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Concurrent Learning for Convergence in Adaptive Control without Persistency of Excitation

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Abstract— **We show that for an adaptive controller that uses recorded and instantaneous data concurrently for adaptation, a verifiable condition on linear independence of the recorded data is sufficient to guarantee exponential tracking error and parameter error convergence. This condition is found to be less restrictive and easier to monitor than a condition on persistently exciting exogenous input signal required by traditional adaptive laws that use only instantaneous data for adaptation.**

I. INTRODUCTION

Adaptive control has been widely studied for control of nonlinear plants with modeling uncertainties with wide ranging applications. Many of these approaches rely on the popular Model Reference Adaptive Control (MRAC) architecture which guarantees that the controlled states track the output of an appropriately chosen reference model. Most MRAC methods achieve this by using a parameterized model of the uncertainty, often referred to as the adaptive element and its parameters referred to as adaptive weights. In MRAC, the adaptive law is designed to update the parameters in the direction of maximum reduction of the instantaneous tracking error cost (e.g. $V(t) = e^T(t)e(t)$). While this approach ensures that the parameters take on values such that the uncertainty is instantaneously suppressed, they do not guarantee the convergence of the parameters to their ideal values unless the system states are Persistently Exciting (PE) [9], [7], [10], [2]. Boyd and Sastry have shown that the condition on PE system states can be related to a PE reference input by noting the following: If the exogenous reference input $r(t)$ contains as many spectral lines as the number of unknown parameters, then the plant states are PE, and the parameter error converges exponentially to zero [3]. However, the condition on persistent excitation of the reference input is restrictive and often infeasible to monitor online. For example, in flight control applications, PE reference inputs may cause nuisance, waste fuel, and may cause undue stress on the aircraft. Furthermore, since the exogenous reference inputs for many online applications are event based and not known a-priori, it is often impossible to monitor online whether a signal is PE. Consequently, parameter convergence is often not guaranteed in practice for many adaptive control applications.

In this paper we present a method that can guarantee exponential tracking error convergence and weight convergence in adaptive control without persistency of excitation. The presented method, termed as Concurrent Learning, uses recorded and current data concurrently for adaption in the framework of MRAC. The concurrent use of past and current data is motivated by the intuitive argument that if the recorded data is made sufficiently rich, and used concurrently for adaptation, then weight convergence can occur without the system states being persistently exciting. In this paper we formalize this intuitive argument and show that if the stored data has as many linearly independent elements as the dimension of the basis of the linearly parameterized uncertainty then exponential parameter and tracking error convergence to zero can be achieved.

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II. ADAPTIVE PARAMETER ESTIMATION WITHOUT PERSISTENCY OF EXCITATION

Adaptive parameter estimation is concerned with using measured output and regressor vectors to form an estimate of unknown system dynamics online. We assume that the unknown system dynamics are linearly parameterized; that is letting $y : \Re^m \to \Re$ denote the output of an unknown model whose regressor vectors $\Phi(x(t)) \in \Re^m$ are known, bounded, and continuously differentiable, and whose unknown parameters are contained in the constant ideal weight vector $W^* \in \Re^m$, the unknown system dynamics are given by:

$$y(t) = W^{*T}\Phi(x(t)). \tag{1}$$

Let $W(t) \in \Re^m$ denote our online estimate of the ideal weights W^*; since for a given x the mapping $\Phi(x)$ is known, then an online estimate of y can be given by the the mapping $\nu : \Re^m \to \Re$ in the following form:

$$\nu(t) = W^T(t)\Phi(x(t)). \tag{2}$$

This results in an approximation error $\epsilon(t) = \nu(t) - y(t)$, which can be represented as $\epsilon(t) = (W - W^*)^T(t)\Phi(x(t))$. Letting $\tilde{W}(t) = W(t) - W^*$ we have,

$$\epsilon(t) = \tilde{W}^T(t)\Phi(x(t)). \tag{3}$$

In the above form it is clear that $\epsilon(t) \to 0$ as $t \to \infty$ if the parameter error $\tilde{W} \to 0$ as $t \to \infty$. Therefore, we wish to design an adaptive law $\dot{W}(t)$, which uses the measurements of $x(t)$, $y(t)$, and the knowledge of the mapping $\Phi(.)$, to ensure $W(t) \to W^*$. Assuming that the full state $x(t)$ is available for measurement, a well known choice for $\dot{W}(t)$ is the following gradient based adaptive law which updates the adaptive weight in the direction of maximum reduction of the instantaneous quadratic cost $V = \epsilon^T(t)\epsilon(t)$[10],[7]:

$$\dot{W}(t) = -\Gamma\Phi(x(t))\epsilon(t). \tag{4}$$

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When using this adaptive law, it is well known that $W(t) \to W^*$ if and only if the vector signal $\Phi(x(t))$ is persistently exciting [10],[2], [7], [1],[9]. Various equivalent definitions of excitation and the persistence of excitation of a bounded vector signal exist in the literature [2],[9], we will use the definitions proposed by Tao in [10]:

Definition 1 A bounded vector signal $\Phi(t)$ is exciting over an interval $[t, t+T]$, $T > 0$ and $t \geq t_0$ if there exists $\gamma > 0$ such that

$$\int_t^{t+T} \Phi(\tau)\Phi^T(\tau)d\tau \geq \gamma I. \quad (5)$$

Definition 2 A bounded vector signal $\Phi(t)$ is persistently exciting if for all $t > t_0$ there exists $T > 0$ and $\gamma > 0$ such that

$$\int_t^{t+T} \Phi(\tau)\Phi^T(\tau)d\tau \geq \gamma I. \quad (6)$$

As an example, consider that in the two dimensional case, vector signals containing a step in every component are exciting, but not persistently exciting; whereas the vector signal $\Phi(t) = [\sin(t), \cos(t)]$ is persistently exciting.

On examining equation 4 we see that the adaptive law uses only instantaneously available information $(x(t), \epsilon(t))$ for adaptation. If the adaptive law used specifically selected and recorded data concurrently with current data for adaptation, and if the recorded data were sufficiently rich, then intuitively it should be possible to guarantee parameter convergence without requiring persistently exciting $\Phi(t)$. We now present a concurrent learning algorithm for adaptive parameter identification that builds on this intuitive concept. Let $j \in \{1, 2, ...p\}$ denote the index of a stored data point x_j, let $\Phi(x_j)$ denote the regressor vector evaluated at point x_j, let $\epsilon_j = \tilde{W}^T\Phi(x_j)$, let $\Gamma > 0$ denote a positive definite learning rate matrix, then the concurrent learning gradient descent algorithm is given as:

$$\dot{W}(t) = -\Gamma\Phi(x(t))\epsilon(t) - \sum_{j=1}^{p} \Gamma\Phi(x_j)\epsilon_j. \quad (7)$$

Without loss of generality, let $\Gamma = I$, then the parameter error dynamics for the concurrent learning gradient descent algorithm can be found by differentiating $\tilde{W}$ and using equation 7:

$$\begin{aligned}\dot{\tilde{W}}(t) &= -\Phi(x(t))\epsilon(t) - \sum_{j=1}^{p} \Phi(x_j)\epsilon_j \\ &= -\Phi(x(t))\Phi^T(x(t))\tilde{W}(t) - \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W}(t) \\ &= -[\Phi(x(t))\Phi^T(x(t)) + \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)]\tilde{W}(t).\end{aligned} \quad (8)$$

This is a linear time varying equation in $\tilde{W}$. We now present a condition on the linear independence of the stored data that characterizes the richness of the recorded data.

Condition 1 The recorded data has as many linearly independent elements as the dimension of $\Phi(x(t))$. That is, if $Z = [\Phi(x_1), ..., \Phi(x_p)]$, then $rank(Z) = m$.

This condition requires that the stored data contain sufficiently different elements to form a basis for the linearly parameterized uncertainty. This condition differs from the condition on PE $\Phi(t)$ in the following ways: 1) This condition applies only to recorded data which is a subset of all past data, whereas persistency of excitation applies also to how $\Phi(t)$ should behave in the future. 3) This condition is conducive to online monitoring since the rank of a matrix can be determined online . 4) It is always possible to record data such that condition 1 is met when the system states are exciting over a finite time interval. 5) It is also possible to meet this condition by selecting and recording data during a normal course of operation over a long period without requiring persistence of excitation.

The following theorem shows that condition 1 is sufficient to guarantee global exponential parameter convergence for concurrent learning gradient descent law of equation 7.

Theorem 1 If the stored data points satisfy condition 1, then $\tilde{W}$ is globally exponentially stable when using the concurrent learning gradient descent weight adaptation law of equation 7.

Proof: Let $V(\tilde{W}) = \frac{1}{2}\tilde{W}(t)^T\tilde{W}(t)$ be a Lyapunov candidate. Since $V(\tilde{W})$ is quadratic, there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha\|\tilde{W}\|^2 \leq V(\tilde{W}) \leq \beta\|\tilde{W}\|^2$. Differentiating w.r.t. time along the trajectories of 8:

$$\begin{aligned}\dot{V}(\tilde{W}) = &-\tilde{W}(t)^T[\Phi(x(t))\Phi^T(x(t)) \\ &+ \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)]\tilde{W}(t).\end{aligned} \quad (9)$$

Let $\Omega(t) = \Phi(x(t))\Phi^T(x(t)) + \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)$, and note that $P = \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j) > 0$ due to condition 1. Hence, $\Omega(t) > 0$ for all t. Furthermore, since $\Phi(x(t))$ is assumed to be continuously differentiable, there exists a $\lambda_m > 0$ such that,

$$\dot{V}(\tilde{W}) \leq -\lambda_m\tilde{W}(t)^T\tilde{W}(t) \leq -\lambda_m\|\tilde{W}\|^2. \quad (10)$$

Hence, using theorem 4.6 from [6] exponential stability of the zero solution $\tilde{W} \equiv 0$ of the parameter error dynamics of equation 8 is established. Furthermore, since the Lyapunov candidate is radially unbounded, the result is global. ■

Remark 1 The above proof shows exponential convergence of parameter estimation error without requiring persistency of excitation in the signal $\Phi(x(t))$. The proof requires that $\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)$ be positive definite, which is guaranteed if condition 1 is satisfied. Furthermore, note that the Lyapunov candidate does not depend on the number of recorded data points.

III. ADAPTIVE CONTROL WITHOUT PERSISTENCY OF EXCITATION

In this section, we consider the problem of tracking error and parameter error convergence in the framework of Model Reference Adaptive Control (MRAC) (see [9], [7], [2] and [10]).

A. Model Reference Adaptive Control

This section discusses the formulation of MRAC. Let $x(t) \in \Re^n$ be the known state vector, let $u \in \Re$ denote the control input, and consider the following system where the uncertainty can be linearly parameterized:

$$\dot{x} = Ax(t) + B(u(t) + \Delta(x(t))), \tag{11}$$

where $A \in \Re^{n \times n}$, $B \in \Re^n$, $B = [0, 0, ..., 1]^T$, and $\Delta(x) \in \Re(x)$ is a continuously differentiable function representing the scalar uncertainty. We assume that the system in 11 is controllable. A reference model can be designed that characterizes the desired response of the system:

$$\dot{x}_{rm} = A_{rm}x_{rm}(t) + B_{rm}r(t), \tag{12}$$

where $A_{rm} \in \Re^{n \times n}$ is a Hurwitz matrix and $r(t)$ denotes a bounded reference signal. A tracking control law consisting of a linear feedback part $u_{pd} = K(x_{rm}(t) - x(t))$, a linear feedforward part $u_{crm} = K_r[x_{rm}^T, r(t)]^T$, and an adaptive part $u_{ad}(x)$ is proposed to have the following form

$$u = u_{crm} + u_{pd} - u_{ad}. \tag{13}$$

Define the tracking error e as $e(t) = x_{rm}(t) - x(t)$; with an appropriate choice of u_{crm} such that $Bu_{crm} = (A_{rm} - A)x_{rm} + B_{rm}r(t)$, the tracking error dynamics are found to have the form

$$\dot{e} = A_m e + B(u_{ad}(x) - \Delta(x)), \tag{14}$$

where the baseline full state feedback controller $u_{pd} = Kx$ is assumed to be designed such that $A_m = A - BK$ is a Hurwitz matrix. Hence for any positive definite matrix $Q \in \Re^{n \times n}$, a positive definite solution $P \in \Re^{n \times n}$ exists to the Lyapunov equation

$$A_m^T P + PA_m + Q = 0. \tag{15}$$

We now state the following assumptions:

Assumption 1 The uncertainty $\Delta(x)$ can be linearly parameterized, that is, there exist a vector of constants $W = [w_1, w_2, ..., w_m]^T$ and a vector of continuously differentiable functions $\Phi(x) = [\phi_1(x), \phi_2(x), ..., \phi_m(x)]^T$ such that

$$\Delta(x) = W^{*T}\Phi(x). \tag{16}$$

Consider the case where the form of the linearly parameterized uncertainty is known, that is the mapping $\Phi(x)$ is known. In this case letting W denote the estimate W^* the adaptive element is chosen as $u_{ad}(x) = W^T\Phi(x)$. For this case it is well known that the adaptive law

$$\dot{W} = -\Gamma_W \Phi(x(t))e^T PB \tag{17}$$

where Γ_W is a positive definite learning rate matrix results in $e(t) \to 0$; however 17 does not guarantee the convergence (or even the boundedness) of W [10]. For this baseline adaptive law, it is also well known that a necessary and sufficient condition for guaranteeing $W(t) \to W$ is that $\Phi(t)$ be persistently exciting [9],[7],[10]. The condition on PE states is required to guarantee parameter error convergence to zero for many classic and recent adaptive control laws as well (e.g. σ-mod [7], e-mod [9], Q-mod [11], L_1 adaptive control [4], and backstepping adaptive control [8]).

B. Proof of Stability

In this section we present two key theorems to guarantee global tracking error and parameter error convergence go to zero when using concurrent learning adaptive control; without requiring persistency of excitation.

Theorem 2 Consider the system in equation 11, the control law of equation 13, the case of structured uncertainty (case 1), and the following weight update law:

$$\dot{W} = -\Gamma_W \Phi(x(t))e^T PB - \sum_{j=1}^{p} \Gamma_W \Phi(x_j)\epsilon_j, \tag{18}$$

and assume that the stored data points satisfy condition 1, then the zero solution $e(t) \equiv 0$ of tracking error dynamics of equation 14 is globally exponentially stable and $W(t) \to W^*$ exponentially.

Proof: Consider the following positive definite and radially unbounded Lyapunov candidate

$$V(e, \tilde{W}) = \frac{1}{2}e^T Pe + \frac{1}{2}\tilde{W}^T {\Gamma_W}^{-1}\tilde{W}. \tag{19}$$

Let $\xi = [e, \tilde{W}]$, and let $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ denote operators that return the smallest and the largest eigenvalue of a matrix, then we have $\frac{1}{2}\min(\lambda_{\min}(P), \lambda_{\min}({\Gamma_W}^{-1}))\|\xi\|^2 \leq V(e, \tilde{W})$
$\leq \frac{1}{2}\max(\lambda_{\max}(P), \lambda_{\max}({\Gamma_W}^{-1}))\|\xi\|^2$. Differentiating 19 along the trajectory of 14, the Lyapunov equation (equation 15), and noting that $\dot{\tilde{W}} = -\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W}(t) - \Gamma_W\Phi(x(t))e^T PB$, we have

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Qe + e^T PB(u_{ad} - \Delta) + \tilde{W}^T(-\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W}(t) - \Gamma_W\Phi(x(t))e^T PB). \tag{20}$$

Canceling like terms and simplifying we have

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Qe - \tilde{W}^T(\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j))\tilde{W}(t). \tag{21}$$

Let $\Omega = \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)$, then due to condition 1 $\Omega > 0$. Then, we have

$$\dot{V}(e, \tilde{W}) \leq -\frac{1}{2}\lambda_{\min}(Q)e^T e - \lambda_{\min}(\Omega)\tilde{W}^T\tilde{W}(t). \tag{22}$$

Hence,

$$\dot{V}(e,\tilde{W}) \leq -\frac{\max(\lambda_{\min}(Q), 2\lambda_{\min}(\Omega))}{\min(\lambda_{\min}(P), \lambda_{\min}(\Gamma_W{}^{-1}))} V(e,\tilde{W}), \quad (23)$$

establishing the exponential stability of the zero solution $e \equiv 0$ and $\tilde{W} \equiv 0$ (using Lyapunov stability theory, see Thm. 3.1 in [6]). Since $V(e,\tilde{W})$ is radially unbounded, the result is global and x tracks x_{ref} exponentially and $W(t) \to W^*$ exponentially as $t \to \infty$. ■

Remark 2 The above proof shows exponential convergence of tracking error $e(t)$ and parameter estimation error $\tilde{W}(t)$ to 0 without requiring persistency of excitation in the signal $\Phi(x(t)$. The only condition required is condition 1, which guarantees that the matrix $\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)$ is positive definite.

Remark 3 The inclusion or removal of new data points in equation 18 does not affect the Lyapunov candidate.

Remark 4 The rate of convergence is determined by the spectral properties of Q, P, Γ_W, and Ω, the first three are dependent on the choice of the linear gains K_p and the learning rates, and the last one is dependent on the choice of the stored data.

Remark 5 For evaluating the adaptive law of equation 18 the term $\epsilon_j = \nu(x_j) - \Delta(x_j)$ is required for the j^{th} data point where $j \in [1, 2, ..p]$. The model error $\Delta(x_j)$ can be observed by noting that

$$\Delta(x_j) = B^T[\dot{x_j} - Ax_j - Bu_j]. \quad (24)$$

Since A, B, x_j, u_j are known, the problem of estimating system uncertainty can be reduced to that of estimation of $\dot{x}$ by using 24. In cases where an explicit measurement for $\dot{x}$ is not available, $\dot{x_j}$ can be estimated using an implementation of a fixed point smoother [5].

In theorem 2 the adaptive law does not prioritize weight updates based on the instantaneous tracking error over the weight updates based on stored data. Such prioritization can be achieved by enforcing separation in the training law by restricting the weight updates based on past data to the nullspace of the weight updates based on current data. To achieve this, we let $\dot{W}_t(t) = \Phi(x(t))e^T PB$ and use the following orthogonal projection operator:

$$W_c(t) = \begin{cases} I - \dot{W}_t(\dot{W}_t(t)^T\dot{W}_t(t))^{-1}\dot{W}_t(t)^T & \text{if } \dot{W}_t(t) \neq 0 \\ I & \text{if } \dot{W}_t(t) = 0 \end{cases} \quad (25)$$

For this case, the following theorem ascertains that global asymptotic stability of the tracking error dynamics and asymptotic convergence of the parameter error to 0 is guaranteed subject to condition 1.

Theorem 3 Consider the system in equation 11, the control law of equation 13, the definition of $W_c(t)$, let for each time t N_Φ be the set containing all $\Phi(x_j) \perp \dot{W}_t(t)$, that is $N_\Phi = \{\Phi(x_j) : W_c(t)\Phi(x_j) = \Phi(x_j)\}$, and consider the following weight update law:

$$\dot{W} = -\Gamma_W \Phi(x(t))e^T PB - \Gamma_W W_c(t) \sum_{j \in N_\Phi} \Phi(x_j)\epsilon_j, \quad (26)$$

furthermore assume that the stored data points $\Phi(x_j)$ satisfy condition 1. Then the zero solution $e(t) \equiv 0$ of tracking error dynamics of equation 14 is globally asymptotically stable and $W(t) \to W^*$.

Proof: : Consider the following positive definite and radially unbounded Lyapunov candidate

$$V(e,\tilde{W}) = \frac{1}{2}e^T Pe + \frac{1}{2}\tilde{W}^T \Gamma_W{}^{-1}\tilde{W}. \quad (27)$$

Differentiating 27 along the trajectory of 14, the Lyapunov equation (equation 15), and noting that $\dot{\tilde{W}} = -\Gamma_W W_c(t) \sum_{j \in N_\Phi} \Phi(x_j)\Phi^T(x_j)\tilde{W}(t) - \Gamma_W \Phi(x(t))e^T PB$, we have

$$\dot{V}(e,\tilde{W}) = -\frac{1}{2}e^T Qe + e^T PB(u_{ad} - \Delta) + \tilde{W}^T(-W_c(t) \sum_{j \in N_\Phi} \Phi(x_j)\Phi^T(x_j)\tilde{W} - \Gamma_W \Phi(x(t))e^T PB). \quad (28)$$

Canceling like terms and simplifying we have

$$\dot{V}(e,\tilde{W}) = -\frac{1}{2}e^T Qe - \tilde{W}^T(W_c(t) \sum_{j \in N_\Phi}^{p} \Phi(x_j)\Phi^T(x_j))\tilde{W}. \quad (29)$$

Note that $\tilde{W} \in \Re^m$ can be written as $\tilde{W}(t) = (I - W_c(t))\tilde{W}(t) + W_c(t)\tilde{W}(t)$, where W_c is the orthogonal projection operator given in equation 25, furthermore $W_c^2(t) = W_c(t)$ and $(I - W_c(t))W_c(t) = 0$. Hence we have:

$$\begin{aligned}\dot{V}(e,\tilde{W}) &= -\frac{1}{2}e^T Qe \\ &- \tilde{W}^T W_c(t) \sum_{j \in N_\Phi}^{p} \Phi(x_j)\Phi^T(x_j)W_c(t)\tilde{W} \\ &- \tilde{W}^T W_c(t) \sum_{j \in N_\Phi}^{p} \Phi(x_j)\Phi^T(x_j)(I - W_c(t))\tilde{W}.\end{aligned} \quad (30)$$

However, since the sum in the last term of $\dot{V}(e,\tilde{W})$ is only performed on the elements in N_Φ we have that for all j $\Phi(x_j) = W_c(t)\Phi(x_j)$, therefore it follows that $\tilde{W}^T W_c(t) \sum_{j \in N_\Phi}^{p} W_c(t)\Phi(x_j)\Phi^T(x_j)W_c(t)(I - W_c(t))\tilde{W} = 0$, hence

$$\begin{aligned}\dot{V}(e,\tilde{W}) &= -\frac{1}{2}e^T Qe \\ &- \tilde{W}^T W_c(t) \sum_{j \in N_\Phi}^{p} \Phi(x_j)\Phi^T(x_j)W_c(t)\tilde{W} \leq 0.\end{aligned} \quad (31)$$

This establishes Lyapunov stability of the zero solution $e \equiv 0$, $\tilde{W} \equiv 0$. To show asymptotic stability, we must show that $\dot{V}(e, \tilde{W}) = 0$ only when $e = 0$ and $\tilde{W} = 0$. Consider the case when $\dot{V}(e, \tilde{W}) = 0$, since Q is positive definite, this means that $e = 0$. Let $e = 0$ and suppose *ad absurdum* there exists a $\tilde{W} \neq 0$ such that $\dot{V}(e, \tilde{W}) = 0$. Since $e = 0$ we have that $\dot{W}_t = 0$, hence from the definition of W_c (equation 25) $W_c = I$. Therefore it follows that the set N_Φ contains all the stored data points, therefore we have that $\tilde{W}^T \sum_{j=0}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W} = 0$. However, since the stored data points satisfy condition 1, $\tilde{W}^T \sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W} > 0$ for all $\tilde{W} \neq 0$, contradicting the claim. Therefore, we have shown that $\dot{V}(e, \tilde{W}) = 0$ only when $e = 0$ and $\tilde{W} = 0$. Thus establishing asymptotic stability of the zero solution $e \equiv 0$ and $\tilde{W} \equiv 0$. Guaranteeing x tracks x_{rm} asymptotically and $W \to W^*$ as $t \to \infty$. Since the Lyapunov candidate is radially unbounded, the result is global. ■

Remark 6 The above proof shows asymptotic convergence of tracking error $e(t)$ and parameter estimation error $\tilde{W}(t)$ without requiring persistency of excitation in the signal $\Phi(x(t))$. The only condition required is condition 1, which guarantees that the matrix $\sum_{j=1}^{p} \Phi(x_j)\Phi^T(x_j)$ is positive definite. Remarks 3 to 5 are also applicable to this theorem.

Remark 7 $\dot{V}(e, \tilde{W})$ will remain negative even when N_Φ is empty at time t if $e \neq 0$. If $e = 0$, N_Φ cannot remain empty due to the definition of W_c. Furthermore, If $e(t) = 0$ or $\Phi(x(t)) = 0$ and $\tilde{W}(t) \neq 0$, $\dot{V}(e, \tilde{W}) = \tilde{W}^T \sum_{j=0}^{p} \Phi(x_j)\Phi^T(x_j)\tilde{W} < 0$ due to condition 1 and the definition of $W_c(t)$ (equation 25). This indicates that parameter convergence will occur even when the tracking error or system states are not PE.

Remark 8 For practical applications the following approximations can be used: $N_\Phi = \{\Phi(x_j) : \|W_c(t)\Phi(x_j) - \Phi(x_j)\| < \beta\}$, where β is a small positive constant, and $W_c(t) = I$ if $|e(t)| < \alpha$ where α is a small positive constant.

IV. NUMERICAL SIMULATIONS

In this section we present results of numerical simulations that support the developed theory.

A. Adaptive Parameter Estimation

In this section we present a simple two dimensional example to illustrate the effect of condition 1. Let t denote the time, dt denote a discrete time interval, and for each $t + dt$ let $\theta(t)$ take on incrementally increasing values from $-\pi$ continuing on to 2π with an increment step equal to dt. Let $y = W^T\Phi(\theta)$ be be the uncertainty to be estimated online with $W = [0.1, 0.6]$ and $\Phi(\theta) = [1, e^{-\|\theta - \pi/2\|^2}]$. We note that y is the output of a RBF Neural Network with a bias term and one neuron. Figure 1 compares the model output y with the estimate ν for the concurrent learning parameter estimation algorithm of theorem 1 and the baseline gradient descent algorithm of equation 4. The concurrent learning gradient descent algorithm outperforms the baseline gradient descent. Figure 2 compares the trajectories of the online estimate of the ideal weights in the weight space. The dotted arrows denote the direction of update based only on current data, whereas the solid arrows denote the direction of weight updates based only on stored data. It can be seen that at the end of the simulation the concurrent learning gradient descent algorithm of theorem 1 arrives at the ideal weights (denoted by $*$) while the baseline gradient algorithm does not. On observing the arrows, we see that the weight updates based on both past and current data combine two linearly independent directions to improve weight convergence. This illustrates the effect of using recorded data when condition 1 is met. For this simulation the learning rate was set to $\Gamma = 5$ for both concurrent learning and baseline gradient descent case. Data points satisfying $\nu(t) - y(t) > 0.05$ were selected for storage and were used by the concurrent learning algorithm.

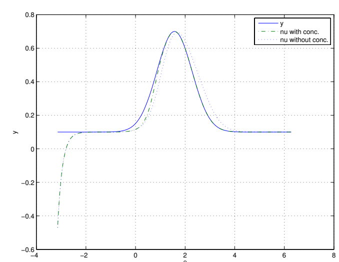

Fig. 1. Performance of online estimators

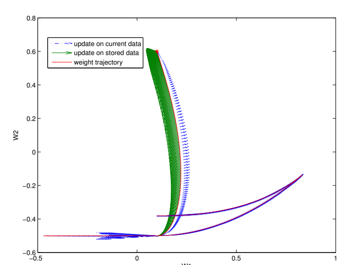

Fig. 2. Comparison of weight trajectories

B. Adaptive Control

In this section we present numerical simulation results of adaptive control of a nonlinear system. Let θ denote the angular position and δ denote the control input, then the unstable dynamics under consideration are given by:

$$\ddot{\theta} = \delta + \sin(\theta) - |\dot{\theta}|\dot{\theta} + 0.5e^{\theta\dot{\theta}}. \quad (32)$$

A second order reference model with natural frequency and damping ration of 1 is used, the linear controller is given by

$K = [1.5, 1.3]$, and the learning rate is set to $\Gamma_W = 3.5$. The initial conditions are set to $x(0) = [\theta(0), \dot{\theta}(0)] = [1, 1]$. The model uncertainty is given by $y = W^{*T}\Phi(x)$ with $W^* = [-1, 1, 0.5]$ and $\Phi(x) = [\sin(\theta), |\dot{\theta}|\dot{\theta}, e^{\theta\dot{\theta}}]$. A step in position ($\theta_c = 1$) is commanded at $t = 20sec$. Figure 3 compares the reference model tracking performance of the baseline adaptive control law of equation 17, the concurrent learning adaptive law of theorem 2, and the concurrent learning adaptive law theorem 3 ($W_c(t)$ as in 25). It can be seen that in both cases the concurrent learning adaptive laws outperform the baseline adaptive law, especially when tracking the step commanded at $t = 20sec$. The reason for this becomes clear when we examine the evolution of weights, for both concurrent learning laws, the weights are very close to their ideal values by this time, whereas for the baseline adaptive law, this is not true. This difference in performance is indicative of the benefit of parameter convergence. We note that in order to make a fair comparison the same learning rate (Γ_W) was used, with this caveat, we note that the concurrent learning adaptive law of theorem 2 outperforms the other two laws. Note that increasing Γ_W for the baseline case may result in an oscillatory response. Furthermore, note that approximately up to 3 seconds the tracking performance of the concurrent learning adaptive law of theorem 3 is similar to that of the baseline adaptive law, indicating that until this time the set N_Φ is empty. As sufficient stored data points become available such that the set N_Φ starts to become nonempty the performance of the concurrent learning adaptive law of theorem 3 approaches that of the concurrent learning adaptive law of theorem 2.

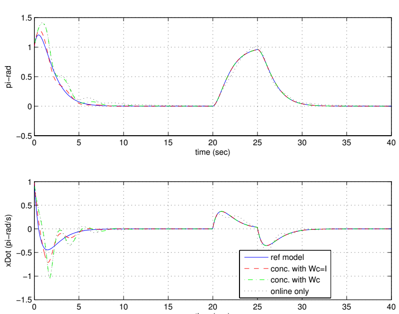

Fig. 3. Tracking performance of adaptive controllers

V. CONCLUSION

We presented a verifiable condition for guaranteeing global exponential stability of tracking error and parameter error dynamics in adaptive control problems when using Concurrent Learning. The presented condition requires that the recorded data have as many linearly independent elements as the dimension of the basis of the uncertainty. We also showed that if the adaptive law is structured such that weight updates on current data are given higher priority by restricting weight updates based on stored data to the nullspace of weight updates based on current data, then the same condition is sufficient to guarantee global asymptotic tracking error and weight convergence. Classical results require the exogenous input signal to have as many spectral lines as the dimension of the basis of the uncertainty (Boyd and Sastry 1986) and are well justified for adaptive controllers that use only current data for adaptation. Our results showed that if both recorded and current data is used concurrently for adaptation, then the condition for convergence relates directly to the spectrum of the recorded data. Such concurrent learning adaptive laws results in great performance benefits, and can guarantee exponential stability and convergence without requiring persistency of excitation.

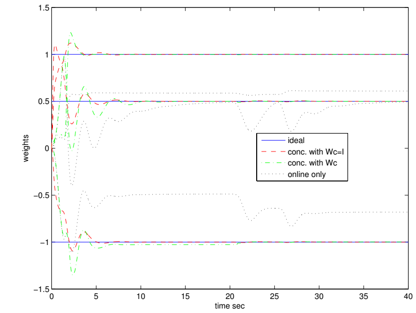

Fig. 4. Evolution of adaptive weights

VI. ACKNOWLEDGMENTS

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