

Stability Analysis of Lur'e-Type Switched Systems

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Abstract—This technical note aims to introduce stability analysis of Lur'e-type switched systems in frequency domain. A new state-input dependent switching function is proposed and it is the key issue to obtain a stability condition that generalizes the celebrated Popov criterion to deal with this class of switched nonlinear systems. Likewise the case of time invariant systems, we propose a frequency domain stability test that is expressed through a convex combination of the subsystems state space matrices. This task is not trivial due to the time-varying nature of the nonlinear systems under consideration. The theory is illustrated by a simple example.

Index Terms—Popov criterion, switched systems.

I. INTRODUCTION

Switched systems constitute an important subclass of hybrid systems characterized by presenting several subsystems and a switching rule that selects, at each time instant, one of them to be connected. In this technical note, the switching rule is the control variable to be determined in order to preserve stability. The books [12], [17] and the papers [5], [13], [16] are useful references for early theoretical developments on this topic.

The stability analysis of continuous-time switched linear systems has been treated by several authors [2], [5], [6] and [10]. The increasing interest in this class of systems is motivated by some of their features. Indeed, the consistency property recently defined in [9] makes clear the importance of the switching rule design, since it enhances the overall performance when compared to that of each isolated subsystem. The stabilization results have been generalized to cope with state feedback control [8], [11] and output feedback control [4], [7].

Due to the success of switched linear systems, the interest in switched nonlinear systems is increasing, as reveal the references [1], [3], [14], [15], [18]–[20] and [22]. A remarkable subclass is the one composed by Lur'e-type switched systems which are characterized by a feedback connection of a switched linear system and a nonlinearity bounded by a sector. For time invariant Lur'e-type systems the stability study has the celebrated Popov criterion as an important result, which yields the stability analysis based on a condition formulated in the frequency domain. Regarding continuous-time Lur'e-type switched systems, the literature presents few results formulated in the time domain for arbitrary switching, see for instance [1], [18] and, to the best of our knowledge, there is no stability test in the frequency domain yielding a stabilizing switching rule. It is worth mentioning that finding a stability test in the frequency domain is far from being trivial since

switched systems are time-varying and, in general, they do not admit a frequency domain representation.

This technical note aims to introduce a stability test of Lur'e-type switched systems in frequency domain. As an initial step, we have generalized the circle criterion to provide stability conditions based on strict positive realness of a certain transfer function. In this case, a state dependent switching rule is determined in order to assure global asymptotical stability of the closed-loop switched nonlinear system. Unfortunately, this kind of switching rule cannot be used to generalize the celebrated Popov criterion to cope with switched systems. To this end, we have to determine a new switching function that depends not only on the state but also on the input of the subsystems. The theory is illustrated by means of an academical example.

The notation is standard. For real matrices or vectors ($'$) indicates transpose. For symmetric matrices, the symbol (\bullet) denotes each of its symmetric blocks. The unitary simplex Λ is composed by all nonnegative vectors $\lambda \in \mathbb{R}^N$ such that $\sum_{j=1}^N \lambda_j = 1$. The convex combination of $\{J_i\}_{i=1}^N$ is denoted by $J_\nu = \sum_{j=1}^N \nu_j J_j$ where $\nu \in \Lambda$.

II. PRELIMINARIES

Consider the following Lur'e-type switched system:

$$\dot{x} = A_\sigma x + H_\sigma w \quad (1)$$

$$y = E_\sigma x + G_\sigma w \quad (2)$$

$$w = -\phi(y) \quad (3)$$

where the vectors $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$ are the state, the input and the output of the switched linear part of the system, respectively. Notice that the input and the output vectors have the same dimensions. The switching function $\sigma(t)$ selects at each $t \geq 0$ a subsystem among those belonging to the set $\mathbb{K} = \{1, \dots, N\}$. The state space realization of each subsystem is defined by matrices (A_i, H_i, E_i, G_i) for all $i \in \mathbb{K}$. The input and the output are coupled by a nonlinear function $\phi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ that belongs to the sector $[0, \kappa]$ for some $\kappa > 0$. It is of the form $\phi(\xi) = [\phi_1(\xi_1) \cdots \phi_m(\xi_m)]'$ where each scalar component satisfies the constraint $(\phi_i(\xi_i) - \kappa \xi_i) \phi_i(\xi_i) \leq 0$ for all $\xi_i \in \mathbb{R}$ and $i \in \mathbb{K}$. As a consequence, functions of this class are such that $(\phi(\xi) - \kappa \xi)' \phi(\xi) \leq 0$ for all $\xi \in \mathbb{R}^m$. Moreover we assume that for $x \in \mathbb{R}^n$ given, the nonlinear equation $y + G_i \phi(y) = E_i x$ admits an unique solution for each $i \in \mathbb{K}$. Notice that this assumption is always fulfilled whenever $G_i = 0, \forall i \in \mathbb{K}$. To ease the notation, the set of all nonlinearities satisfying these algebraic conditions is denoted by Φ . Clearly $\phi(0) = 0$ whenever $\phi \in \Phi$ which implies that $x = 0$ is an equilibrium point of the switched nonlinear system (1)–(3).

Let us consider the class \mathcal{M} of Metzler matrices $\Pi \in \mathbb{R}^N \times \mathbb{R}^N$ with nonnegative off diagonal elements $\pi_{ji} \geq 0, \forall j \neq i \in \mathbb{K} \times \mathbb{K}$ satisfying $\sum_{j \in \mathbb{K}} \pi_{ji} = 0, \forall i \in \mathbb{K}$. From the fact that $e' \Pi = 0$ with $e' = [1 \ 1 \ \cdots \ 1] \in \mathbb{R}^N$, matrix Π has a null eigenvalue which is the one of maximum real part. The Frobenius–Perron Theorem indicates that the eigenvector $v \in \mathbb{R}^N$ associated to the null eigenvalue of Π is nonnegative. Hence, the usual normalization $\sum_{i \in \mathbb{K}} v_i = 1$ makes $v \in \Lambda$. In addition, for an arbitrary $\nu \in \Lambda$ the matrix $\bar{\Pi} = -I + \nu e'$ is a Metzler matrix of class \mathcal{M} .

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Theorem 1: Let the symmetric matrices $Q_i \in \mathbb{R}^{n \times n}$, $\forall i \in \mathbb{K}$ be given. The following statements are equivalent:

- a) There exist matrices $W_i > 0$, $i \in \mathbb{K}$ and a Metzler matrix $\Pi \in \mathcal{M}$ satisfying

$$Q_i + \sum_{j \in \mathbb{K}} \pi_{ji} W_j < 0, \quad i \in \mathbb{K}. \quad (4)$$

- b) There exist symmetric matrices R_i , $i \in \mathbb{K}$ and $\nu \in \Lambda$ satisfying $R_\nu = 0$ and

$$Q_i + R_i < 0, \quad i \in \mathbb{K}. \quad (5)$$

Proof: Assume that a) holds. Setting $R_i = \sum_{j \in \mathbb{K}} \pi_{ji} W_j$ for all $i \in \mathbb{K}$, it is obvious that inequality (5) is satisfied. On the other hand, choosing $\nu \in \Lambda$ the eigenvector associated to the null eigenvalue of $\Pi \in \mathcal{M}$ it is seen that $\sum_{i \in \mathbb{K}} \pi_{ji} \nu_i = 0$ for each $j \in \mathbb{K}$. Consequently

$$\begin{aligned} R_\nu &= \sum_{i \in \mathbb{K}} \nu_i \sum_{j \in \mathbb{K}} \pi_{ji} W_j \\ &= \sum_{j \in \mathbb{K}} \left(\sum_{i \in \mathbb{K}} \pi_{ji} \nu_i \right) W_j = 0 \end{aligned} \quad (6)$$

and part b) holds as well. Conversely, assume that part b) holds. Setting $\Pi = -I + \nu e' \in \mathcal{M}$ it is seen that $\sum_{j \in \mathbb{K}} \pi_{ji} W_j = \sum_{j \in \mathbb{K}} \nu_j W_j - W_i$ for each $i \in \mathbb{K}$. Hence, it remains to determine $W_i > 0$, $i \in \mathbb{K}$ such that $\sum_{j \in \mathbb{K}} \nu_j W_j - W_i = R_i$ for each $i \in \mathbb{K}$. We argue that a possible solution to these equations is given by

$$W_i = W_N + (R_N - R_i), \quad i \in \mathbb{K} \quad (7)$$

with W_N arbitrary. Indeed, from the fact that $R_\nu = 0$ we obtain

$$\begin{aligned} \sum_{j \in \mathbb{K}} \nu_j W_j - W_i &= - \sum_{j \in \mathbb{K}} \nu_j R_j + R_i \\ &= R_i \end{aligned} \quad (8)$$

for each $i \in \mathbb{K}$. Finally, considering $W_N > 0$ large enough, the determination of W_i from (7) provides $W_i > 0$ for all $i \in \mathbb{K}$ and the claim follows. \square

From the numerical viewpoint both inequalities are nonconvex but (5) appears to be simpler to handle since whenever $\nu \in \Lambda$ is fixed it becomes linear. Clearly, from the proof of Theorem 1, the positivity of matrices W_i , $\forall i \in \mathbb{K}$ can be removed and the result remains valid.

A. Switching Function Design

Consider the switched linear system (1) with zero input

$$\dot{x} = A_\sigma x, \quad x(0) = x_0 \quad (9)$$

where, as before, $x \in \mathbb{R}^n$ is the state vector.

Lemma 1 ([6]): Suppose there exist $P_i > 0$ and $\Pi \in \mathcal{M}$ such that the Lyapunov–Metzler inequalities

$$A'_i P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j < 0, \quad i \in \mathbb{K} \quad (10)$$

hold. With the state dependent switching function $u(x) = \arg \min_{i \in \mathbb{K}} x' P_i x$, system (9) is globally asymptotically stable. The min-type function $v(x) = \min_{i \in \mathbb{K}} x' P_i x$ is a Lyapunov function for the closed-loop system controlled by $\sigma(t) = u(x(t))$.

This is a well known result that can be generalized in several directions as, for instance, to cope with performance indexes similar to \mathcal{H}_2 and \mathcal{H}_∞ norms. It can be used on control and filter design as

well. In our present context, the result of Lemma 1 is now slightly modified to get a new and alternative stability condition.

Lemma 2: Suppose there exist $P > 0$, symmetric matrices W_i , $\forall i \in \mathbb{K}$ and $\Pi \in \mathcal{M}$ such that the modified Lyapunov–Metzler inequalities

$$A'_i P + P A_i + \sum_{j \in \mathbb{K}} \pi_{ji} W_j < 0, \quad i \in \mathbb{K} \quad (11)$$

hold. With the state dependent switching function $u(x) = \arg \min_{i \in \mathbb{K}} x' W_i x$, system (9) is globally asymptotically stable. The quadratic function $v(x) = x' P x$ is a Lyapunov function for the closed-loop system controlled by $\sigma(t) = u(x(t))$.

Proof: The proof follows from Lemma 1 by adopting a parameterized Metzler matrix $\Pi(\mu) = \mu \Pi \in \mathcal{M}$ where $\Pi \in \mathcal{M}$ and $\mu > 0$. We search for a feasible solution of (10) of the form $P_i = P + \mu^{-1} W_i$ with $P > 0$ and W_i symmetric for all $i \in \mathbb{K}$. Making $\mu \rightarrow +\infty$ we see that $P_i \rightarrow P$ and $\sum_{j \in \mathbb{K}} \pi_{ji}(\mu) P_j \rightarrow \sum_{j \in \mathbb{K}} \pi_{ji} W_j$, $i \in \mathbb{K}$ which implies that (10) reduces to (11) as μ goes to infinity. Moreover, $u(x) = \arg \min_{i \in \mathbb{K}} x' P_i x = \arg \min_{i \in \mathbb{K}} x' W_i x$ and the proof is complete. \square

Perhaps one of the most interesting features of Lemma 2 is that it decouples the Lyapunov function $v(x)$ from the switching function $u(x)$, a fact that is not apparent in the original Lyapunov–Metzler inequalities (10).

Corollary 1: Suppose there exist $P > 0$, symmetric matrices R_i , $\forall i \in \mathbb{K}$ and $\nu \in \Lambda$ such that $R_\nu = 0$ and the linear matrix inequalities

$$A'_i P + P A_i + R_i < 0, \quad i \in \mathbb{K} \quad (12)$$

hold. With the state dependent switching function $u(x) = \arg \max_{i \in \mathbb{K}} x' R_i x$, the system (9) is globally asymptotically stable.

Proof: From Theorem 1, we see that inequalities (11) and (12) together with $R_\nu = 0$ are equivalent. It remains to determine the switching function using (7), that is $u(x) = \arg \min_{i \in \mathbb{K}} x' W_i x = \arg \max_{i \in \mathbb{K}} x' R_i x$ which is the desired result. \square

We are now in position to prove the most important result of this section that constitutes the theoretical basis for the development of a frequency domain stability test for the Lur'e-type switched system (1)–(3).

Theorem 2: The stability conditions given in Lemma 2 and Corollary 1 are feasible if and only if there exists $\nu \in \Lambda$ such that A_ν is Hurwitz.

Proof: Since the conditions of Lemma 2 and Corollary 1 are identical we have to prove that the property is valid only for one of them. Assume (12) holds for $P > 0$ and $R_\nu = 0$. Multiplying each inequality (12) by $\nu_i \geq 0$ and summing up for all $i \in \mathbb{K}$ we get $A'_\nu P + P A_\nu < 0$ which implies that A_ν is Hurwitz. Conversely, from the fact that A_ν is Hurwitz there exist $P > 0$ and $S > 0$ such that $A'_\nu P + P A_\nu = -S$ which can be rewritten as $R_\nu = 0$ with $-R_i = A'_i P + P A_i + S$. Eliminating $S > 0$, this equality becomes $A'_i P + P A_i + R_i < 0$ for each $i \in \mathbb{K}$ and the proof is complete. \square

Since Lemma 2 and Corollary 1 are particular cases of Lemma 1, it is clear that the existence of $\nu \in \Lambda$ such that A_ν is Hurwitz is only sufficient for the solvability of the stability conditions given in Lemma 1. Hence, they are more general and less restrictive.

III. MAIN RESULTS

This section is devoted to design a state dependent switching function such that the origin of the Lur'e-type switched system (1)–(3) is globally asymptotically stable. It is shown how to construct a new switching function depending, simultaneously, on the state and on the input of the plant that allows the interpretation of the stability criterion in the frequency domain.

A. The Generalized Circle Criterion

The circle criterion is now generalized to cope with Lur'e-type switched systems. First, we adopt a min-type Lyapunov function that makes clear the class of systems for which the stability conditions can be expressed in the frequency domain.

Theorem 3: Define matrices $E_{\kappa i} = \kappa E_i$ and $G_{\kappa i} = I + \kappa G_i$ for all $i \in \mathbb{K}$. If there exist $P_i > 0$, $\forall i \in \mathbb{K}$ and $\Pi \in \mathcal{M}$ satisfying the matrix inequalities

$$\begin{bmatrix} A_i' P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j & P_i H_i - E_{\kappa i}' \\ \bullet & -G_{\kappa i} - G_{\kappa i}' \end{bmatrix} < 0 \quad (13)$$

for all $i \in \mathbb{K}$, then with the state dependent switching function $u(x) = \arg \min_{i \in \mathbb{K}} x' P_i x$ and $\sigma(t) = u(x(t))$ the system (1)–(3) is globally asymptotically stable.

Proof: Consider the positive definite continuous Lyapunov function candidate $v(x) = \min_{i \in \mathbb{K}} x' P_i x$. At time $t \geq 0$ the switching function becomes $\sigma(t) = i \in I(x(t))$ where $I(x) = \{i : v(x) = x' P_i x\}$ and the Dini derivative satisfies

$$\begin{aligned} D^+ v(x) &= \min_{\ell \in I(x)} 2x' P_\ell (A_i x + H_i w) \\ &< - \sum_{j \in \mathbb{K}} \pi_{ji} x' P_j x + 2(w + \kappa y)' w \\ &\leq 2(\phi(y) - \kappa y)' \phi(y) \end{aligned} \quad (14)$$

where the first inequality follows from (13) and the last one follows from the fact that $x' P_j x \geq x' P_i x$, $\forall j \in \mathbb{K}$ whenever $i \in I(x)$. Consequently, using the fact that $\phi \in \Phi$ the conclusion is $D^+ v(x) < 0$ for all $x \neq 0 \in \mathbb{R}^n$, concluding thus the proof. \square

The stability condition given in Theorem 3 is nonconvex due to the products of variables appearing in the first diagonal block. However, this term is essential since it allows us to handle open-loop unstable models. Similarly to Corollary 1 the feasibility of (13) is ensured by the feasibility of $R_\nu = 0$ and

$$\begin{bmatrix} A_i' P + P A_i + R_i & P H_i - E_{\kappa i}' \\ \bullet & -G_{\kappa i} - G_{\kappa i}' \end{bmatrix} < 0, \quad i \in \mathbb{K}. \quad (15)$$

Notice, however, that matrix R_i is concentrated in the first diagonal block of the LMI (15) which implies that only a subclass of Lur'e type switched systems may be considered.

Theorem 4: Assume that $(H_i, E_i, G_i) = (H, E, G)$ for all $i \in \mathbb{K}$. The stability condition of Theorem 3 holds whenever there exists $P > 0$ such that

$$\begin{bmatrix} A_\nu' P + P A_\nu & P H - E_{\kappa}' \\ \bullet & -G_{\kappa} - G_{\kappa}' \end{bmatrix} < 0 \quad (16)$$

for some $\nu \in \Lambda$.

Proof: Applying the Schur Complement to inequality (16) it follows that $A_\nu' P + P A_\nu + Q = -S$ where $S > 0$, $Q = (P H - E_{\kappa}') (G_{\kappa} + G_{\kappa}')^{-1} (P H - E_{\kappa}')'$ and $G_{\kappa} + G_{\kappa}' > 0$. Consequently, using the fact that $\nu \in \Lambda$, it can be rewritten as $R_\nu = 0$ where $-R_i > A_i' P + P A_i + Q$ for all $i \in \mathbb{K}$. Hence, substituting Q and performing again the Schur Complement we obtain the inequality (15) and the claim follows. \square

It is clear that if some matrix of the triplet (H, E, G) depends on the index $i \in \mathbb{K}$ the result does not hold anymore. This theorem is important to get a frequency domain interpretation of the stability condition provided by Theorem 3 because it is well known that $P > 0$ satisfying (16) exists if and only if the system $(A_\nu, H, E_\kappa, G_\kappa)$ is Extended Strictly Positive Real (ESPR) or, equivalently, if and only

if the transfer function $T(s, \nu) = E(sI - A_\nu)^{-1} H + G$ satisfies the frequency domain constraint

$$T(-j\omega, \nu)' + T(j\omega, \nu) > -2\kappa^{-1} I, \quad \forall \omega \in \mathbb{R} \quad (17)$$

which, in the scalar case, reduces to $\text{Re}(T(j\omega, \nu)) > -\kappa^{-1}$ for all $\omega \in \mathbb{R}$. Consequently, we are interested in determining $\nu \in \Lambda$ which produces the maximum sector provided by the following min-max optimization problem:

$$\max_{\nu \in \Lambda_a} \min_{\omega \geq 0} \text{Re}(T(j\omega, \nu)) \quad (18)$$

where Λ_a is the subset of Λ composed by all $\lambda \in \Lambda$ such that A_λ is Hurwitz. Once $\nu \in \Lambda_a$ is determined all other matrix variables needed to implement the max-type switching function $u(x) = \arg \max_{i \in \mathbb{K}} x' R_i x$ follow from the solution of the LMI (15) together with $R_\nu = 0$.

To deal with more general Lur'e-type systems we have to search for a switching function of the form $\sigma(t) = u(x(t), w(t))$. To this end, we need to introduce an additional assumption involving the system (1)–(3). Actually, at each $t \geq 0$ we have to determine both the output and the switching function from $y + G_\sigma \phi(y) = E_\sigma x(t)$ and $\sigma = u(x(t), -\phi(y))$. From the first equation y depends on σ making the second one an implicit equation that must admit a solution. The easier way to circumvent this difficulty is to impose $(E_i, G_i) = (E, G)$, $\forall i \in \mathbb{K}$.

Theorem 5: Assume that $(E_i, G_i) = (E, G)$, $\forall i \in \mathbb{K}$ and define matrices $E_\kappa = \kappa E$ and $G_\kappa = I + \kappa G$. If there exist $P > 0$, symmetric matrices R_i , $\forall i \in \mathbb{K}$ and $\nu \in \Lambda$ satisfying $R_\nu = 0$ and the linear matrix inequalities

$$\begin{bmatrix} A_i' P + P A_i & P H_i - E_{\kappa}' \\ \bullet & -G_{\kappa} - G_{\kappa}' \end{bmatrix} + R_i < 0, \quad i \in \mathbb{K} \quad (19)$$

then with the state-input dependent switching function

$$u(x, w) = \arg \max_{i \in \mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' R_i \begin{bmatrix} x \\ w \end{bmatrix} \quad (20)$$

the system (1)–(3) is globally asymptotically stable.

Proof: We consider the Lyapunov function candidate $v(x) = x' P x$ and that at an arbitrary $t \geq 0$ we have $\sigma(t) = u(x(t), w(t)) = i \in \mathbb{K}$. Performing the time derivative along a trajectory of the system (1)–(3) it follows that $\dot{v}(x) < 2(\phi(y) - \kappa y)' \phi(y)$ where we have used the fact that $R_\nu = 0$. Consequently, taking into account that $\phi \in \Phi$ the conclusion is $\dot{v}(x) < 0$ for all $(x, w) \neq 0$, concluding thus the proof. \square

The implementation of the switching rule (20) needs the online measurement not only of the state $x(t) \in \mathbb{R}^n$ but also of the input $w(t) \in \mathbb{R}^m$. In practice, this does not seem to be a problem since w is the output of nonlinear actuators that, in general, can be measured. Due to the dimension of R_i , the stability conditions of Theorem 5 are feasible provided that there exist $P > 0$ and $\nu \in \Lambda$ such that

$$\begin{bmatrix} A_\nu' P + P A_\nu & P H_\nu - E_{\kappa}' \\ \bullet & -G_{\kappa} - G_{\kappa}' \end{bmatrix} < 0 \quad (21)$$

or equivalently in the frequency domain, the transfer function $T(s, \nu) = E(sI - A_\nu)^{-1} H_\nu + G$ must satisfy (17). Adopting the same reasoning, the results of [21] which deal with \mathcal{H}_∞ performance can be generalized to cope with a wider class of switched linear systems.

B. The Generalized Popov Criterion

The celebrated Popov criterion is one of the most valuable theoretical issues on stability theory of Lur'e-type systems. Our purpose now is to generalize it to Lur'e-type switched systems. To ease the presentation we assume that $G_i = 0$ for all $i \in \mathbb{K}$.

Theorem 6: Let $\theta \geq 0$. Define matrices $E_{\theta i} = \kappa E_i + \theta E_i A_i$ and $G_{\theta i} = I + \theta E_i H_i$ for all $i \in \mathbb{K}$. If there exist $P_i > 0$ and $\Pi \in \mathcal{M}$ satisfying the matrix inequalities

$$\begin{bmatrix} A_i' P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j & P_i H_i - E_{\theta i}' \\ \bullet & -G_{\theta i} - G_{\theta i}' \end{bmatrix} < 0 \quad (22)$$

for all $i \in \mathbb{K}$ then with the state dependent switching function $u(x) = \arg \min_{i \in \mathbb{K}} x' P_i x$ and $\sigma(t) = u(x(t))$ the system (1)–(3) is globally asymptotically stable.

Proof: Consider the Lyapunov function candidate

$$V(x) = v(x) + 2\theta \sum_{k=1}^m \int_0^{y_k} \phi_k(\xi_k) d\xi_k \quad (23)$$

where $v(x) = \min_{i \in \mathbb{K}} x' P_i x$. The rest of the proof follows the same pattern of that of Theorem 3, being thus omitted. \square

The existence of a solution for inequality (22) cannot be expressed in the frequency domain. Actually, even though we make the assumption that $(E_i, H_i) = (E, H)$, $\forall i \in \mathbb{K}$, the matrix $E_{\theta i} = \kappa E + \theta E A_i$ is always index dependent. Fortunately, as before, adopting a state-input dependent switching function the next result circumvents this difficulty.

Theorem 7: Let $\theta \geq 0$. Assume that $E_i = E$ and define matrices $E_{\theta i} = \kappa E + \theta E A_i$ and $G_{\theta i} = I + \theta E H_i$ for all $i \in \mathbb{K}$. If there exist $P > 0$, symmetric matrices R_i , $\forall i \in \mathbb{K}$ and $\nu \in \Lambda$ satisfying $R_\nu = 0$ and the matrix inequalities

$$\begin{bmatrix} A_i' P + P A_i & P H_i - E_{\theta i}' \\ \bullet & -G_{\theta i} - G_{\theta i}' \end{bmatrix} + R_i < 0, \quad i \in \mathbb{K} \quad (24)$$

then with the state-input dependent switching function (20) the system (1)–(3) is globally asymptotically stable.

Proof: We consider the Lyapunov function candidate (23) with $v(x) = x' P x$ and that at an arbitrary $t \geq 0$ we have $\sigma(t) = i \in \mathbb{K}$. Performing the time derivative along a trajectory of the switched system (1)–(3) it follows that:

$$\begin{aligned} \dot{V}(x) &= 2x' P (A_i x + H_i w) + 2\theta \phi(y)' \dot{y} \\ &< -\max_{j \in \mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' R_j \begin{bmatrix} x \\ w \end{bmatrix} + 2(w + \kappa y)' w \\ &\leq -\begin{bmatrix} x \\ w \end{bmatrix}' R_\nu \begin{bmatrix} x \\ w \end{bmatrix} + 2(w + \kappa y)' w \\ &\leq 2(\phi(y) - \kappa y)' \phi(y) \end{aligned} \quad (25)$$

where we have used the switching function (20), the inequality (24) and the fact that $R_\nu = 0$. Taking into account that $\phi \in \Phi$ the conclusion is $\dot{V}(x) < 0$ for all $(x, w) \neq 0$, concluding thus the proof. \square

Adopting the same reasoning as before, the inequalities (24) are feasible whenever there exist $P > 0$ and $\nu \in \Lambda$ such that

$$\begin{bmatrix} A_\nu' P + P A_\nu & P H_\nu - E_{\theta \nu}' \\ \bullet & -G_{\theta \nu} - G_{\theta \nu}' \end{bmatrix} < 0 \quad (26)$$

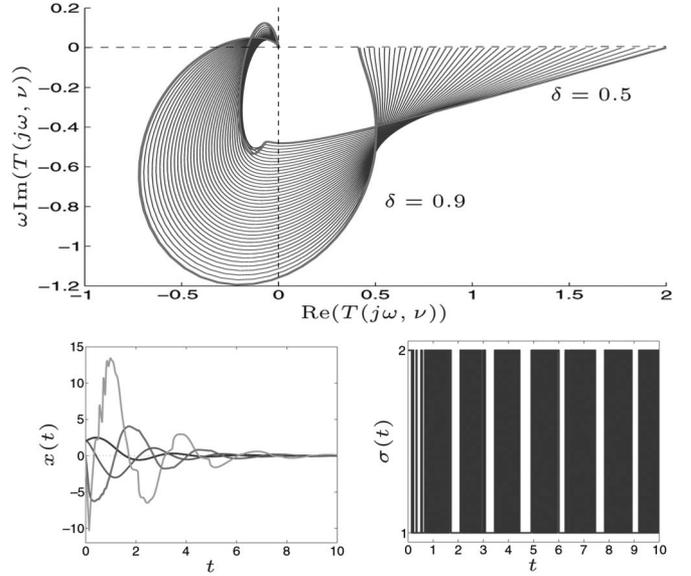


Fig. 1. Generalized Popov criterion, state trajectories and switching function.

where $E_{\theta \nu} = \kappa E + \theta E A_\nu$ and $G_{\theta \nu} = I + \theta E H_\nu$. This LMI is solvable with respect to $P > 0$ if and only if the parameterized transfer function $F(s, \nu) = E_{\theta \nu} (sI - A_\nu)^{-1} H_\nu + G_{\theta \nu}$ is ESPR. Fortunately, after some algebraic manipulations we obtain $F(s, \nu) = (\kappa + \theta s)T(s, \nu) + I$ and, consequently, in the scalar case, the transfer function $F(s, \nu)$ is ESPR if and only if

$$\operatorname{Re}(T(j\omega, \nu)) - \left(\frac{\theta}{\kappa}\right) \omega \operatorname{Im}(T(j\omega, \nu)) > -\kappa^{-1} \quad (27)$$

holds for all $\omega \in \mathbb{R}$. The values of $\theta > 0$, $\kappa > 0$ and $\nu \in \Lambda$ are calculated by selecting $\nu \in \Lambda$ and drawing the curves in the plane $\operatorname{Re}(T(j\omega, \nu)) \times \omega \operatorname{Im}(T(j\omega, \nu))$ for all $\omega \geq 0$. From this diagram, the crossing points in the (x, y) axis provide $(-1/\kappa, 1/\theta)$ such that (27) is satisfied. The vector $\nu \in \Lambda$ is selected so as the value of $\kappa > 0$ is maximized which yields the maximum sector of allowable nonlinearities $\phi \in \Phi$.

C. Illustrative Example

Let us consider the following example with two strictly proper subsystems:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -14 & -13 & -12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & -14 & -13 & 6 \end{bmatrix}$$

$H_1 = [0 \ 0 \ 0 \ 1]'$, $H_2 = [0 \ 0 \ 1 \ 2]'$ and $E_1 = E_2 = [2 \ 1 \ 0 \ 0]'$. Denoting $\nu = [\delta \ (1 - \delta)]'$, it can be verified that A_ν is Hurwitz for all $0.4 \leq \delta \leq 1.0$. Fig. 1 shows the Popov plot for $0.5 \leq \delta \leq 0.9$. For the extreme values in this interval we have obtained $(\kappa, \theta) \approx (6.25, 2.02)$ for $\delta = 0.5$ and $(\kappa, \theta) \approx (3.09, 6.45)$ for $\delta = 0.9$. Hence, adopting $\nu = [0.5 \ 0.5]'$, we have applied Theorem 7 to obtain matrices R_1 and R_2 . The adopted nonlinearity $\phi(y) = (\kappa/2)y(1 - \sin(5y))$ with $\kappa = 6.25$ satisfies the sector condition. Fig. 1 shows the time evolution of the state variables starting from the initial condition $x(0) = [2 \ 2 \ 0 \ 0]'$ as well as the switching rule $\sigma(t)$. This example illustrates that the state-input dependent switching function provided in Theorem 7 is very effective for stabilization. It is also seen the occurrence of intermittent stable sliding modes.

IV. CONCLUSION

In this technical note, we have generalized the celebrated Popov criterion to cope with nonlinear Lur'e-type switched systems. The key issue was the proposition of a new state-input dependent switching rule. The main result was derived from the conditions for the existence of a solution to the Lyapunov-Metzler inequalities which made possible the derivation of a stability criterion based on strict positive realness of some classes of transfer functions associated to the isolated subsystems. Hence, likewise the case of time invariant systems we have obtained a frequency domain stability test expressed by a convex combination of the subsystems state space matrices. A simple numerical example illustrated the validity of the proposed design method. As a perspective for future work we would like to mention the discrete-time counterpart of the present results.

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