, I.R. and Slotine, J.J.E. (2014a). Output-feedback control of nonlinear systems using control contraction metrics and convex optimization. In *Proc. of Australian Control Conference*, 215–220.

Manchester, I.R. and Slotine, J.J.E. (2014b). Control contraction metrics and universal stabilizability. In *Proc. of IFAC World Congress*, volume 19, 8223–8228.

Pavlov, A., Pogromsky, A., van de Wouw, N., Nijmeijer, H., and Rooda, K. (2004). Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters*, 52(3), 257–261.

Pavlov, A., Pogromsky, A., van de Wouw, N., and Nijmeijer, H. (2007). On convergence properties of piecewise affine systems. *International Journal of Control*, 80(8), 1233–1247.

Piiroinen, P.T. and Kuznetsov, Y.A. (2008). An event-driven method to simulate Filippov systems with accurate computing of sliding motions. *ACM Transactions on Mathematical Software*, 34(3), 13:1–13:24.

Rajamani, R. (1998). Observers for Lipschitz nonlinear systems. *IEEE Transactions on Automatic Control*, 43(3), 397–401.

Russo, G., di Bernardo, M., and Sontag, E.D. (2010). Global entrainment of transcriptional systems to periodic inputs. *PLoS Comput Biol*, 6(4), e1000739.

Sotomayor, J. and Teixeira, M.A. (1996). Regularization of discontinuous vector fields. In *Proc. of International Conference on Differential Equations, Lisbon*, 207–223.

Utkin, V.I. (1992). *Sliding modes in control and optimization*. Springer.

van de Wouw, N. and Pavlov, A. (2008). Tracking and synchronisation for a class of PWA systems. *Automatica*, 44(11), 2909–2915.

Vidyasagar, M. (2002). *Nonlinear systems analysis*. SIAM, 2nd edition.

Zemouche, A. and Boutayeb, M. (2013). On LMI conditions to design observers for Lipschitz nonlinear systems. *Automatica*, 49(2), 585 – 591.

Appendix A

K-reachable sets

Let $K > 0$ be any positive real number. A subset $\mathcal{C} \subseteq \mathbb{R}^n$ is *K-reachable* if for any two points x_0 and y_0 in $\mathcal{C}$ there is some continuously differentiable curve $\gamma : [0,1] \to \mathcal{C}$ such that $\gamma(0) = x_0$, $\gamma(1) = y_0$ and $|\gamma'(r)| \leq K|y_0 - x_0|$, $\forall r$. For convex sets $\mathcal{C}$, we may pick $\gamma(r) = x_0 + r(y_0 - x_0)$, so $\gamma'(r) = y_0 - x_0$ and we can take $K = 1$. Thus, convex sets are 1-reachable, and it is easy to show that the converse holds.

Matrix measure

The *matrix measure* (Dahlquist, 1958; Vidyasagar, 2002) associated to a matrix $A \in \mathbb{R}^{n \times n}$ is the function $\mu(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}$ defined as

$$\mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}.$$

See (Vidyasagar, 2002; Desoer and Haneda, 1972) for a list of properties of this measure. The most common used matrix measures are those associated to the ℓ^1-norm, the Euclidean norm and the uniform norm, and they are defined as follows:

$$\mu_1(A) = \max_j \left[a_{jj} + \sum_{i \neq j} |a_{ij}| \right],$$

$$\mu_2(A) = \lambda_{max} \left(\frac{A + A^{\mathrm{T}}}{2} \right),$$

$$\mu_\infty(A) = \max_i \left[a_{ii} + \sum_{j \neq i} |a_{ij}| \right].$$