Combining iISS and ISS With Respect to Small Inputs: The Strong iISS Property

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***Abstract*—This technical note studies the notion of Strong iISS, which is defined as the combination of input-to-state stability (ISS) with respect to small inputs, and integral input-to-state stability (iISS). This notion characterizes the robustness property that the state remains bounded as long as the magnitude of exogenous inputs is reasonably small, but may diverge for stronger disturbances. We provide several Lyapunov-based sufficient conditions for Strong iISS. One of them relies on iISS Lyapunov functions admitting a radially non-vanishing (class $\mathcal{K}$) dissipation rate. Although such dissipation inequality appears natural in view of the existing Lyapunov characterization of iISS and ISS, we show through a counter-example that it is not a necessary condition for Strong iISS. Less conservative conditions are then provided, as well as tools to estimate the tolerated input magnitude that preserves solutions' boundedness.**

***Index Terms*—Nonlinear dynamical systems, robustness, stability analysis.**

I. Introduction

Since its introduction by Sontag in 1989, input-to-state stability (ISS) [11] has become one of the key concepts in the analysis and control of nonlinear systems. The ISS property requires that the norm of solutions be upper-bounded by a vanishing transient term depending on the initial state, plus a term which is somewhat proportional to the magnitude of the input signal applied to the system. A strength of ISS stands in its Lyapunov characterization: any ISS system admits a storage function with a class $\mathcal{K}_\infty$ dissipation rate [14]. ISS provides a crucial robustness feature: for any bounded input, the resulting solutions are bounded as well (BIBS property): see [13] for a survey.

Despite its indubitable success, the solutions' boundedness under arbitrary bounded input (possibly of large amplitude) makes ISS a very strong requirement in many applications. Indeed, several practical systems do provide some robustness to inputs of "reasonable" amplitude, but generate unbounded behaviours when the applied disturbance is too intense. To overcome this intrinsic limitation, the ISS property has been relaxed following two main philosophies. The first one stands in a local version of ISS (namely LISS [15]), confining both the state and the input signal to a compact neighbourhood of the origin [7]. However, LISS being actually equivalent to asymptotic stability of the system without input (0-AS), the practical relevance of LISS is conditioned by the estimate of the domain of attraction and of the size of tolerated inputs [8]. The second one is known as integral input-to-state stability (iISS [12]). Instead of bounding the solution's norm by the amplitude of the applied input, it involves a term that quantifies the energy fed into the system. Its Lyapunov characterization [2] only differs from that of ISS by the fact that the dissipation rate is allowed to be any continuous positive definite function (rather than a class $\mathcal{K}_\infty$ function). The fact that the dissipation rate is not necessarily unbounded, and may even vanish for large values of the state norm, explains why iISS does not guarantee the BIBS property; in other words, some iISS systems may be destabilized by inputs of arbitrarily small amplitude.

The objective of this note is to introduce and characterize an intermediate property, halfway between the robustness strengths of ISS and the generality of iISS. More precisely, we define the notion of *Strong iISS* as the combination of ISS with respect to small inputs, and iISS. This combination ensures that the solutions of any Strongly iISS system are globally bounded as long as the amplitude of the input signal is below a specific input threshold, and, above this threshold, they inherit all properties of iISS systems.

We start by providing the necessary definitions (Section II). In Section III, we demonstrate by means of a counter-example that the intuitive characterization of Strong iISS by means of a Lyapunov function with a class $\mathcal{K}$ dissipation rate (i.e., not necessarily unbounded, but surely non vanishing for large states) is not valid in general. Alternative Lyapunov conditions for Strong iISS are provided in Section IV: in particular we show that, if ISS with respect to small inputs and forward completeness can be shown based on the same Lyapunov function, then the system is Strongly iISS. Specific results for bilinear systems are also provided. Some conclusive remarks and a comparative table between the robustness features guaranteed by iISS, Strong iISS and ISS are provided in Section V. Proofs are provided in Section VI.

Notation: We use the standard definitions for $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ comparison functions. A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}$ if it is continuous and positive definite. For a nondecreasing continuous function $\gamma:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$, $\gamma(\infty)\in\mathbb{R}_{\geq 0}\cup\{\infty\}$ denotes the quantity $\lim_{s\to+\infty}\gamma(s)$. Given $x\in\mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given $A\in\mathbb{R}^{n\times n}$, $|A|:=\max_{|x|=1}|Ax|$, i.e. $|A|$ is the largest singular value of A. Given $\delta\geq 0$, $\mathcal{B}_\delta:=\{x\in\mathbb{R}^n:|x|\leq\delta\}$. Given a positive integer m, $\mathcal{U}^m$ denotes the set of all measurable locally essentially bounded functions $u:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$. For a given $u\in\mathcal{U}^m$ and a set $\mathcal{I}\subset\mathbb{R}_{\geq 0}$, $\|u\|:=\operatorname{ess\,sup}_{t\geq 0}|u(t)|$ and $\|u\|_{\mathcal{I}}:=\operatorname{ess\,sup}_{t\in\mathcal{I}}|u(t)|$. Given a constant $R>0$, we let $\mathcal{U}^m_{<R}$ denote the set $\{u\in\mathcal{U}^m:\|u\|<R\}$. $\mathrm{sat}:\mathbb{R}\to\mathbb{R}$ is the scalar saturation function defined as $\mathrm{sat}(s)=\min\{1;|s|\}\mathrm{sign}(s)$ for all $s\in\mathbb{R}$. Given $a\in\mathbb{R}\cup\{\infty\}$, and two continuous function $f,g:\mathbb{R}\to\mathbb{R}$, we write $g(s)=\mathcal{O}(f(s))$ as $s\to a$ if $\limsup_{s\to a}|g(s)/f(s)|<\infty$. A function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ is called a *storage function* if it is continuously differentiable and satisfies $V(0)=0$ and $V(x)>0$ for all $x\neq 0$. It is said to be *proper* if, in addition, the set $\{x\in\mathbb{R}^n:V(x)\leq c\}$ is compact for any fixed $c\geq 0$.

II. Definitions

We consider nonlinear systems with exogenous inputs:

$$\dot{x}=f(x,u), \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^m$ is the input and $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ denotes a locally Lipschitz function satisfying $f(0,0)=0$. Given $x_0\in\mathbb{R}^n$ and an input signal $u\in\mathcal{U}^m$, the solution of (1) starting at x_0 at time $t=0$ is referred to as $x(\cdot;x_0,u)$ (or simply $x(\cdot)$ when the context is clear) on the time domain where it is defined. We start by recalling the definition of iISS, orginally introduced in [12]: the system

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(1) is said to be *Integral Input-to-State Stable* if there exist a class $\mathcal{KL}$ function β and class $\mathcal{K}_\infty$ functions μ_1, μ_2 such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}^m$, its solution satisfies, for all $t \geq 0$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|u(s)|)\, ds\right).$$

It is well known that iISS guarantees to (1) some robustness with respect to a class of inputs with finite energy. In particular, it was shown in [12, Prop. 6] that if the above estimate holds then, for all $u \in \mathcal{U}^m$ satisfying $\int_0^\infty \mu_2(|u(s)|)ds < \infty$, all solutions of (1) converge to the origin. In most practical applications, however, more stringent robustness properties are usually desired. Two particularly relevant features are the guaranteed boundedness of solutions in response to specific bounded inputs (*bounded input-bounded state* (BIBS) property, [13]), and the fact that solutions converge to zero if the input vanishes (*converging input-converging state* (CICS) property, [13]). It is known that iISS systems may generate unbounded solutions in response to arbitrarily small inputs. For instance, the system $\dot{x} = -(x/(1+x^2)) + u$ is iISS, but given any constant input $u^\star > 0$ (even arbitrarily small), any solution starting from an initial condition $x_0 \geq u^\star$ grows unbounded. In the same way, vanishing inputs may generate unbounded solutions for iISS system; see e.g. [3], [4].

In summary, iISS ensures neither BIBS nor CICS. Nonetheless, many iISS systems do exhibit some robustness to vanishing inputs, or inputs with sufficiently small magnitude. This is obviously the case of all ISS systems [11], which are known to generate bounded solutions for any bounded inputs. A lot of dynamical systems can stand exogenous inputs that are sufficiently small, but exhibit non proper behaviors for too large inputs. A natural way to describe this limited robustness property is to consider ISS only for small inputs.

Definition 1 (ISS wrt Small Inputs): The system (1) is said to be *Input-to-State Stable with respect to small inputs* if there exist a constant $R > 0$ (referred to as an input threshold) and functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}^m_{<R}$, its solution satisfies, for all $t \geq 0$

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \mu(\|u\|). \tag{2}$$

ISS with respect to small inputs constitutes a stronger requirement than the property referred to as local ISS (LISS) in e.g. [7], [15]. Indeed, it imposes that (2) holds for small inputs, but over the whole state space $\mathbb{R}^n$, whereas LISS is a local property both in the state and in the input. In particular, LISS does not ensure global asymptotic stability of the origin for the unperturbed dynamics (0-GAS), while ISS with respect to small inputs does. Formally, LISS is actually equivalent to 0-AS. It is thus of limited interest for robustness analysis purposes, unless both the domain of attraction and the magnitude of tolerated inputs can be estimated [8].

If the state estimate (2) holds for all $u \in \mathcal{U}^m$, then we recover the classical definition of ISS [11]. However, given a finite R, the above property does not provide any information on the behavior of the system when the magnitude of the input signal u overpasses R. In particular, the system's solutions are not even guaranteed to exist at all time when $\|u\| > R$, as illustrated by the following example.

Example 1: Consider the scalar system $\dot{x} = -x + u + \xi(u)x^2$, where $\xi : \mathbb{R} \to \mathbb{R}$ denotes any locally Lipschitz function satisfying $\xi(u) = 0$ for all $u \in [-1; 1]$ and $|\xi(u)| \geq 1$ for all $|u| \geq 2$. For instance, one can pick $\xi(u) := \mathrm{sat}(u-2) + \mathrm{sat}(u+2)$. This system can easily be shown to be ISS with respect to small inputs, namely $u \in \mathcal{U}^1_{<1}$. However, for any constant $u \geq 2$, the solution satisfies $\dot{x}(t) \geq -x(t) + x(t)^2$, which has finite escape time for any initial state greater than 1.

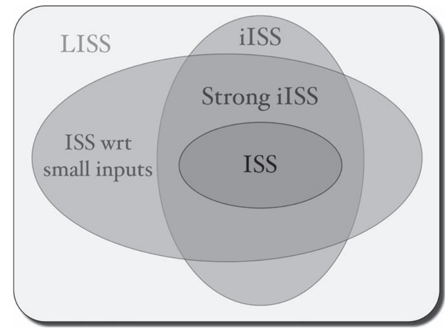

Fig. 1. Schematic hierarchy between ISS-related concepts.

Therefore, while ISS with respect to small inputs guarantees interesting robustness properties when the system is perturbed by sufficiently small inputs, an additional requirement is needed to ensure at least the forward completeness of the system for larger inputs. The aim of this technical note is to study a property half way between the strength of ISS and the generality of iISS: the Strong iISS.

Definition 2 (Strong iISS): The system (1) is said to be *Strongly iISS* if it is both iISS, and ISS with respect to small inputs. In other words, there exist $R > 0$, $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$ such that, for all $u \in \mathcal{U}^m$, all $x_0 \in \mathbb{R}^n$ and all $t \geq 0$, its solution satisfies the following two properties:

$$|x(t)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|u(s)|)\, ds\right) \tag{3}$$

$$\|u\| < R \Rightarrow |x(t)| \leq \beta(|x_0|, t) + \mu(\|u\|). \tag{4}$$

The constant R is called an *input threshold* and the function μ is referred to as a *supply gain* for (1).

We stress that, for a given system (1), the choice of the input threshold R is not unique; the above functions β and μ may actually depend on the choice of R.

As an immediate consequence of this definition, we can see from (3) that Strong iISS implies the existence of solutions of (1) for any input $u \in \mathcal{U}^m$ (forward completeness). Elementary considerations from (4) also show that, like ISS but unlike iISS, the Strong iISS property guarantees the CICS property. It also ensures the BIBS for all inputs of magnitude smaller than the input threshold R. More generally, in response to any $u \in \mathcal{U}^m_{<R}$, Strong iISS ensures the asymptotic convergence of solutions to a neighborhood of the origin whose size is "proportional" to the magnitude of the applied input. This is known as the *asymptotic gain* (AG) property in [15] and can be formally defined as the existence of a positive definite continuous function $\sigma : [0; R) \to \mathbb{R}_{\geq 0}$ such that solutions of (1) satisfy

$$\limsup_{t \to \infty} |x(t; x_0, u)| \leq \sigma(\|u\|), \tag{5}$$

provided that $u \in \mathcal{U}^m_{<R}$, $R > 0$. ISS is known to be equivalent to 0-GAS plus AG for all $u \in \mathcal{U}^m$ [15, Th. 1]. Similarly, the next proposition states that Strong iISS is equivalent to iISS plus AG for small inputs. The proof is straightforward, and is therefore omitted.

Proposition 1 (iISS+AG): The system (1) is Strongly iISS with input threshold $R > 0$ if and only if it is iISS and its solutions satisfy the asymptotic gain property (5) for all $u \in \mathcal{U}^m_{<R}$, where $\sigma \in \mathcal{K}_\infty$.

In a nutshell, ISS is thus a special case of Strong iISS, which itself combines ISS with respect to small inputs and iISS, which both are special cases of LISS. Fig. 1 summarizes these inclusions.

The rest of this technical note aims at providing some insights into the Strong iISS property, to confront it with existing robustness properties, and to study its behavior under systems interconnection.

III. Class $\mathcal{K}$ Dissipation Rate

A. Sufficient Condition

It is well known that both ISS and iISS of (1) are equivalent to the existence of a proper storage function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and a class $\mathcal{K}_\infty$ function γ satisfying for all $x\in\mathbb{R}^n$ and all $u\in\mathbb{R}^m$

$$\frac{\partial V}{\partial x}(x)f(x,u)\leq -\alpha\left(|x|\right)+\gamma\left(|u|\right). \tag{6}$$

The function α is often referred to as a *dissipation rate* associated to the storage function V. If $\alpha\in\mathcal{K}_\infty$, then we recover the ISS characterization [14]. If $\alpha\in\mathcal{PD}$, then the above dissipation inequality is equivalent to the iISS of (1) [2]. Among all $\mathcal{PD}$ functions α, we may distinguish those satisfying $\liminf_{s\to\infty}\alpha(s)=0$ (*vanishing* $\mathcal{PD}$ *functions*) from those satisfying $\liminf_{s\to\infty}\alpha(s)>0$ (*non-vanishing* ones). A natural conjecture is then that, when α is a non-vanishing $\mathcal{PD}$ function, the estimate (6) is equivalent to Strong iISS. Indeed, such a dissipation rate would imply the decrease of V for large values of the state if the input is of sufficiently small amplitude. The following statement, proved in Section VI-A, establishes that the existence of such a dissipation rate does ensure Strong iISS.

Theorem 1 (Non-Vanishing Dissipation Rate $\Rightarrow$ Strong iISS): Assume that there exists a proper storage function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and a function $\gamma\in\mathcal{K}_\infty$ satisfying, for all $x\in\mathbb{R}^n$ and all $u\in\mathbb{R}^m$

$$\frac{\partial V}{\partial x}(x)f(x,u)\leq -W(x)+\gamma\left(|u|\right), \tag{7}$$

where $W:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ is a continuous positive definite function satisfying $W_\infty:=\liminf_{|x|\to\infty}W(x)>0$. Then the system (1) is Strongly iISS with input threshold $R=\gamma^{-1}(W_\infty)$.

We mention the fact that, for the case when $R=\infty$, we recover the Lyapunov characterization of ISS [14]. We also stress that any continuous positive definite function $W:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ satisfying $\liminf_{|x|\to\infty}W(x)=W_\infty>0$ can be lower bounded by a class $\mathcal{K}$ function α satisfying $\alpha(\infty)=W_\infty$. Thus Theorem 1 essentially requires a class $\mathcal{K}$ dissipation rate.

B. A Counter-Example

In view of the Lyapunov characterizations of iISS and ISS, a natural conjecture would be that the existence of a class $\mathcal{K}$ dissipation rate is not only sufficient but also necessary for Strong iISS. While this conjecture may sound quite intuitive, it happens to be wrong as shown by the following counter-example.

Example 2 (Strong iISS $\not\Rightarrow$ $\mathcal{K}$ Rate): Consider the scalar system

$$\dot{x}=-\frac{x}{1+x^2}\left[1-|x|\left(u^2-|u|\right)\right]. \tag{8}$$

We claim that: a) this system is Strongly iISS; but b) given any continuous positive definite function $W:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ satisfying $\liminf_{|x|\to\infty}W(x)>0$ and any $\gamma\in\mathcal{K}_\infty$, no differentiable function $V:\mathbb{R}\to\mathbb{R}_{\geq 0}$ may satisfy (7). Thus, (8) admits no storage function with a class $\mathcal{K}$ dissipation rate.

To prove Item a), notice that iISS can be established using the proper storage function $V_1(x):=(1/2)\ln(1+x^2)$, whereas ISS with respect to inputs in $\mathcal{U}^1_{<1}$ can be shown considering $V_2(x):=x^4/4$. For Item b), consider any differentiable function $V:\mathbb{R}\to\mathbb{R}_{\geq 0}$. Then it holds that

$$\dot{V}=-\frac{\partial V}{\partial x}(x)\frac{x}{1+x^2}+\frac{\partial V}{\partial x}(x)\frac{x|x|}{1+x^2}\left(u^2-|u|\right). \tag{9}$$

Therefore, the dissipation inequality (7) may only be satisfied with a non-vanishing function W if the term $(\partial V/\partial x)(x)(x/(1+x^2))$ does not vanish for large state values (as it is the only negative term in (9) whenever $x\geq 0$ and $|u|\geq 1$). This imposes in particular that $\liminf_{x\to\infty}(1/x)(\partial V/\partial x)(x)\geq c$, for some constant $c>0$. Considering $u=2$, it then follows from (9) that $\liminf_{x\to+\infty}\dot{V}\geq\lim_{x\to+\infty}-c+2cx=+\infty$, whereas (7) imposes $\dot{V}\leq\gamma(2)$ for all $x\in\mathbb{R}$. This contradiction establishes Item b).

Example 2 thus disallows the converse of Theorem 1. In other words, the class of iISS systems with $\mathcal{K}$ dissipation rate is a strict subset of the class of Strongly iISS systems. Section IV is devoted to the development of less conservative conditions for Strong iISS.

C. Estimating the Input Threshold

Even though Theorem 1 does not constitute a necessary and sufficient characterization of Strong iISS, it yet happens to be very handy in practice. As we will see in Section IV-C, a typical way of invoking this result is through a quadratic Lyapunov function scaled by a logarithmic function. The following result focuses on such storage functions, and provides an explicit estimate of the input threshold. For inputs below this threshold, it also gives an estimate of the asymptotic gain, as recalled in (5). Its proof is provided in Section VI-B.

Theorem 2 (Input Threshold Estimate): Let $P,Q\in\mathbb{R}^{n\times n}$ denote two symmetric positive definite matrices and assume that the function $V(x):=\ln(1+x^TPx)$ satisfies, for all $x\in\mathbb{R}^n$ and all $u\in\mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x)f(x,u)\leq -\frac{x^TQx}{1+x^TPx}+\gamma\left(|u|\right),$$

where $\gamma\in\mathcal{K}_\infty$. Then the system (1) is Strongly iISS and its input threshold can be picked as

$$R=\gamma^{-1}(\lambda_0), \tag{10}$$

where $\lambda_0>0$ denotes the smallest solution of the algebraic equation $\det(Q-\lambda P)=0$. Furthermore, given any $u\in\mathcal{U}^m_{<R}$, the asymptotic gain property (5) holds with a function σ defined as

$$\sigma(s):=\sqrt{\frac{1}{p_m}\ln\left(\frac{\lambda_0}{\lambda_0-\gamma(s)}\right)},\quad \forall s\in[0;R), \tag{11}$$

where $p_m>0$ is the smallest eigenvalue of P.

IV. Alternative Conditions for Strong iISS

A. Characterization Using Two Storage Functions

We start by stressing that the existing Lyapunov characterizations for ISS [14] and iISS [2] can be readily employed to provide a characterization of Strong iISS based on two storage functions. More precisely, based on those works, it is straightforward to show that (1) is Strongly iISS with input threshold $R>0$ if and only if there exist two proper storage functions V and W, $\rho \in \mathcal{PD}$, and $\alpha, \eta, \gamma \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\rho(|x|) + \eta(|u|) \tag{12a}$$

$$|u| < R \Rightarrow \frac{\partial W}{\partial x} f(x,u) \leq -\alpha(|x|) + \gamma(|u|). \tag{12b}$$

Condition (12b) can be equivalently stated as

$$|u| < \kappa(|x|) \quad \Rightarrow \quad \frac{\partial \tilde{W}}{\partial x}(x) f(x,u) \leq -\tilde{\rho}(|x|), \tag{13}$$

with $\kappa \in \mathcal{K}$, $\tilde{\rho} \in \mathcal{PD}$, and a proper storage function $\tilde{W}$.

The combined properties (12a) and (13) can easily be checked on Example 2 with the functions $V(x) = \tilde{W}(x) = (1/2)\ln(1+x^2)$ by picking $\rho(s) = \tilde{\rho}(s) = s^2/(1+s^2)^2$, $\eta(s) = s$, and $\kappa(s) = \mathrm{sat}(s)$ for all $s \geq 0$. It is worth stressing that, for this example, (12a) and (13) can be established through a unique storage function.

Even though these combined dissipation inequalities do constitute a necessary and sufficient condition for Strong iISS, their practical exploitation is not as handy as dissipation inequalities involving a unique storage function. Indeed, condition (12a) is valid for all x and u but does not guarantee the state boundedness even to small inputs, while conditions (12b) or (13) provide no information on the behavior of W for large inputs. This "switching" between subsets may complicate the analysis. It is not clear at this stage whether Strong iISS systematically guarantees the existence of a single storage function satisfying both (12a) and (13).

The rest of this section is thus devoted to the development of alternative conditions for Strong iISS, relying on a single Lyapunov function. We have not yet succeeded in establishing the necessity of any of the conditions we present below; the characterization of Strong iISS using a single storage function thus remains an open question.

B. Link With Forward Completeness

As illustrated by Example 1, unlike ISS with respect to small inputs, Strong iISS guarantees forward complete even for large inputs. The next result states that if both ISS with respect to small inputs and forward completeness can be shown through the same Lyapunov function, then the system is Strongly iISS.

Theorem 3: Assume that there exist a proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, some functions $\alpha, \gamma \in \mathcal{K}$, two constants $R_x, R_u \geq 0$ and two continuous nondecreasing functions $\nu_1, \nu_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$|u| \leq R_u \Rightarrow \frac{\partial V}{\partial x} f(x,u) \leq -\alpha(|x|) + \gamma(|u|) \tag{14}$$

$$|x| \geq R_x \Rightarrow \frac{\partial V}{\partial x} f(x,u) \leq \nu_1(|u|) V + \nu_2(|u|). \tag{15}$$

Then the system (1) is Strongly iISS.

Note that (14) immediately implies that (1) is ISS with respect to small inputs [14]; the contribution of the above result is mostly to establish iISS. Its proof is provided in Section VI-C.

In view of [1], condition (15) constitutes a characterization of the forward completeness of (1). Hence, Theorem 3 basically states that if the forward completeness and the ISS with respect to small inputs can be established through the study of the same storage function, then the system is also iISS (thus Strongly iISS). An extension of the counter-example in [2, Sec. V] shows that, in general, ISS with respect to small inputs and forward completeness are not enough to guarantee iISS (thus not Strong iISS either); the use of a common Lyapunov function for the two properties is thus crucial here.

The assumptions of Theorem 1 clearly guarantee those of Theorem 3. In addition, the Strong iISS of the system in Example 2 can be checked with Theorem 3 with the function $V(x) = (1/2)\ln(1+x^2)$. Hence, Theorem 3 is also strictly less conservative than Theorem 1.

Moreover, Theorem 3 allows to recover and extend a result originally presented in [12].

Corollary 1: Assume that there exist a proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\lambda, \gamma \in \mathcal{K}_\infty$, and a constant $q > 0$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -(q - \lambda(|u|)) V(x) + \gamma(|u|). \tag{16}$$

Then the system (1) is Strongly iISS.

This dissipation inequality was used in [12, Th. 2] to establish iISS. Corollary 1 states that Strong iISS actually holds under the same condition, thus guaranteeing the converging input-converging state property and the asymptotic gain property (5) with respect to small inputs without any additional assumption. Corollary 1 is a consequence of Theorem 3; its proof is provided in Section VI-D.

C. Bilinear Systems

In this section, we focus on the particular class of bilinear systems, that is systems of the form:

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i\right) x + Bu, \tag{17}$$

where $A \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \{1,\ldots,m\}$, and $u = (u_1,\ldots,u_m)^T$. A necessary and sufficient condition (namely, A Hurwitz) for the iISS of such systems was established in [12, Th. 5]. The following statement, proved in Section VI-E, establishes that Strong iISS actually holds under the same condition, thus additionally ensuring the asymptotic gain property (5) for inputs below a certain threshold.

Corollary 2 (Strong iISS of Bilinear Systems): The bilinear system (17) is Strongly iISS if and only if the matrix A is Hurwitz. In that case, letting $P = P^T > 0$ and $Q = Q^T > 0$ denote any matrices satisfying $A^T P + PA \leq -Q$, an input threshold for (17) is

$$R = \frac{\lambda_0 p_m / (2 + \sqrt{p_m})}{|P| \max\left\{b_M; \sum_{i=1}^m |A_i|\right\}},$$

where λ_0 is the smallest solution of $\det(Q - \lambda P) = 0$, p_m is the smallest eigenvalue of P and b_M is the largest singular value of B.

D. Exploiting Zero-Output Dissipativity

Based on the notion of *zero-output smooth dissipativity*, we can derive an alternative growth order condition on the supply functions for Strong iISS. This notion, introduced in [2], imposes the existence of a proper storage function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial W}{\partial x}(x) f(x,u) \leq \sigma(|u|), \tag{18}$$

where $\sigma \in \mathcal{K}$. The following result is a consequence of Theorem 1 that exploits this property.

Proposition 2: Assume that (1) is zero-output smooth dissipative and that there exist a differentiable function $V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and

	iISS	Strong iISS	ISS
0-GAS	Yes	Yes	Yes
BIBS	No	For $\|u\| < R$	Yes
CICS	No	Yes	Yes
Cascade preservation	Under growth conditions	Yes	Yes

Fig. 2. Features guaranteed by iISS, Strong iISS and ISS: global asymptotic stability without inputs (0-GAS), bounded input-bounded state (BIBS), converging input-converging state (CICS), and preservation under cascade.

$\kappa_0, \alpha_0, \eta_0, \gamma_0 \in \mathcal{K}$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$V_0(x) \geq \kappa_0\left(|x|\right) \tag{19}$$

$$\frac{\partial V_0}{\partial x} f(x,u) \leq -\alpha_0(V_0) + \eta_0(V_0)\gamma_0\left(|u|\right), \tag{20}$$

with η_0 and α_0 satisfying

$$\eta_0(s) = \mathcal{O}\left(\alpha_0(s)\right) \quad \text{as } s \to \sup_{x\in\mathbb{R}^n} V_0(x). \tag{21}$$

Then there exists a proper storage function V satisfying (6) with a $\mathcal{K}$ dissipation rate α and $\gamma \in \mathcal{K}_\infty$. In particular, (1) is Strongly iISS.

The proof of this result is provided in Section VI-F. Is is worth noting that (20) is actually a direct consequence of 0-GAS of (1), as stated in [2, Lemma IV.10]. The main requirement of the above definition therefore stands on the growth constraint (21). Note that, due to the continuity of η_0 and α_0 on $\mathbb{R}_{\geq 0}$, this condition is trivial if V_0 is bounded. Notice indeed that the function V_0 is not required to be a proper storage function. In particular, it is not necessarily radially unbounded. Condition (19) only imposes that V_0 is positive for all $x \in \mathbb{R}^n \setminus \{0\}$ and that it does not vanish for large values of the state norm. If, however, V_0 is a proper storage function, then the assumption of zero-output smooth dissipativity can be replaced by a growth condition on η_0 independent of α_0, as summarized by the following statement whose proof is provided in Section VI-G.

Corollary 3: Assume that (20) and (21) hold for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, with a proper storage function $V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}$ functions κ_0, α_0, η_0 and γ_0. Assume also that

$$\int_0^{+\infty} \frac{ds}{1+\eta_0(s)} = +\infty. \tag{22}$$

Then there exists a proper storage function V satisfying (6) with a $\mathcal{K}$ dissipation rate α and $\gamma \in \mathcal{K}_\infty$. In particular, (1) is Strongly iISS.

V. Conclusion

We have introduced the notion of Strong iISS and showed that it constitutes an interesting compromise between the strengths of ISS and the generality of iISS. After showing that the natural Lyapunov characterization through a class $\mathcal{K}$ dissipation rate fails to be a necessary requirement for this property, we have presented alternative Lyapunov tools that may be used to establish Strong iISS and estimate the maximum input magnitude that can be tolerated without compromising the solutions' boundedness. We have applied these results to bilinear systems. We have also stressed the interest of providing necessary and sufficient conditions for Strong iISS in terms of a single Lyapunov function for future research. We can summarize the respective features of iISS, Strong iISS and ISS by the comparative table of Fig. 2. The fact that cascades of Strongly iISS systems are themselves Strongly iISS is stated in [5] and will be the subject of another technical note [6].

VI. Proofs

A. Proof of Theorem 1

As stressed below Theorem 1, any continuous positive definite function W satisfying $W_\infty := \liminf_{|x|\to\infty} W(x) > 0$ can be lower bounded by a $\alpha \in \mathcal{K}$ satisfying $\alpha(\infty) = W_\infty$. It then holds that $(\partial V/\partial x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|)$. Since $\alpha \in \mathcal{K} \subset \mathcal{PD}$, iISS of (1) directly follows from the classical iISS characterization [2]. In addition, let $R = \gamma^{-1}(W_\infty) = \gamma^{-1} \circ \alpha(\infty) > 0$ as proposed in the statement. Note that α is then invertible over $[0; \gamma(R))$. Given any fixed input $u \in \mathcal{U}_{<R}$, let

$$\varepsilon\left(\|u\|\right) := \frac{1}{2}\left(1 - \frac{\gamma\left(\|u\|\right)}{\alpha(\infty)}\right). \tag{23}$$

Then α is invertible over $[0; \gamma(\|u\|)/(1-\varepsilon(\|u\|))]$, and it holds that

$$|x| \geq \alpha^{-1}\left(\frac{\gamma\left(\|u\|\right)}{1-\varepsilon\left(\|u\|\right)}\right) \Rightarrow \frac{\partial V}{\partial x} f(x,u) \leq -\varepsilon\left(\|u\|\right)\alpha\left(|x|\right).$$

The constant $\varepsilon(\|u\|)$ being positive since $u \in \mathcal{U}_{<R}$, [10, Th. 4.18] guarantees that the solutions of (1) are bounded and that

$$\limsup_{t\to\infty} |x(t;x_0,u)| \leq \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \mu\left(\|u\|\right), \quad \forall x_0 \in \mathbb{R}^n, \tag{24}$$

where $\mu(\|u\|) := \alpha^{-1}(\gamma(\|u\|)/(1-\varepsilon(\|u\|)))$ and $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ satisfy $\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$ for all $x \in \mathbb{R}^n$, as ensured by the fact that V is a proper storage function. Noticing that, from (23),

$$\mu\left(\|u\|\right) = \alpha^{-1}\left(\frac{2\gamma\left(\|u\|\right)\alpha(\infty)}{\alpha(\infty) + \gamma\left(\|u\|\right)}\right),$$

and recalling that $\alpha(\infty) > \gamma(\|u\|)$, we get that $\mu(\|u\|) \leq \alpha^{-1} \circ \gamma(\|u\|)$. We then obtain from (24) that $\limsup_{t\to\infty} |x(t;x_0,u)| \leq \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1} \circ \gamma(\|u\|)$. This corresponds to the asymptotic gain property [15] recalled in (5). Noticing that the system is also 0-GAS (as it is iISS), we conclude with Proposition 1 that the system is Strongly iISS with input threshold R.

B. Proof of Theorem 2

The proof relies on the following result, which gives the minimal value of the ratio between two quadratic forms based on the characteristic value of the associated matrix pencil [9, Ch. X-6].

Lemma 1 ([9]): For any symmetric matrices $P, Q \in \mathbb{R}^{n\times n}$, with Q positive definite, it holds that

$$\min_{x\in\mathbb{R}^n} \frac{x^T Q x}{x^T P x} = \min\left\{\lambda \in \mathbb{R} : \det(Q - \lambda P) = 0\right\}.$$

Consider the function $V(x) = \ln(1 + x^T P x)$. By assumption, its derivative along the solutions of (1) reads $\dot{V} \leq -x^T Q x/(1 + x^T P x) + \gamma(|u|)$. First notice that the Strong iISS follows directly from Theorem 1 by observing that the function $x \mapsto x^T Q x/(1 + x^T P x)$ can be lower bounded by a class $\mathcal{K}$ function of $|x|$. Furthermore, invoking Lemma 1, it holds that $x^T Q x \geq \lambda_0 x^T P x$ for all $x \in \mathbb{R}^n$, where $\lambda_0 > 0$ is defined in the statement of Theorem 2, and consequently $\dot{V} \leq -\lambda_0 x^T P x/(1 + x^T P x) + \gamma(|u|)$. Noticing that $x^T P x = e^{V(x)} - 1$, it follows that $\dot{V} < 0$ whenever $\lambda_0(1 - e^{-V(x)}) > \gamma(|u|)$. In view of the assumption (10) made on R, for any $|u| < R$, it holds that $\lambda_0 - \gamma(|u|) > 0$. It follows that

$$V(x) > \ln\left(\frac{\lambda_0}{\lambda_0 - \gamma\left(|u|\right)}\right) \quad \Rightarrow \quad \dot{V} < 0.$$

Consequently, for any $u \in \mathcal{U}^m_{<R}$, the set $\{x \in \mathbb{R}^n : V(x) \leq \ln[\lambda_0/(\lambda_0 - \gamma(\|u\|))]\}$ is globally attractive for (1). Observing that $x^T Px \geq p_m|x|$, we conclude that the ball $\mathcal{B}_\delta$, with δ defined in (11), is globally attractive for (1), thus establishing the asymptotic gain property (5). Since the system (1) is 0-GAS (as it is Strongly iISS), we conclude with Proposition 1 that it is Strongly iISS with input threshold R.

C. Proof of Theorem 3

ISS w.r.t. small inputs being straightforward from (14), there is only iISS to show. We proceed by establishing zero-output dissipativity [2]. To that aim, first notice that (14) ensures that $(\partial V/\partial x)(x)f(x,0) \leq -\alpha(|x|)$. Following the same reasoning as in the proof of [2, Lemma IV.10], this ensures the existence of $\lambda_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous and increasing and $\lambda_2 \in \mathcal{K}$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, $(\partial V/\partial x)(x)f(x,u) \leq -\alpha(|x|) + \lambda_1(|x|)\lambda_2(|u|)$. In particular

$$|x| \leq R_x \Rightarrow \frac{\partial V}{\partial x} f(x,u) \leq \lambda_1(R_x)\lambda_2(|u|). \tag{25}$$

Now, let $W(x) := \ln(1 + V(x))$ for all $x \in \mathbb{R}^n$. Equation (15) then ensures that, for all $|x| \geq R_x$,

$$\frac{\partial W}{\partial x} f(x,u) \leq \frac{\nu_1(|u|)\,V}{1+V} + \frac{\nu_2(|u|)}{1+V} \leq \nu_1(|u|) + \nu_2(|u|). \tag{26}$$

In the same way, (25) yields

$$|x| \leq R_x \Rightarrow \frac{\partial W}{\partial x} f(x,u) \leq \lambda_1(R_x)\lambda_2(|u|). \tag{27}$$

It follows from (26) and (27) that $(\partial W/\partial x)f(x,u) \leq \nu(|u|)$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, where $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the non-decreasing continuous function defined as $\nu(\cdot) := \bar{\eta}\nu_1(\cdot) + \nu_2(\cdot) + \lambda_1(R_x)\lambda_2(\cdot)$. Since ν is not necessarily a class $\mathcal{K}$ function, zero-output dissipativity does not follow yet. Nonetheless, (14) guarantees that $(\partial W/\partial x)f(x,u) \leq \gamma(|u|)$ whenever $|u| \leq R_u$. Therefore, defining

$$\tilde{\gamma}(s) := \begin{cases} \gamma(s) & \text{if} \quad s \leq R_u/2 \\ as + b & \text{if} \quad s \in \left(\frac{R_u}{2}; R_u\right] \\ \nu(s) + 2\gamma(s) & \text{if} \quad s > R_u, \end{cases}$$

where a and b are conveniently chosen to ensure continuity, $\tilde{\gamma}$ is a class $\mathcal{K}$ function and it holds that $(\partial W/\partial x)f(x,u) \leq \tilde{\gamma}(|u|)$. It follows that (1) is zero-output dissipative. Consequently, recalling that it is 0-GAS in view of (14) and invoking [2, Th. 1], it is also iISS. The conclusion follows.

D. Proof of Corollary 1

Since V is a proper storage function, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$ for all $x \in \mathbb{R}^n$. In view of (16), Let $R_u := \lambda^{-1}(q/2)$. Then it holds that

$$|u| \leq R_u \Rightarrow \dot{V} \leq -\frac{q}{2}V(x) + \gamma(|u|) \leq -\frac{q}{2}\underline{\alpha}(|x|) + \gamma(|u|),$$

which makes (14) satisfied with $\alpha(\cdot) = q\underline{\alpha}(\cdot)/2$. Moreover, (16) also ensures that $\dot{V} \leq \lambda(|u|)V(x) + \gamma(|u|)$. This, in turn, establishes (15) with $\nu_1(\cdot) = \lambda(\cdot)$ and $\nu_2(\cdot) = \gamma(\cdot)$ and for any $R_x \geq 0$. The conditions of Theorem 3 are all fulfilled and the conclusion follows.

E. Proof of Corollary 2

The necessity part of the statement is immediate: if the matrix A is not Hurwitz then the system is not 0-GAS and consequently not Strongly iISS either. To show the sufficiency, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the proper storage function defined as $V(x) = \ln(1 + x^T Px)$. Its derivative along the solutions of (17) reads

$$\begin{aligned}\dot{V} &= \frac{x^T(A^T P + PA)x}{1 + x^T Px} + \frac{2x^T P\left(Bu + \sum_{i=1}^m u_i A_i x\right)}{1 + x^T Px} \\ &\leq -\frac{x^T Qx}{1 + x^T Px} + \frac{2|x|\left|P\left(Bu + \sum_{i=1}^m u_i A_i x\right)\right|}{1 + p_m|x|^2}.\end{aligned}$$

Noticing that $|P(Bu + \sum_{i=1}^m u_i A_i x)| \leq |P| \max\{b_M; \sum_{i=1}^m |A_i|\}(1 + |x|)|u|$, we get that

$$\dot{V} \leq \frac{-x^T Qx}{1 + x^T Px} + |P| \max\left\{b_M; \sum_{i=1}^m |A_i|\right\} \frac{2(|x| + |x|^2)}{1 + p_m|x|^2}|u|.$$

Using the fact that $s/(1 + p_m s^2) \leq 1/2\sqrt{p_m}$ and $s^2/(1 + p_m s^2) \leq 1/p_m$ for all $s \geq 0$, we get that

$$\dot{V} \leq \frac{-x^T Qx}{1 + x^T Px} + |P| \max\left\{b_M; \sum_{i=1}^m |A_i|\right\} \frac{\sqrt{p_m} + 2}{p_m}|u|.$$

Theorem 2 can then readily be invoked to conclude.

F. Proof of Proposition 2

From the zero-output dissipativity, there exists a proper storage function W and $\sigma \in \mathcal{K}$ such that (18) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Let $V(x) := W(x) + \int_{V_0(0)}^{V_0(x)} ds/(1 + \eta_0(s))$ for each $x \in \mathbb{R}^n$. Since W is a proper storage function and $\eta_0 \in \mathcal{K}$, V is also a proper storage function and its derivative yields

$$\begin{aligned}\dot{V} &= \dot{W} + \frac{\dot{V}_0}{1 + \eta_0(V_0)} \\ &\leq \sigma(|u|) - \frac{\alpha_0(V_0)}{1 + \eta_0(V_0)} + \frac{\eta_0(V_0)}{1 + \eta_0(V_0)}\gamma_0(|u|) \\ &\leq -\frac{\alpha_0(V_0)}{1 + \eta_0(V_0)} + \sigma(|u|) + \gamma_0(|u|).\end{aligned}$$

The function $s \mapsto \alpha_0(s)/(1 + \eta_0(s))$ is a $\mathcal{PD}$ function. In addition, (21) ensures that $\liminf_{s\to\infty}(\alpha_0(s)/(1 + \eta_0(s))) > 0$. Consequently, there exists $\alpha \in \mathcal{K}$ such that $\alpha_0(s)/(1 + \eta_0(s)) \geq \alpha(s)$ for all $s \in \mathbb{R}_{\geq 0}$. It then follows from (19) that $\dot{V} \leq -\alpha(V_0) + \sigma(|u|) + \gamma_0(|u|) \leq -\alpha \circ \kappa_0(|x|) + \sigma(|u|) + \gamma_0(|u|)$. The conclusion then follows from Theorem 1 since $\alpha \circ \kappa_0$ is a class $\mathcal{K}$ function (as the composition of $\mathcal{K}$ functions) and $\sigma + \gamma_0$ can be upper bounded by a $\mathcal{K}_\infty$ function.

G. Proof of Corollary 3

Let $V(x) := \int_0^{V_0(x)} ds/(1 + \eta_0(s))$, for all $x \in \mathbb{R}^n$. This function is continuously differentiable and positive definite. In addition, (22) ensures that it is also radially unbounded. In other words, V is a proper storage function and the rest of the proof follows along the lines of that of Proposition 2.

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