International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

A necessary and sufficient condition for input-to-state stability of quantised feedback systems

J. Wang

To cite this article: J. Wang (2017) A necessary and sufficient condition for input-to-state stability of quantised feedback systems, International Journal of Control, 90:9, 1846-1860, DOI: 10.1080/00207179.2016.1226517

To link to this article: http://dx.doi.org/10.1080/00207179.2016.1226517

Accepted author version posted online: 23 Aug 2016.
Published online: 08 Sep 2016.

INTERNATIONAL JOURNAL OF CONTROL, 2017
VOL. 90, NO. 9, 1846–1860
http://dx.doi.org/10.1080/00207179.2016.1226517

Taylor & Francis
Taylor & Francis Group

A necessary and sufficient condition for input-to-state stability of quantised feedback systems

J. Wang

School of Mathematics and Physics, Bohai University, Jinzhou, Liaoning, P.R. China

ABSTRACT
For zooming-out/in method used in the design of quantised feedback systems, the property of the duration of zoom-out mode (this duration is defined as capture time) is essential to input-to-state stability (ISS) of systems. This paper shows that a necessary and sufficient condition of achieving ISS with respect to external disturbances for quantised feedback systems is that capture time under the proposed coding scheme is uniformly bounded. It further shows that the coding scheme under which capture time is only bounded and not uniformly bounded cannot guarantee ISS of systems. A coding scheme is designed for uniformly bounded capture time and therefore achieves ISS of systems.

ARTICLE HISTORY
Received 7 December 2015
Accepted 16 August 2016

KEYWORDS
Capture time; uniformly bounded; input-to-state stability; quantised control

1. Introduction

In this paper, we consider an input-to-state stabilisation problem with a communication channel of finite data rate connecting the measurement sensor to the controller. Our task is to present a necessary and sufficient condition for input-to-state stability (ISS) of a continuous linear time-invariant system with unknown disturbances and design the coding scheme to satisfy the presented condition. Accordingly, ISS of the system is achieved.

Feedback control of systems with quantised state measurements is a very active and expanding research area motivated by numerous applications, where communication between the plant and the controller is limited due to capacity or security constraints (see, e.g. Brockett & Liberzon, 2000; Ceragioli, De Persis, & Frasca, 2011; Corradinia & Orlandob, 2008; Delchamps, 1990; Elia & Mitter, 2001; Fu & Xie, 2005; Hespanha, Ortega, & Vasudevan, 2002; Ishii & Francis, 2002; Kang & Ishii, 2015; Liberzon, 2014; Liberzon & Nesic, 2007; Matveev & Savkin, 2006; Nair & Evans, 2004; Picasso & Colaneri, 2008; Saldi, Linder, & Yüksel, 2015; Savkin & Cheng, 2007; Sharon & Liberzon, 2012; Tatikonda & Mitter, 2004; Wang & Yan, 2014; Wong & Brockett, 1999, and the references therein). In many engineering applications, since external disturbances harass the system, some papers address the stability of quantised feedback systems with disturbances, differing mainly in the stability property they aim to achieve and in their assumptions on external disturbances (Gurt & Nair, 2007; Hespanha et al., 2002; Liberzon & Nesic, 2007; Martins, Dahleh, & Elia, 2006; Matveev & Savkin, 2006; Nair & Evans, 2004; Niu & Ho, 2014; Sharon & Liberzon, 2012; Tatikonda & Mitter, 2004, and the references therein). In Hespanha et al. (2002), Tatikonda and Mitter (2004), and Martins et al. (2006), state boundedness in the presence of bounded disturbances is achieved by using the knowledge of a disturbance bound. InMatveev and Savkin (2006), a stabilisation problem of stochastic linear plants is considered involving stochastic communication channel and a controller is designed to bound the plant's state in probability. In Nair and Evans (2004), mean square stability in the stochastic setting is obtained by utilising statistical information about the disturbance (a bound on its appropriate moment). In Gurt and Nair (2007), performance analysis is considered for bit-rate-limited stochastic control systems with quantised state feedback and a quantisation scheme is proposed for computing an a priori bound on the mean square state. In Niu and Ho (2014), an adaptive quantiser is proposed to attain H_∞ disturbance attenuation performance. These four latter papers use (and prove) stochastic stability notions. Deterministic stability for unknown bounded disturbances is shown in Sharon and Liberzon (2012), Liberzon and Nesic (2007) and Kameneva and Nesic (2008). In Sharon and Liberzon (2012) and Liberzon and Nesic (2007), the deterministic stability considered is ISS. In Kameneva and Nesic (2008), l_2 stabilisation for quantised linear systems

CONTACT J. Wang wjth100@sina.com

is obtained. In contrast to these works, we are concerned with a necessary and sufficient condition for ISS of systems with quantised state measurements and unknown disturbances.

The focus of the present work is on achieving a necessary and sufficient condition for ISS of quantised feedback systems with unknown disturbances. The condition is achieved under the coding scheme based on spherical polar coordinates. Compared to the coding scheme under Cartesian coordinates, spherical polar coordinate coding scheme helps to develop a desired relation between the quantised data and the corresponding quantisation error. This relation shows that the magnitude of the quantised data is proportional to an upper bound of the magnitude of the corresponding quantisation error, which facilitates the stability analysis of systems and the achievement of the necessary and sufficient condition. We utilise an ISS-like property (see Sontag & Wang, 1996) which involves bounded nonlinear gains from the initial state and the supremum norm of the disturbance to the supremum norm of the state and also from the supremum limit of the disturbance to the supremum limit of the state. The contributions of this paper are as follows:

(1) It shows that the property of capture time (the duration of zoom-out mode) is essential to ISS of systems. And it proves that the quantised feedback system is input-to-state stable with respect to external disturbances if and only if capture time under the presented coding scheme is uniformly bounded. A coding scheme under which capture time is uniformly bounded is designed to achieve ISS for the system.
(2) Furthermore, it proves that the coding scheme under which capture time is only bounded and not uniformly bounded fails to provide ISS for the system. To show this, under only bounded capture time, a sequence of bounded disturbances can be constructed to drive the state to outside any large region containing the origin, which implies that the system is not input-to-state stable with respect to the disturbance of this kind.
(3) In the above results, the coding scheme is of infinite data rate, so, as the final result, a coding scheme of finite data rate is designed for uniformly bounded capture time and therefore achieves ISS for the system.

In what follows, $||\cdot||$ denotes the Euclidean norm for a vector and the corresponding matrix induced norm for a matrix, $\delta_{\min}(\cdot)$ denotes the minimum singular value of a matrix and $||\cdot||_I$ denotes the supremum norm of a signal on an interval I. A continuous function γ: $R_{\geq 0} \to R_{\geq 0}$ is of class $\mathcal{K}_\infty$ ($\gamma \in \mathcal{K}_\infty$) if it is zero at zero, strictly increasing, and unbounded. $\lceil\cdot\rceil$ denotes the ceiling function. $A \overset{(*)}{=} B$ denotes that $A = B$ according to the expression (*), and similar meanings are for $A \overset{(*)}{\leq} B$, $A \overset{(*)}{\geq} B$ and so on. O denotes a column vector with all zero elements and appropriate dimensions. ≔ denotes 'defined as'.

2. Problem statement

We consider the following linear continuous time invariant system:

$$\dot{X} = AX + BU + \varpi, \tag{1}$$

where $X \in \mathbb{R}^d$ and $U \in \mathbb{R}^m$ are state and input, respectively, $\varpi \in \mathbb{R}^s$ is an unknown disturbance, assumed to be Lebesgue-measurable and locally bounded, A and B are system matrices with appropriate dimensions and (A, B) is controllable. To avoid triviality, assume that A is not Hurwitz stable.

A noiseless digital channel is located between sensor and controller, which can transmit one code word at each time step. We are concerned with input-to-state stabilisation problem of the system over the digital channel in the following sense: there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial condition $X(0)$ and every disturbance ϖ, we have

$$||X(t)|| \leq \gamma_1(||X(0)||) + \gamma_2(||\varpi||_{[0,\infty)}), \forall t \geq 0$$

and

$$\limsup_{t\to\infty} ||X(t)|| \leq \gamma_3\Big(\limsup_{t\to\infty} ||\varpi(t)||\Big).$$

For system (1), the corresponding discrete-time system is

$$\widehat{X}_{k+1} = \widehat{G}\widehat{X}_k + \widehat{H}U_k + \widehat{\omega}_k \tag{2}$$

where $\widehat{X}_k = X(kT_s)$, $U_k = U(kT_s)$, $\widehat{G} = e^{AT_s}$, $\widehat{H} = \int_0^{T_s} e^{At}Bdt$, $\widehat{\omega}_k = \int_{kT_s}^{(k+1)T_s} e^{A((k+1)T_s-t)}\varpi(t)dt$, T_s is the sample time. We introduce a discrete-time version of the definition of ISS. This will suffice for our analysis since the discrete-time ISS can be used to prove an appropriate version of continuous-time ISS that takes inter-sample behaviour into account and the stability bound valid only at the sampling instants can be extended to all $t > 0$. For similar results, see Nesic, Teel, and Sontag (1999). System (2) is said to be of ISS if there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that the solutions of the system satisfy the following for all $\widehat{X}_0$ and all $\widehat{\omega}$:

$$||\widehat{X}_k|| \leq \gamma_1(||\widehat{X}_0||) + \gamma_2(||\widehat{\omega}||_{[0,\infty)}), \forall k \geq 0 \tag{3}$$

and

$$\limsup_{k\to\infty} ||\widehat{X}_k|| \leq \gamma_3\Big(\limsup_{k\to\infty} ||\widehat{\omega}_k||\Big). \tag{4}$$

The aim of this paper is to present a necessary and sufficient condition of achieving ISS of system (2) with respect to unknown disturbances. Based on this condition, the coding scheme is designed to achieve ISS of the system.

3. Quantiser based on spherical polar coordinates

The encoder and the decoder in the paper will be restricted to use the quantiser based on spherical polar coordinates (Wang & Yan, 2014). As shown later, under spherical polar coordinates, the coding scheme facilitates the stability analysis of systems. Let the vector $X = [x_1\ x_2\ \cdots\ x_{d-1}\ x_d]^{\mathrm{T}} \in \mathbb{R}^d$, where the notation `T' means transpose. Then, we call the column $[x_1\ x_2\ \cdots\ x_{d-1}\ x_d]^{\mathrm{T}}$ as the Cartesian rectangular coordinate of X. The vector can also be represented using spherical polar coordinates

$$\begin{bmatrix} r \\ \theta_1 \\ \vdots \\ \theta_{d-2} \\ \theta_{d-1} \end{bmatrix} \in \mathbb{B}^d := \left\{ \begin{bmatrix} r \\ \theta_1 \\ \vdots \\ \theta_{d-2} \\ \theta_{d-1} \end{bmatrix} : 0 \leq r < \infty, 0 \leq \theta_1, \theta_2, \ldots, \theta_{d-2} \leq \pi, 0 \leq \theta_{d-1} \leq 2\pi \right\}$$

via the coordinate transformation pair

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ &\vdots \\ x_{d-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\theta_{d-1} \\ x_d &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\sin\theta_{d-1} \end{aligned}$$

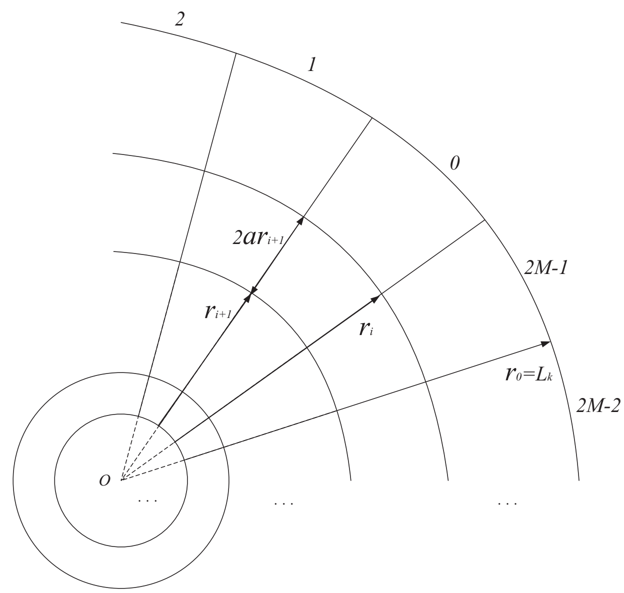

Figure 1. Partition in two-dimensional space.

and

$$\begin{aligned} r &= \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_d)^2} \\ \theta_1 &= \arccos\frac{x_1}{\sqrt{(x_1)^2 + \cdots + (x_d)^2}} \\ \theta_2 &= \arccos\frac{x_2}{\sqrt{(x_2)^2 + \cdots + (x_d)^2}} \\ &\vdots \\ \theta_{d-2} &= \arccos\frac{x_{d-2}}{\sqrt{(x_{d-2})^2 + \cdots + (x_d)^2}} \\ \theta_{d-1} &= \begin{cases} \arccos\dfrac{x_{d-1}}{\sqrt{(x_{d-1})^2 + (x_d)^2}}, & \text{if } x_d \geq 0, \\ 2\pi - \arccos\dfrac{x_{d-1}}{\sqrt{(x_{d-1})^2 + (x_d)^2}}, & \text{if } x_d < 0. \end{cases} \end{aligned}$$

Definition 3.1: A quantiser based on spherical polar coordinates (for abbreviation, a quantiser) at time k is a trituple (L_k, a, M), where the real number $L_k > 0$ represents the radius of the support ball at time k, the real number $a > 0$ regulates the proportional coefficient, and the positive integer $M \geq 2$ represents the number of the angles into which the angle of radian π is equally partitioned. This quantiser partitions the support

$$\Lambda_k = \left\{X \in \mathbb{R}^d : r \leq L_k\right\}$$

into quantisation blocks as follows:

the sets $\{X \in \mathbb{R}^d : \frac{L_k}{(1+2a)^{i+1}} < r \leq \frac{L_k}{(1+2a)^i}, j_n\frac{\pi}{M} < \theta_n \leq (j_n+1)\frac{\pi}{M}, n = 1, \ldots, d-2, s\frac{\pi}{M} < \theta_{d-1} \leq (s+1)\frac{\pi}{M}\}$, indexed by $(i, j_1, \ldots, j_{d-2}, s)$, $i = 0, 1, 2, \ldots, j_n = 0, \ldots, M-1$ for $n = 1, \ldots, d-2$, and $s = 0, \ldots, 2M-1$.

For each k, let $r_i(k) = \frac{L_k}{(1+2a)^i}$, $i = 0, 1, 2, \ldots$, so $\frac{r_i}{r_{i+1}} = 1 + 2a$. See Figure 1 for an illustration in the case of two dimension. Since there are infinite quantisation blocks in the support by Definition 3.1, the quantiser needs an infinite data rate. The quantiser with infinite data rate is adopted first to highlight the main results. In Section 5, a quantiser of finite data rate will be proposed.

3.1 Estimate of quantisation error

We take $\overline{X}_k$ with the spherical polar coordinates

$$r = \frac{(1+a)L_k}{(1+2a)^{i+1}},$$
$$\theta_i = \left(j_i + \frac{1}{2}\right)\frac{\pi}{M}, \text{ for } i = 1, \ldots, d-2,$$
$$\theta_{d-1} = \left(s + \frac{1}{2}\right)\frac{\pi}{M} \tag{5}$$

as the estimate of X_k which is in the quantisation block indexed by $(i, j_1, \ldots, j_{d-2}, s)$ (see Definition 3.1). We estimate the quantisation error norm $||X_k - \overline{X}_k||$ for X_k in the region Λ_k.

Lemma 3.1 (Wang & Yan, 2014)**:** *Let* (L_k, a, M) *be a quantiser in Definition 3.1; let* Λ_k *be the support. Then,*

$$||X_k - \overline{X}_k|| \le \eta||X_k||$$

for any $X_k \in \Lambda_k$*, where*

$$\eta = a + (d-1)\frac{\pi}{2M}. \tag{6}$$

From Lemma 3.1, the relation between the state X_k in Λ_k and the corresponding quantisation error $e_k = \overline{X}_k - X_k$ reflects a fact that the quantiser resolution will become fine as $||X_k||$ tends to 0 and coarse as $||X_k||$ is far from it.

3.2 Coding scheme based on spherical polar coordinates

In this paper, quantised state feedback controller $U_k = \widehat{K}\overline{\widehat{X}}_k$ is used for system (2), where $\widehat{K}$ is state feedback matrix and $\overline{\widehat{X}}_k$ is the estimate of $\widehat{X}_k$. Hence, the resulting closed-loop system is

$$\widehat{X}_{k+1} = \widehat{G}\widehat{X}_k + \widehat{H}\widehat{K}\overline{\widehat{X}}_k + \widehat{\omega}_k \tag{7}$$

If the state $\widehat{X}_k$ is quantised by the quantiser (L_k, a, M) directly, then by Lemma 3.1, we have $||\widehat{X}_k - \overline{\widehat{X}}_k|| \le \eta||\widehat{X}_k||$. Hence, from (7),

$$||\widehat{X}_{k+1}|| \le (||\widehat{G} + \widehat{H}\widehat{K}|| + \eta||\widehat{H}\widehat{K}||)||\widehat{X}_k|| + ||\widehat{\omega}_k||$$

Obviously, if K and η are selected to satisfy $||\widehat{G} + \widehat{H}\widehat{K}|| + \eta||\widehat{H}\widehat{K}|| < 1$, then this will facilitate the stability analysis of the system. However, for a given $\widehat{K}$, even though the absolute value of each eigenvalue of $\widehat{G} + \widehat{H}\widehat{K}$ is less than one, i.e. $\widehat{G} + \widehat{H}\widehat{K}$ is Schur stable, $||\widehat{G} + \widehat{H}\widehat{K}||$ may be greater than one. Then, the inequality $||\widehat{G} + \widehat{H}\widehat{K}|| + \eta||\widehat{H}\widehat{K}|| < 1$ will have no solution. To deal with this, as Wang and Yan (2014), let $\widehat{X}_k = PX_k$, then from (2), we have

$$X_{k+1} = GX_k + HU_k + \omega_k \tag{8}$$

where P is an invertible matrix, $G = P^{-1}\widehat{G}P$, $H = P^{-1}\widehat{H}$ and $\omega_k = P^{-1}\widehat{\omega}_k$. Applying the controller $U_k = \widehat{K}\overline{\widehat{X}}_k = K\overline{X}_k$ to system (8), we have

$$X_{k+1} = GX_k + HK\overline{X}_k + \omega_k \tag{9}$$

where $K = \widehat{K}P$, $\overline{X}_k$ is the estimate of the state X_k and $\overline{\widehat{X}}_k = P\overline{X}_k$. If the state X_k is quantised by the quantiser (L_k, a, M) instead of the state $\widehat{X}_k$, then by Lemma 3.1, we have $||X_k - \overline{X}_k|| \le \eta||X_k||$. Hence, from (9),

$$||X_{k+1}|| \le (||G + HK|| + \eta||HK||)||X_k|| + ||\omega_k||$$

It is proven in Wang and Yan (2014) that if $(\widehat{G}, \widehat{H})$ is controllable, then there exists an invertible matrix P and a state feedback matrix K such that $||G + HK|| < 1$. The method to obtain K, P and η is given in Wang and Yan (2014). Obviously, system (2) is of ISS if and only if system (8) is of ISS, so in the sequel, we only need to guarantee ISS of system (8).

Let the initial radius of support Λ_0 be a positive number L_0, which is known to the encoder and the decoder.

- Encoding:

At time k, let $X_k = P^{-1}\widehat{X}_k$. If $||X_k|| \le L_k$, the encoder labels quantisation blocks in support Λ_k and encodes X_k as $V(k)$, where $V(k)$ is the label of the quantisation block in which X_k is; if $||X_k|| > L_k$, then encodes X_k as $V(k) = \phi$. Update the parameter L_k of quantiser (L_k, a, M) as

$$L_{k+1} = \begin{cases} \mathcal{E}_k^{in}(L_k, X_k, s_k) & \text{if } ||X_k|| \le L_k \\ \mathcal{E}_k^{\text{out}}(L_k, X_k, s_k) & \text{if } ||X_k|| > L_k, \end{cases} \tag{10}$$

where $\mathcal{E}_k^{in}$ and $\mathcal{E}_k^{\text{out}}$ are functions of L_k, X_k and s_k, and satisfy

$$1 > \frac{\mathcal{E}_k^{in}}{L_k} \geq \mu_m \text{ and } ||G|| + g > \frac{\mathcal{E}_k^{\text{out}}}{L_k} > ||G|| \quad (11)$$

with $\mu_m > \mu$ defined as (17), $g > 0$ and $s_k \in \{0, 1\}$ is a mark used when necessary.

- Decoding:

At time k, if $V(k)$ labels the quantisation block indexed by $(i, j_1, \dots, j_{d-2}, s)$, the decoder evaluates $\overline{X}_k$ with the spherical polar coordinates

$$r = \frac{(1+a)}{(1+2a)^{i+1}} L_k,\ \theta_n = \left(j_n + \frac{1}{2}\right)\frac{\pi}{M},$$
$$n = 1, \dots, d-2,\ \theta_{d-1} = \left(s + \frac{1}{2}\right)\frac{\pi}{M}; \quad (12)$$

and let $\overline{\overline{X}}_k = P\overline{X}_k$; if $V(k) = \phi$, then evaluates $\overline{X}_k = O \in \mathbb{R}^d$ and let $\overline{\overline{X}}_k = O \in \mathbb{R}^d$. Update the parameter L_k of quantiser (L_k, a, M) as

$$L_{k+1} = \begin{cases} \mathcal{D}_k^{in}(L_k, V(k), s_k) & \text{if } V(k) \neq \phi \\ \mathcal{D}_k^{\text{out}}(L_k, V(k), s_k) & \text{if } V(k) = \phi, \end{cases} \quad (13)$$

where $\mathcal{D}_k^{in}$ and $\mathcal{D}_k^{\text{out}}$ are functions of L_k, $V(k)$ and s_k, and satisfy $\mathcal{D}_k^{in} = \mathcal{E}_k^{in}$ and $\mathcal{D}_k^{\text{out}} = \mathcal{E}_k^{\text{out}}$, $k = 0, 1, \dots$.

4. Main results

According to the above coding scheme, the control input

$$U_k = \begin{cases} K\overline{X}_k, & \text{if } ||X_k|| \leq L_k \\ O, & \text{if } ||X_k|| > L_k, \end{cases} \quad (14)$$

where K is state feedback gain matrix, so system (8) with the control input (14) is

$$X_{k+1} = \begin{cases} GX_k + HK\overline{X}_k + \omega_k, & ||X_k|| \leq L_k \\ GX_k + \omega_k, & ||X_k|| > L_k. \end{cases} \quad (15)$$

By Lemma 3.1, we have $||X_k - \overline{X}_k|| \leq \eta ||X_k||$ for $||X_k|| \leq L_k$, so

$$||X_{k+1}|| \leq \begin{cases} \mu||X_k|| + ||\omega_k||, & ||X_k|| \leq L_k \\ ||G||||X_k|| + ||\omega_k||, & ||X_k|| > L_k, \end{cases} \quad (16)$$

where

$$\mu = ||G + HK|| + \eta ||HK||, \quad (17)$$

K and η are chosen to make $\mu < 1$. For the method to design controller K and η, see Wang and Yan (2014).

In the following, by zoom-out mode at time k and zoom-in mode at time k, we mean that the state of the system meets $||X_k|| > L_k$ and $||X_k|| \leq L_k$, respectively. From (14), during the zoom-out mode, zero input is applied to the system and the system is open loop, while during the zoom-in mode, feedback is applied to the system and the system is closed-loop. L_k is updated as (10) or (13) for these two modes.

Definition 4.1 (Capture time)**:** $k_1 - k_0$ is said to be capture time at time k_0, denoted by $\mathcal{T}_{k_0}$, if at time k_0 the system enters into zoom-out mode ($||X_{k_0}|| > L_{k_0}$) and at time k_1 the system first enters into zoom-in mode ($||X_{k_1}|| \leq L_{k_1}$) from the zoom-out mode after k_0.

Obviously, capture time is the duration of a zoom-out mode.

Definition 4.2 (Uniformly bounded)**:** Capture time is said to be uniformly bounded if there exist T and $k_N > 0$ such that $\mathcal{T}_k \leq T$ for $k > k_N$, where k is the time when the system enters into zoom-out mode and T is a uniform bound of capture time.

Theorem 4.1: *Consider the system consisting of plant (8) and controller (14). Let K be such that G + HK is Schur stable. Then, under the above coding scheme, the system is input-to-state stable with respect to unknown disturbances if and only if capture time is uniformly bounded.*

The proof of Theorem 4.1 will rely on the following lemmas.

Lemma 4.1: *Consider the system consisting of plant (8) and controller (14). Under the above coding scheme, if capture time is uniformly bounded with uniform bound $T \geq 1$ and there exists an integer $k_T < \infty$ such that*

$$k_T = \min\Bigg\{k \geq 0 : ||X_{k_T}|| \leq L_{k_T}, L_{k_T} > \left(\frac{||G||^T}{\mu_m - \mu} + \frac{1 - ||G||^T}{1 - ||G||}\right)||\omega||_{[0,\infty)}\Bigg\} \quad (18)$$

then, $||X_k|| \leq L_{k_T}$ and $L_k \leq L_{k_T}$ for $k \geq 0$, where μ_m and μ are defined as (11) and (17), respectively.

Proof: By (10), (11) and (18), we have $||X_k|| \leq L_{k_T}$ and $L_k \leq L_{k_T}$ for $k \in [0, k_T)$. There remains to prove that $||X_k|| \leq L_{k_T}$ and $L_k \leq L_{k_T}$ for $k \geq k_T$.

Without loss of generality, if $||X_{k_T}|| \in [\frac{||\omega||_{[0,\infty)}}{\mu_m - \mu}, L_{k_T}]$, then $L_{k_T} \geq ||X_{k_T}|| \geq \frac{||\omega||_{[0,\infty)}}{\mu_m - \mu}$ and $L_{k_T}\mu + ||\omega||_{[0,\infty)} <$

$L_{k_T}\mu_m$. And from (16), we have

$$\begin{aligned} ||X_{k_T+1}|| &\le \mu||X_{k_T}|| + ||\omega_{k_T}|| \\ &\le \mu L_{k_T} + ||\omega||_{[0,\infty)} \\ &< L_{k_T}\mu_m \overset{(11)}{\le} \mathcal{E}_{k_T}^{in} \\ &\overset{(10)}{=} L_{k_T+1} < L_{k_T}, \end{aligned}$$

that is, at $k_T + 1$, the system still remains in zoom-in mode. By mathematic induction, there exists $k'_T > k_T$ such that $||X_k|| \in [\frac{||\omega||_{[0,\infty)}}{\mu_m-\mu}, L_{k_T}]$ and the system is in zoom-in mode for $k \in [k_T, k'_T)$, so $L_k \le L_{k_T}$ for $k \in [k_T, k'_T)$. If $k'_T = \infty$, then the proof is completed; if $k'_T < \infty$, then $||X_{k'_T}|| < \frac{||\omega||_{[0,\infty)}}{\mu_m-\mu}$.

In the case of $||X_{k'_T}|| < \frac{||\omega||_{[0,\infty)}}{\mu_m-\mu}$, if after k'_T the system remains in zoom-in mode, then since for $k \ge k'_T$, $L_k \le L_{k'_T} < L_{k_T}$ and

$$\begin{aligned} ||X_{k+1}|| &\overset{(16)}{<} \mu||X_k|| + ||\omega_k|| \\ &< \mu\frac{||\omega||_{[0,\infty)}}{\mu_m-\mu} + ||\omega||_{[0,\infty)} \\ &= \mu_m\frac{||\omega||_{[0,\infty)}}{\mu_m-\mu} \\ &< \frac{||\omega||_{[0,\infty)}}{\mu_m-\mu} \\ &< L_{k_T}, \end{aligned} \tag{19}$$

by mathematical induction, the proof is completed; if there exists $k''_T > k'_T$ such that the system is in zoom-in mode at $k''_T - 1$ and enters into zoom-out mode at k''_T, then $||X_{k''_T}|| \overset{(19)}{<} \frac{||\omega||_{[0,\infty)}}{\mu_m-\mu}$ and

$$\begin{aligned} ||X_{K''_T+T}|| &\overset{(16)}{\le} ||G||^T||X_{K''_T}|| + \sum_{i=K''_T}^{K''_T+T-1} ||G||^{K''_T+T-1-i}||\omega_i|| \\ &< \frac{||G||^T||\omega||_{[0,\infty)}}{\mu_m-\mu} + \sum_{i=K''_T}^{K''_T+T-1} ||G||^{K''_T+T-1-i}||\omega||_{[0,\infty)} \\ &= ||\omega||_{[0,\infty)}\left(\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||}\right) \end{aligned}$$

By (18), we have $||X_{k''_T+T}|| < L_{k_T}$. Since capture time is uniformly bounded with uniform bound T, let $L_{k''_T+T} = L_{k_T}$. Then, $||X_k|| \le L_{k_T}$, $L_k \le L_{k_T}$ for $k \in [k''_T, k''_T + T]$ and at $k''_T + T$, the system enters into zoom-in mode again. Repeat the above procedure, the proof is completed. ∎

Lemma 4.2: *Consider the system consisting of plant (8) and controller (14). Under the above coding scheme, if capture time is uniformly bounded with uniform bound $T \ge 1$, then $||X_k|| \le L_{k_T}$ and $L_k \le L_{k_T}$ for $k \ge 0$.*

Proof: If there exists k_T in Lemma 4.1, then by Lemma 4.1, the proof is completed. If there exists no such a k_T, then we should consider three cases at any $k \ge 0$:

(a) the system is in zoom-in mode and $L_k \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$;

or (b) the system is in zoom-out mode and $L_k \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$;

or (c) the system is in zoom-out mode and $L_k > (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$.

If Case (c) occurs, by the coding scheme, L_k will increase until the system enters zoom-in mode at $k_T > k$ and L_{k_T} satisfies (18), which counters the assumption. Hence, only Case (a) or (b) can occur. For Case (b), we have $||X_k|| > L_k$. If $||X_k|| > (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$, then there exists $k_T > k$ such that $L_{k_T} > (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$ and $||X_{k_T}|| \le L_{k_T}$, which counters the assumption. Therefore, we have $||X_k|| \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$ and $L_k \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)}$ for $k > 0$ in Case (b) or (a). Thus, $||X_k|| \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)} < L_{k_T}$ and $L_k \le (\frac{||G||^T}{\mu_m-\mu} + \frac{1-||G||^T}{1-||G||})||\omega||_{[0,\infty)} < L_{k_T}$, $k \ge 0$. ∎

Lemma 4.3: *Under the above coding scheme, if capture time is uniformly bounded and at time k_0 the system enters into zoom-out mode, then*

$$||X_k|| < \psi^1_{\text{out}}(||X_{k_0}||) + \psi^2_{\text{out}}(||\omega||_{[k_0,k-1]}),\ k_1 \ge k > k_0, \tag{20}$$

where ψ^1_{out} and $\psi^2_{\text{out}} \in \mathcal{K}_\infty$ and are independent of X_{k_0}, k_1 is the time when the system first enters into zoom-in mode after k_0, that is, $\mathcal{T}_{k_0} = k_1 - k_0$.

Proof: Since the system is open loop at $k \in (k_0, k_1]$, $X_k = G^{k-k_0}X_{k_0} + \sum_{i=k_0}^{k-1} G^{k-1-i}\omega_i$, $k_1 \ge k > k_0$. And since capture time is uniformly bounded, there exist T and $k_N > 0$ such that $\mathcal{T}_k < T$ at $k > k_N$. So, $||X_k|| < ||G||^{T'}X_{k_0} + \sum_{i=k_0}^{k-1}||G||^{k-1-i}||\omega_i||$, $k_1 \ge k > k_0$, where $T' = \max\{\max\{\mathcal{T}_k : k \le k_N\}, T\}$. Let $\psi^1_{\text{out}}(||X_{k_0}||) = ||G||^{T'}X_{k_0}$ and $\psi^2_{\text{out}}(||\omega||_{[k_0,k-1]}) = \sum_{i=k_0}^{k-1}||G||^{k-1-i}||\omega_i||$, then ψ^1_{out} and ψ^2_{out} are independent of X_{k_0} and (20) holds with ψ^1_{out} and $\psi^2_{\text{out}} \in \mathcal{K}_\infty$. ∎

Lemma 4.4: *Under the above coding scheme, the system will not enter into zoom-out mode at time k if at time $k - 1$ the system is in zoom-in mode and the disturbance ω_{k-1} meets $||\omega_{k-1}|| \le (\frac{\mathcal{E}^{in}_{k-1}}{L_{k-1}} - \mu)L_{k-1}$.*

Proof: This follows since

$$\begin{aligned} ||X_k|| &\overset{(16)}{\leq} \mu||X_{k-1}|| + ||\omega_{k-1}|| \\ &\leq \mu L_{k-1} + \left(\frac{\mathcal{E}_{k-1}^{in}}{L_{k-1}} - \mu\right) L_{k-1} \\ &= \mathcal{E}_{k-1}^{in} \overset{(10)}{=} L_k. \end{aligned}$$

■

Proof of Theorem 1: (Sufficiency) Assume that capture time is uniformly bounded. First, we prove that (3) holds.

Without loss of generality, let the system enters into zoom-out mode at time k_{2i} with $k_0 = 0$ and zoom-in mode at time k_{2i+1}, $i = 0, 1, \ldots$, then, by Lemma 4.3, we have

$$||X_k|| < \psi_{\text{out}}^1(||X_{k_{2i}}||) + \psi_{\text{out}}^2(||\omega||_{[k_{2i},k-1]}), k \in (k_{2i}, k_{2i+1}] \quad (21)$$

where ψ_{out}^1 and $\psi_{\text{out}}^2 \in \mathcal{K}_\infty$ and are independent of $X_{k_{2i}}$, $i = 0, 1, \ldots$. When the system enters into zoom-in mode at time k_{2i+1}, we have $||X_{k_{2i+1}}|| \leq L_{k_{2i+1}}$ and

$$\begin{aligned} ||X_k|| &< L_k \\ &\overset{(11)}{<} L_{k_{2i+1}}, k \in (k_{2i+1}, k_{2i+2} - 1] \end{aligned} \quad (22)$$

Furthermore, since the system is in zoom-out mode for $k \in [k_{2i}, k_{2i+1})$,

$$\begin{aligned} ||X_{k_{2i+1}-1}|| &> L_{k_{2i+1}-1} \\ &\overset{(11)}{>} \frac{L_{k_{2i+1}}}{||G|| + g} \\ &\overset{(22)}{>} \frac{||X_k||}{||G|| + g}, \quad k \in (k_{2i+1}, k_{2i+2} - 1] \end{aligned}$$

So,

$$\begin{aligned} ||X_k|| &< (||G|| + g)||X_{k_{2i+1}-1}|| \\ &\overset{(21)}{<} (||G|| + g)(\psi_{\text{out}}^1(||X_{k_{2i}}||) \\ &\quad + \psi_{\text{out}}^2(||\omega||_{[k_{2i},k-1]})), k \in (k_{2i+1}, k_{2i+2} - 1] \end{aligned} \quad (23)$$

and

$$\begin{aligned} ||X_{k_{2i+2}}|| &\overset{(16)}{<} \mu||X_{k_{2i+2}-1}|| + ||\omega_{k_{2i+2}-1}|| \\ &< ||G||||X_{k_{2i+2}-1}|| + ||\omega_{k_{2i+2}-1}|| \\ &\overset{(23)}{<} ||G||(||G|| + g)(\psi_{\text{out}}^1(||X_{k_{2i}}||) \\ &\quad + \psi_{\text{out}}^2(||\omega||_{[k_{2i},k_{2i+2}-2]})) + ||\omega_{k_{2i+2}-1}|| \\ &:= \psi(||X_{k_{2i}}||, ||\omega||_{[k_{2i},k_{2i+2}-1]}) \end{aligned} \quad (24)$$

where $\psi \in \mathcal{K}_\infty$ is independent of $X_{k_{2i}}$, $i = 0, 1, \ldots$. From (21), (23) and (24), we have

$$||X_k|| < \psi(||X_{k_{2i}}||, ||\omega||_{[k_{2i},k-1]}), k \in (k_{2i}, k_{2i+2}] \quad (25)$$

Define

$$\begin{aligned} &\psi^{(i)}(||X_{k_0}||, ||\omega||_{[k_0,k_{2i+2}-1]}) \\ &= \psi(\cdots\psi(\psi(||X_{k_0}||, ||\omega||_{[k_0,k_2-1]}), \\ &\quad ||\omega||_{[k_2,k_4-1]}), \ldots, ||\omega||_{[k_{2i},k_{2i+2}-1]}) \end{aligned}$$

and $I = \min\{i : \psi^{(i)}(||X_{k_0}||, ||\omega||_{[k_0,k_{2i+2}-1]}) \geq L_{k_T}\}$. If $I < \infty$, then from Lemma 4.2,

$$\begin{aligned} ||X_k|| &< \psi^{(I)}(||X_{k_0}||, ||\omega||_{[k_0,k_{2I+2}-1]}) \\ &:= \gamma_1'(||X_{k_0}||) + \gamma_2'(||\omega||_{[k_0,\infty)}), \forall k \geq k_0 \end{aligned}$$

where γ_1' and $\gamma_2' \in \mathcal{K}_\infty$ and are independent of X_{k_0}. If $I = \infty$, that is, $\max_i\{\psi^{(i)}(||X_{k_0}||, ||\omega||_{[k_0,k_{2i+2}-1]})\} < L_{k_T}$, then

$$\begin{aligned} ||X_k|| &< \max_i\{\psi^{(i)}(||X_{k_0}||, ||\omega||_{[k_0,k_{2i+2}-1]})\} \\ &:= \gamma_1''(||X_{k_0}||) + \gamma_2''(||\omega||_{[k_0,\infty)}), \forall k \geq k_0 \end{aligned}$$

where γ_1'' and $\gamma_2'' \in \mathcal{K}_\infty$ and are independent of X_{k_0}. Let $\gamma_1 = \max\{\gamma_1', \gamma_1''\}$ and $\gamma_2 = \max\{\gamma_2', \gamma_2''\}$, so

$$||X_k|| < \gamma_1(||X_{k_0}||) + \gamma_2(||\omega||_{[k_0,\infty)}), \forall k \geq k_0$$

where γ_1 and $\gamma_2 \in \mathcal{K}_\infty$ and are independent of X_{k_0}.

Next, we prove that (4) holds. When the system enters into zoom-out mode at time k_{2i}, by the coding scheme, the system is in zoom-in mode at time $k_{2i} - 1$. From Lemma 4.4, we have $||X_{k_{2i}-1}|| < L_{k_{2i}-1} < \frac{||\omega_{k_{2i}-1}||}{\frac{\mathcal{E}_{k-1}^{in}}{L_{k-1}} - \mu} \overset{(11)}{<} \frac{||\omega_{k_{2i}-1}||}{\mu_m - \mu}$, so

$$\begin{aligned} ||X_{k_{2i}}|| &\overset{(16)}{\leq} \mu||X_{k_{2i}-1}|| + ||\omega_{k_{2i}-1}|| \\ &\leq \mu\frac{||\omega_{k_{2i}-1}||}{\mu_m - \mu} + ||\omega_{k_{2i}-1}|| \\ &\leq \frac{\mu_m}{\mu_m - \mu}||\omega_{k_{2i}-1}||. \end{aligned}$$

Thus, from (25),

$$\begin{aligned} ||X_k|| &< \psi(\frac{\mu_m}{\mu_m - \mu}||\omega_{k_{2i}-1}||, ||\omega||_{[k_{2i},k-1]}) \\ &:= \psi'(||\omega||_{[k_{2i}-1,k-1]}), k \in (k_{2i}, k_{2i+2}], \end{aligned}$$

so, $\limsup_{k\to\infty}||X_k|| \leq \gamma_3(\limsup_{k\to\infty}||\omega_k||)$, where $\gamma_3 = \psi' \in \mathcal{K}_\infty$.

(Necessity) If the system is input-to-state stable, but capture time is not uniformly bounded, that is, for given T and $k_N > 0$, there exists $k > k_N$ such that $\mathcal{T}_k > T$, then, since T and k_N are arbitrary and G is not Schur stable, $||X_{k+T}|| = ||G^T X_k + \sum_{i=k}^{T-1} G^{T-1-i}\omega_i||$ will tend to infinity for ω_i with $||\omega_i|| < \infty$ as T tends to infinity, which implies that there exists no function γ_3 for (4). ■

Definition 4.3 (Bounded)**:** Capture time is said to be bounded if for given time $k > 0$ when the system enters into zoom-out mode, there exists $T > 0$ such that $\mathcal{T}_k \le T$.

It should be pointed out that with the coding scheme under which capture time is only bounded, the system fails to be input-to-state stable.

Theorem 4.2: *Consider the system consisting of plant* (8) *and controller* (14). *Suppose that G has a real eigenvalue r with* $|r| > 1$ *and let K be such that G + HK is Schur stable, then the above coding scheme under which capture time is only bounded fails to guarantee input-to-state stability of the system.*

Proof: Without loss of generality, let the system be in zoom-in mode at time $k = 0$. It will suffice to show that a bounded disturbance sequence $\{\omega_k, k = 0, 1, 2, \dots\}$ and a time subsequence $\{k_i, i = 0, 1, 2, \dots\}$ can be constructed such that under the disturbance the system enters into zoom-in mode at k_{2i} and zoom-out mode at $k_{2i+1} + 1$, $i = 0, 1, 2, \dots$, with $k_0 = 0$; however, the capture time is only bounded but not uniformly bounded. Hence, by Theorem 4.1, the proof is completed.

Let

$$\mathcal{E}_{in}(k'', k', L_{k'}) = \mathcal{E}^{in}_{k''-1}(\mathcal{E}^{in}_{k''-2}(\cdots \mathcal{E}^{in}_{k'+1}(\mathcal{E}^{in}_{k'}(L_{k'}, X_{k'}, s_{k'}), X_{k'+1}, s_{k'+1}), \dots,), X_{k''-1}, s_{k''-1})$$

and

$$\begin{aligned}&\mathcal{E}_{\text{out}}(k'', k', L_{k'})\\ &= \mathcal{E}^{\text{out}}_{k''-1}(\mathcal{E}^{\text{out}}_{k''-2}(\cdots \mathcal{E}^{\text{out}}_{k'+1}(\mathcal{E}^{\text{out}}_{k'}(L_{k'}, X_{k'}, s_{k'}), X_{k'+1}, s_{k'+1}), \dots,), X_{k''-1}, s_{k''-1})\end{aligned}$$

Construct a time subsequence $\{k_i, i = 0, 1, 2, \dots\}$ with $k_0 = 0$,

$$k_{2i+1} = \min\left\{k : \mathcal{E}_{in}(k, k_{2i}, L_{k_{2i}}) < \frac{\hat{\varepsilon}}{w}\right\} \tag{26}$$

$$k_{2i+2} = \min\{k : |r|^{k-(k_{2i+1}+1)}\hat{\varepsilon} < \mathcal{E}_{\text{out}}(k, k_{2i+1}+1, W_{k_{2i}})\},\quad i = 0, 1, \dots \tag{27}$$

and a bounded disturbance sequence $\{\omega_k, k = 0, 1, 2, \dots\}$ with

$$\omega_k = \begin{cases} -(G + HK)X_k - HKe_k + \hat{\varepsilon}\zeta, & k = k_{2i+1}\\ 0, & k \neq k_{2i+1}\end{cases}\quad i = 0, 1, \dots, \tag{28}$$

where $e_k = \overline{X}_k - X_k$, $\hat{\varepsilon} > 0$, ζ is a eigenvector of G with $||\zeta|| = 1$ and its corresponding eigenvalue r, $w > 1$ meets

$$\mathcal{E}_{\text{out}}(k_{2i+1} + 1 + T_{2i+1}, k_{2i+1} + 1, \frac{\hat{\varepsilon}}{w}) = |r|^{T_{2i+1}}\hat{\varepsilon} \tag{29}$$

with $T_{2i+1} = T > 0$ and

$$W_{k_{2i}} = \mathcal{E}_{in}(k_{2i+1} + 1, k_{2i}, L_{k_{2i}}). \tag{30}$$

We will prove that under the above disturbance sequence, for any $T > 0$, the capture time at time $k_{2i+1} + 1$, $i = 0, 1, \dots$, is more than T, that is, $\mathcal{T}_{k_{2i+1}+1} > T$, $i = 0, 1, \dots$, so capture time is not uniformly bounded. However, $\mathcal{T}_{k_{2i+1}+1} < \infty$, that is, capture time is bounded.

(1) Since the system is in zoom-in mode at $k = 0$, $||X_0|| < L_0$. And since the system is closed-loop in this mode and $\omega_k = 0$ for $k \in [k_0, k_1)$ by (28), for such $X_0 \in \mathbb{R}^d$, we have $||X_{k_1}|| < \mathcal{E}_{in}(k_1, 0, L_0) \overset{(26)}{<} \frac{\hat{\varepsilon}}{w}$ and $W_0 = \mathcal{E}_{in}(k_1 + 1, 0, L_0) < \frac{\hat{\varepsilon}}{w}$. By (28) $\omega_{k_1} = -(G + HK)X_{k_1} - HKe_{k_1} + \hat{\varepsilon}\zeta$ and $||X_{k_1+1}|| = ||(G + HK)X_{k_1} + HKe_{k_1} + \omega_{k_1}|| = ||\hat{\varepsilon}\zeta|| = \hat{\varepsilon} > W_0 = L_{k_1+1}$, so the system enters into zoom-out mode at $k_1 + 1$. By the coding scheme and (28), $u_k = 0$ and $\omega_k = 0$, $k \in [k_1 + 1, k_2)$, where $k_2 = \min\{k : |r|^{k-(k_1+1)}\hat{\varepsilon} < \mathcal{E}_{\text{out}}(k, k_1 + 1, W_0)\}$ by (27), so we have

$$\begin{aligned}X_{k_2} &= G^{k_2-(k_1+1)}X_{k_1+1} = G^{k_2-(k_1+1)}\hat{\varepsilon}\zeta = r^{k_2-(k_1+1)}\hat{\varepsilon}\zeta\\ L_{k_2} &= \mathcal{E}_{\text{out}}(k_2, k_1 + 1, \mathcal{E}_{in}(k_1 + 1, 0, L_0))\\ &= \mathcal{E}_{\text{out}}(k_2, k_1 + 1, W_0).\end{aligned}$$

This implies that $\|X_{k_2}\| < L_{k_2}$, that is, the system enters into zoom-in mode at k_2 again. By (26), (27), (29) and $W_0 < \frac{\hat{\varepsilon}}{w}$, $\mathcal{T}_{k_1+1} = k_2 - (k_1 + 1) > T_1 = T$, but $\mathcal{T}_{k_1+1} = k_2 - (k_1 + 1) < \infty$.

(2) Suppose that $\|X_{k_{2i}}\| < L_{k_{2i}}$, that is, the system is in zoom-in mode at k_{2i}, we prove that $\infty > \mathcal{T}_{k_{2i+1}+1} > T$. Since the system is closed-loop in this mode and $\omega_k = 0$ for $k \in [k_{2i}, k_{2i+1})$ by (28), we have

$$\begin{aligned}\|X_{k_{2i+1}}\| &< L_{k_{2i+1}}\\ &= \mathcal{E}_{in}(k_{2i+1}, k_{2i}, L_{k_{2i}})\\ &\overset{(26)}{<} \frac{\hat{\varepsilon}}{w}\end{aligned}$$

and

$$\begin{aligned}\|X_{k_{2i+1}+1}\| &= \|(G + HK)X_{k_{2i+1}} + HKe_{k_{2i+1}} + \omega_{k_{2i+1}}\|\\ &\overset{(28)}{=} ||\hat{\varepsilon}\zeta|| > \frac{\hat{\varepsilon}}{w}\\ &\overset{(26)}{>} \mathcal{E}_{in}(k_{2i+1}, k_{2i}, L_{k_{2i}}) > \mathcal{E}_{in}(k_{2i+1} + 1, k_{2i}, L_{k_{2i}})\\ &\overset{(30)}{=} W_{k_{2i}} = L_{k_{2i+1}+1}\end{aligned} \tag{31}$$

This implies that the system enters into zoom-out mode at $k_{2i+1}+1$.

By the coding scheme, the system is open loop after $k_{2i+1}+1$, and by (28), we have

$$\begin{aligned} X_{k_{2i+2}} &= G^{k_{2i+2}-(k_{2i+1}+1)} X_{k_{2i+1}+1} = G^{k_{2i+2}-(k_{2i+1}+1)}\hat{\varepsilon}\zeta = r^{k_{2i+2}-(k_{2i+1}+1)}\hat{\varepsilon}\zeta \\ L_{k_{2i+2}} &= \mathcal{E}_{\text{out}}(k_{2i+2}, k_{2i+1}+1, W_{k_{2i}}) \end{aligned} \tag{32}$$

From (32) and (27),

$$\begin{aligned} \|X_{k_{2i+2}}\| &= |r|^{k_{2i+2}-(k_{2i+1}+1)}\hat{\varepsilon} \\ &< \mathcal{E}_{\text{out}}(k_{2i+2}, k_{2i+1}+1, W_{k_{2i}}) \\ &= L_{k_{2i+2}}, \end{aligned}$$

so the system enters into zoom-in mode at k_{2i+2}. From (26), (27), (29) and $W_{k_{2i}} \overset{(31)}{<} \frac{\hat{\varepsilon}}{w}$, we have $\mathcal{T}_{k_{2i+1}+1} = k_{2i+2} - (k_{2i+1}+1) > T_{2i+1} = T$, but $\mathcal{T}_{k_{2i+1}+1} = k_{2i+2} - (k_{2i+1}+1) < \infty$. ■

Under the above coding scheme, whether capture time is uniformly bounded or not depends on the update rules (10) and (13) of L_k. In the following example, the system is not of ISS since the update rule does not guarantee that capture time is uniformly bounded.

Example 4.1: To illustrate Theorem 4.2, consider a linearised model of single inverted pendulum around equilibrium point with parameters

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{pmatrix},$$

where the mass of the cart $M = 1.096$ kg, the mass of the pendulum $m = 0.109$ kg, the length from the centre of the pendulum to the pivot $l = 0.25$ m and equals to the half length of the pendulum, $I = 0.034$ kg m^2 is the moment of inertia of the pendulum, g is the gravity acceleration, $b = 0.1$ N/m/s is the slide friction coefficient between the cart and the rail. $X = [x_1, x_2, x_3, x_4]^T$, of which x_1 and x_2 denote the position of the cart from the rail origin and its velocity, respectively, x_3 is the angular displacement measured from upright position and x_4 is the angular velocity of the pendulum.

The controller

$$K = \begin{pmatrix} 12.5369 & 22.5648 & -66.4509 & -63.1470 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0.1136 & -0.4595 & 0.2305 & -0.7130 \\ 0 & 1.5829 & 3.9566 & 7.7772 \\ 0 & 0 & 2.0867 & -2.5012 \\ 0 & 0 & 0 & 26.8472 \end{pmatrix}$$

are obtained as Wang and Yan (2014) and the parameters of quantiser $M = 175$, $\alpha = 0.023$ and $L_0 = 6$. The initial state $X(0) = (0.1, -0.5, 0.8, -0.7)^T$. At time $k = 0$, the system is in zoom-in mode and closed-loop since $\|X_0 = P^{-1}X(0)\| < L_0$. Let $\mathcal{E}_k^{in} = 0.62L_k$ and $\mathcal{E}_k^{\text{out}} = 17.1L_k$ in (10) meeting (11). The sample time $T_s = 0.1$ s. We do not follow exactly the fixed T_{2i+1} in (29) in the proof of Theorem 4.2, but let $T_{2i+1} = 2.5i + 2$, $i = 0, 1, \dots$, in (29). The simulation result shows that the system exhibits more and more large overshoots (Figure 2 (a)) in response to the bounded disturbance (Figure 2(b)) with $\hat{\varepsilon} = 0.05$ in (28) and confirms that for any T, there is an i such that $\mathcal{T}_{k_{2i+1}+1} > T$, so capture time is not uniformly bounded and the system is not input-to-state stable.

Next, to achieve ISS of the system, we give an update rule of L_k with uniform bounded capture time to the above coding scheme. Let L_0 be a positive number and let $L_{\max} = L_0$ and $s_0 = 0$.

Coding scheme with uniform bounded capture time:

- Encoding:
  At time k, let $X_k = P^{-1}\widehat{X}_k$. If $\|X_k\| \le L_k$, then the encoder labels quantisation blocks in Λ_k and encodes X_k as codeword $V(k)$, the label of the quantisation block in which X_k is, and updates the parameter L_k of quantiser (L_k, a, M) as $L_{k+1} = L_k\Omega_{in}$ with $\Omega_{in} \in (\mu_m, 1)$ and $s_{k+1} = 0$;
  if $\|X_k\| > L_k$ and $s_k = 0$, then the encoder encodes $X(k)$ as $V(k) = \phi$, and updates $L_{k+1} = L_{\max}$ and $s_{k+1} = 1$;
  if $\|X_k\| > L_k$ and $s_k = 1$, then the encoder encodes $X(k)$ as $V(k) = \phi$, and updates $L_{k+1} = L_k\Omega_{\text{out}}$ with $\Omega_{\text{out}} > \|G\|$, $s_{k+1} = 1$ and $L_{\max} = \max\{L_{\max}, L_{k+1}\}$.

That is,

$$L_{k+1} = \begin{cases} L_k\Omega_{in} & \text{if } \|X_k\| \le L_k \\ L_{\max} & \text{if } \|X_k\| > L_k,\ s_k = 0 \\ L_k\Omega_{\text{out}} & \text{if } \|X_k\| > L_k,\ s_k = 1 \end{cases}$$

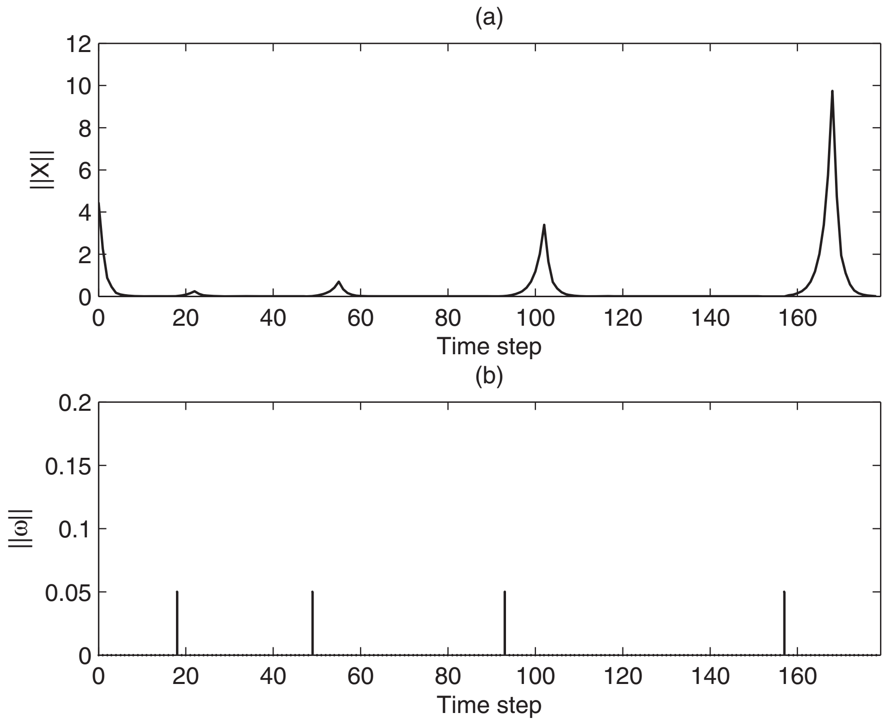

Figure 2. Simulation results for Example I.

$$s_{k+1} = \begin{cases} 0 \text{ if } ||X_k|| \le L_k \\ 1 \text{ if } ||X_k|| > L_k \end{cases}$$

$$L_{\max} = \max\{L_{\max}, L_{k+1}\} \text{ if } ||X_k|| > L_k, s_k = 1$$

- Decoding:
  At time k, if $V(k)$ labels the quantisation block indexed by $(i, j_1, \dots, j_{d-2}, s)$, then the decoder evaluates $\overline{X}_k$ with the spherical polar coordinates (12), updates the parameter L_k of quantiser (L_k, a, M) as $L_{k+1} = L_k\Omega_{in}$ and $s_{k+1} = 0$, and let $\overline{\overline{X}}_k = P\overline{X}_k$.
  if $V(k) = \phi$ and $s_k = 0$, then $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$, and updates $L_{k+1} = L_{\max}$ and $s_{k+1} = 1$;
  if $V(k) = \phi$ and $s_k = 1$, then $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$, and updates $L_{k+1} = L_k\Omega_{\text{out}}$, $s_{k+1} = 1$ and $L_{\max} = \max\{L_{\max}, L_{k+1}\}$.

That is,

$$L_{k+1} = \begin{cases} L_k\Omega_{in} & \text{if } V(k) \text{ is the label of the quantised block containing } X_k \\ L_{\max} & \text{if } V(k) = \phi, s_k = 0 \\ L_k\Omega_{\text{out}} & \text{if } V(k) = \phi, s_k = 1 \end{cases}$$

$$s_{k+1} = \begin{cases} 0 & \text{if } V(k) \text{ is the label of the quantised block containing } X_k \\ 1 & \text{if } V(k) = \phi \end{cases}$$

$$L_{\max} = \max\{L_{\max}, L_{k+1}\} \text{ if } V(k) = \phi, s_k = 1$$

Thus $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$ if $||X_k|| > L_k$.

Remark 4.1: (1) When the system is in zoom-out mode ($||X_k|| > L_k$), $s_k = 0$ denotes that the system first enters into zoom-out mode from zoom-in mode; $s_k = 1$ denotes that the system consecutively enters into zoom-out mode more than once from zoom-in mode. (2) From the encoding and the decoding procedure, we notice that at any time k, $L_{\max}$ memorises the maximum of the radius of the support ball up to k. From Lemma 4.2, the radius of the support ball is finite, so the coding scheme guarantees that capture time is uniformly bounded by letting $L_{k+1} = L_{\max}$ if the system first enters into zoom-out mode at k from zoom-in mode.

Example 4.2: To illustrate Theorem 4.1, we apply the coding scheme with uniform bounded capture time to the same system as in Example 4.1. The controller has the same feedback gain matrix, the disturbance behaves in the same way and the quantiser has the same parameters as in Example 4.1. $\Omega_{in} = 0.62$, $\Omega_{\text{out}} = 17.1$. The simulation result shows that the system exhibits non-increasing overshoots (Figure 3(a)) in response to the bounded disturbance (Figure 3(b)) with $\hat{\varepsilon} = 2$ in (28) and is input-to-state stable.

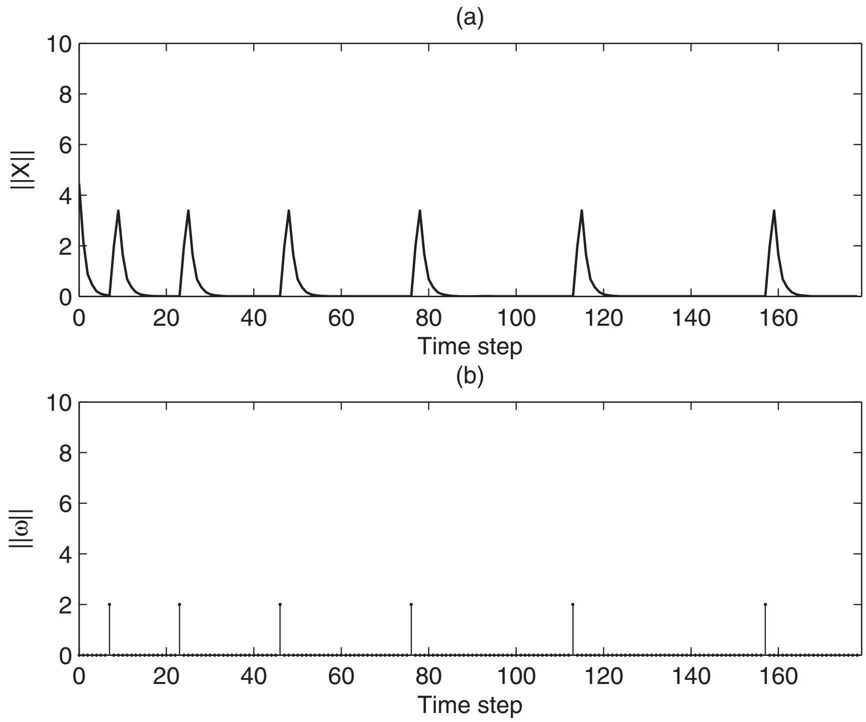

Figure 3. Simulation results for Example 2.

5. Input-to-state stability under coding scheme with finite data rate

5.1 Coding scheme of finite data rate

In the above section, to highlight the main results, a general coding scheme is presented with infinite data rate. Next, we proceed to present a specific version with finite data rate based on the above coding scheme.

Definition 5.1: A quantiser based on spherical polar coordinate at time k is a four-tuple (L_k, N, a, M), where the real number $L_k > 0$ represents the radius of the support ball at time k, the positive integer $N \geq 2$ represents the number of the proportional concentric balls, the real number $a > 0$ regulates the proportional coefficient, and the positive integer $M \geq 2$ represents the number of the angles into which the angle of radian π is equally partitioned. This quantiser partitions the support

$$\Lambda_k = \left\{X \in \mathbb{R}^d : r \leq L_k\right\}$$

into $2(N-1)M^{d-1}+1$ quantisation blocks as follows:

(1) the sets $\{X \in \mathbb{R}^d : \frac{L_k}{(1+2a)^{N-1-i}} < r \leq \frac{L_k}{(1+2a)^{N-2-i}}, j_n\frac{\pi}{M} < \theta_n \leq (j_n+1)\frac{\pi}{M}, n = 1, \ldots, d-2, s\frac{\pi}{M} < \theta_{d-1} \leq (s+1)\frac{\pi}{M}\}$, indexed by $(i, j_1, \ldots, j_{d-2}, s)$, $i = 0, \ldots, N-2$, $j_n = 0, \ldots, M-1$ for $n = 1, \ldots, d-2$, and $s = 0, \ldots, 2M-1$, whose number is $(N-1)\cdot M^{d-2}\cdot 2M = 2(N-1)M^{d-1}$; and

(2) the set $\left\{X \in \mathbb{R}^d : r \leq \frac{L_k}{(1+2a)^{N-1}}\right\}$.

Correspondingly, we divide zoom-in mode ($||X_k|| \leq L_k$) in the coding scheme in Section 3.2 into two submodes, namely, measurement-update mode and measurement-wait mode. By measurement-update mode, we means that $\frac{L_k}{(1+2a)^{N-1}} < ||X_k|| \leq L_k$. In this mode, the system is closed loop and the quantisation region is reduced to achieve convergence of the estimation error. And by measurement-wait mode, we means that $||X_k|| \leq \frac{L_k}{(1+2a)^{N-1}}$. In this mode, no feedback input is applied to the system and the system is open loop like zoom-out mode and waits for going back to measurement-update mode or zoom-out mode. The difference between zoom-out mode and measurement-wait mode is to update L_k and mark s_k; for details, see the following encoding and decoding procedure.

For each k, we denote $S_i(k) = \{X \in \mathbb{R}^d : r \leq \frac{L_k}{(1+2a)^{N-i}}\}$, $i = 1, \ldots, N$.

Let the initial L_0 be a positive number and let $L_{\max} = L_0$ and a mark $s_0 = 0$.

- Encoding:
  At time k, let $X_k = P^{-1}\widehat{X}_k$. If $X_k \in \Lambda_k \backslash S_1(k)$, that is, the system is in measurement-update mode, then the encoder labels $2(N-1)M^{d-1}$ quantisation blocks in $\Lambda_k \backslash S_1(k)$ and encodes X_k as code word $V(k)$, the label of the quantisation block in which X_k

is, and updates the parameter L_k of quantiser (L_k, N, a, M) as $L_{k+1} = L_k\Omega_{in}$ with Ω_{in} meeting (36) and $s_{k+1} = 0$;
if $X_k \in S_1(k)$, that is, the system is in measurement-wait mode, then the encoder encodes $X(k)$ as $V(k) = \phi_0$, and updates $L_{k+1} = L_k\Omega_{in}$ and $s_{k+1} = 0$;
if $||X_k|| > L_k$, that is, the system is in zoom-out mode, and $s_k = 0$, then the encoder encodes $X(k)$ as $V(k) = \phi_1$, and updates $L_{k+1} = L_{\max}$ and $s_{k+1} = 1$;
if $||X_k|| > L_k$ and $s_k = 1$, then the encoder encodes $X(k)$ as $V(k) = \phi_1$, and updates $L_{k+1} = L_k\Omega_{\text{out}}$ with $\Omega_{\text{out}} > ||G||$, $s_{k+1} = 1$ and $L_{\max} = \max\{L_{\max}, L_{k+1}\}$.

That is,

$$V(k) = \begin{cases} \text{The label of the quantised block containing } X_k & \text{if } X_k \in \Lambda_k/S_1(k) \\ \phi_0 & \text{if } X_k \in S_1(k) \\ \phi_1 & \text{if } ||X_k|| > L_k \end{cases}$$

$$L_{k+1} = \begin{cases} L_k\Omega_{in} & \text{if } ||X_k|| \le L_k \\ L_{\max} & \text{if } ||X_k|| > L_k,\ s_k = 0 \\ L_k\Omega_{\text{out}} & \text{if } ||X_k|| > L_k,\ s_k = 1 \end{cases} \quad (33)$$

$$s_{k+1} = \begin{cases} 0 & \text{if } ||X_k|| \le L_k \\ 1 & \text{if } ||X_k|| > L_k \end{cases}$$

$$L_{\max} = \max\{L_{\max}, L_{k+1}\} \text{ if } ||X_k|| > L_k,\ s_k = 1$$

Thus, we have $2(N-1)M^{d-1} + 2$ code words, i.e. ϕ_0, ϕ_1 and $2(N-1)M^{d-1}$ labels, so the data rate

$$R = \log_2 \left\lceil 2(N-1)M^{d-1} + 2 \right\rceil. \quad (34)$$

- Decoding:
  At time k, if $V(k)$ labels the quantisation block indexed by $(i, j_1, \ldots, j_{d-2}, s)$, then the decoder evaluates $\overline{X}_k$ with the spherical polar coordinates

$$r = \frac{(1+a)}{(1+2a)^{N-1-i}}L_k,\ \theta_n = \left(j_n + \frac{1}{2}\right)\frac{\pi}{M},$$
$$n = 1, \ldots, d-2,\ \theta_{d-1} = \left(s + \frac{1}{2}\right)\frac{\pi}{M}, \quad (35)$$

updates the parameter L_k of quantiser (L_k, N, a, M) as $L_{k+1} = L_k\Omega_{in}$ and $s_{k+1} = 0$, and let $\overline{\overline{X}}_k = P\overline{X}_k$.
if $V(k) = \phi_0$, then $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$, and updates $L_{k+1} = L_k\Omega_{in}$ and $s_{k+1} = 0$;
if $V(k) = \phi_1$ and $s_k = 0$, then $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$, and updates $L_{k+1} = L_{\max}$ and $s_{k+1} = 1$;
if $V(k) = \phi_1$ and $s_k = 1$, then $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$, and updates $L_{k+1} = L_k\Omega_{\text{out}}$, $s_{k+1} = 1$ and $L_{\max} = \max\{L_{\max}, L_{k+1}\}$.

That is,

$$\overline{X}_k = \begin{cases} (35) & \text{if } V(k) \text{ is the label of the quantised block containing } X_k \\ O & \text{if } V(k) = \phi_0 \text{ or } \phi_1 \end{cases}$$

$$L_{k+1} = \begin{cases} L_k\Omega_{in} & \text{if } V(k) = \phi_0 \text{ or the label of the quantised block containing } X_k \\ L_{\max} & \text{if } V(k) = \phi_1,\ s_k = 0 \\ L_k\Omega_{\text{out}} & \text{if } V(k) = \phi_1,\ s_k = 1 \end{cases}$$

$$s_{k+1} = \begin{cases} 0 & \text{if } V(k) = \phi_0 \text{ or the label of the quantised block containing } X_k \\ 1 & \text{if } V(k) = \phi_1 \end{cases}$$

$$L_{\max} = \max\{L_{\max}, L_{k+1}\} \text{ if } V(k) = \phi_1,\ s_k = 1$$

Thus, $\overline{X}_k = O$ and $\overline{\overline{X}}_k = O$ if $X_k \in S_1(k)$ or $||X_k|| > L_k$.
Let N, a, Ω_{in} and μ meet

$$1 > \Omega_{in} > \max\left\{\mu, \frac{||G||}{(1+2a)^{N-1}}\right\} \quad (36)$$

Remark 5.1: (1) When the system is in zoom-out mode ($||X_k|| > L_k$), $s_k = 0$ denotes that the system first enters into zoom-out mode from measurement-update or measurement-wait mode; $s_k = 1$ denotes that the system consecutively enters into zoom-out mode more than once from the other two modes. (2) From the encoding and the decoding procedure, we notice that at any time k, $L_{\max}$ memorises the maximum of the radius of the support ball up to k. From Lemma 5.2, the radius of the support ball is finite, so the coding scheme guarantees that the capture time is uniformly bounded by letting $L_{k+1} = L_{\max}$ if the system first enters into zoom-out mode at k from the other two modes.

5.2 Input-to-state stability under coding scheme of finite data rate

Under the above coding scheme, let the control input

$$U_k = \begin{cases} K\overline{X}_k, & \frac{L_k}{(1+2a)^{N-1}} < ||X_k|| \le L_k \\ O, & ||X_k|| \le \frac{L_k}{(1+2a)^{N-1}} \\ O, & ||X_k|| > L_k, \end{cases} \quad (37)$$

where K is feedback gain matrix, so system (8) with controller (37) is

$$X_{k+1} = \begin{cases} GX_k + HK\overline{X}_k + \omega_k, & \frac{L_k}{(1+2a)^{N-1}} < ||X_k|| \le L_k \\ GX_k + \omega_k, & ||X_k|| \le \frac{L_k}{(1+2a)^{N-1}} \\ GX_k + \omega_k, & ||X_k|| > L_k. \end{cases}$$

By Lemma 3.1, we have $||X_k - \overline{X}_k|| \leq \eta ||X_k||$ for $X_k \in \Lambda_k \backslash S_1(k)$, so

$$||X_{k+1}|| \leq \begin{cases} \mu||X_k|| + ||\omega_k||, & \frac{L_k}{(1+2a)^{N-1}} < ||X_k|| \leq L_k \\ ||G||||X_k|| + ||\omega_k||, & ||X_k|| \leq \frac{L_k}{(1+2a)^{N-1}} \\ ||G||||X_k|| + ||\omega_k||, & ||X_k|| > L_k, \end{cases} \tag{38}$$

where μ is defined as (17).

Lemma 5.1: *Consider the system consisting of plant (8) and controller (37). Under the coding scheme in Section 5.1, if there exists an integer $k_T < \infty$ such that*

$$k_T = \min\{k \geq 0 : ||X_k|| \leq L_k, L_k > \overline{L}\} \tag{39}$$

then, $||X_k|| \leq L_{k_T}$ and $||L_k|| \leq L_{k_T}$ for $k \geq 0$, where

$$\overline{L} = \max\left\{ \frac{||\omega||_{[0,\infty)}}{\Omega_{in}^{\delta-1}} \left(\Omega_{in} - \frac{||G||}{(1+2a)^{N-1}}\right)^{-1}, \frac{||\omega||_{[0,\infty)}}{\Omega_{in}^{\delta-1}} (\Omega_{in} - \mu)^{-1}, (1+||G||)||\omega||_{[0,\infty)}(1 - \Omega_{in}^{\delta+1}||G||)^{-1} \right\}$$

δ is a positive integer.

Proof: By (33) and (39), we have $||X_k|| \leq L_{k_T}$ and $||L_k|| \leq L_{k_T}$ for $k \in [0, k_T)$. There remains to prove that $||X_k|| \leq L_{k_T}$ and $||L_k|| \leq L_{k_T}$ for $k \geq k_T$.

We first prove that the system is not in zoom-out mode at $k \in [k_T, k_T + \delta]$ by mathematical induction, that is, $||X_k|| \in [0, L_k]$ for $k \in [k_T, k_T + \delta]$. So, $L_k = L_{k_T}\Omega_{in}^{k-k_T}$ for $k \in [k_T, k_T + \delta]$ by the coding scheme.

At $k = k_T$, the system is not in zoom-out mode since $||X_{k_T}|| \leq L_{k_T}$.

Assume that $||X_k|| \in [0, L_k]$ at time $k \in [k_T, k_T + \delta - 1]$, we prove that $||X_{k+1}|| \in [0, L_{k+1}]$. Consider two cases:

(1) If $||X_k|| \in [0, \frac{L_k}{(1+2a)^{N-1}}]$, then according to the coding scheme, the system is in measurement-wait mode, so by (38),

$$\begin{aligned} ||X_{k+1}|| &< ||G||||X_k|| + ||\omega_k|| \\ &< \frac{||G||L_k}{(1+2a)^{N-1}} + ||\omega||_{[0,\infty)} \\ &= \frac{||G||L_{k_T}\Omega_{in}^{k-k_T}}{(1+2a)^{N-1}} + ||\omega||_{[0,\infty)}, \\ & \qquad k \in [k_T, k_T + \delta - 1]. \end{aligned} \tag{40}$$

From (39), we have $L_{k_T} > \frac{||\omega||_{[0,\infty)}}{\Omega_{in}^{\delta-1}}(\Omega_{in} - \frac{||G||}{(1+2a)^{N-1}})^{-1}$, so $\frac{||G||L_{k_T}\Omega_{in}^{\delta-1}}{(1+2a)^{N-1}} + ||\omega||_{[0,\infty)} < L_{k_T}\Omega_{in}^{\delta}$. And from (40), we have

$$\begin{aligned} ||X_{k+1}|| &< L_{k_T}\Omega_{in}^{k+1-k_T} \\ &= L_{k+1}. \end{aligned}$$

(2) If $||X_k|| \in (\frac{L_k}{(1+2a)^{N-1}}, L_k]$, then according to the coding scheme, the system is in measurement-update mode, so by (38),

$$\begin{aligned} ||X_{k+1}|| &< \mu||X_k|| + ||\omega_k|| \\ &< \mu L_k + ||\omega||_{[0,\infty)} \\ &= \mu L_{k_T}\Omega_{in}^{k-k_T} + ||\omega||_{[0,\infty)}, k \in [k_T, k_T + \delta - 1]. \end{aligned} \tag{41}$$

From (39), we have $L_{k_T} > \frac{||\omega||_{[0,\infty)}}{\Omega_{in}^{\delta-1}}(\Omega_{in} - \mu)^{-1}$, so $\mu L_{k_T}\Omega_{in}^{\delta-1} + ||\omega||_{[0,\infty)} < L_{k_T}\Omega_{in}^{\delta}$. And from (41),

$$\begin{aligned} ||X_{k+1}|| &< L_{k_T}\Omega_{in}^{k-k_T+1} \\ &= L_{k+1}. \end{aligned}$$

Therefore, $||X_k|| \leq L_k, k \in [k_T, k_T + \delta]$, that is, the system is not in zoom-out mode at time $k \in [k_T, k_T + \delta]$. So $L_k \leq L_{k_T}, k \in [k_T, k_T + \delta]$.

If after $k_T + \delta$ the system does not enter into zoom-out mode, that is, $||X_k|| \leq L_k = L_{k_T}\Omega_{in}^{k-k_T}, k > k_T + \delta$, the proof is completed. If at time $k' - 1 > k_T + \delta$ the system is not in zoom-out mode, but enters into zoom-out mode at k' after $k_T + \delta$ first, that is,

$$||X_{k'-1}|| \leq L_{k'-1} = L_{k_T}\Omega_{in}^{k'-1-k_T} < L_{k_T}\Omega_{in}^{\delta}, \tag{42}$$

but $||X_{k'}|| > L_{k'}$, then,

$$\begin{aligned} ||X_{k'}|| &\overset{(38)}{<} \begin{cases} \mu||X_{k'-1}|| + ||\omega_k||, & ||X_{k'-1}|| \in \left(\frac{L_{k'-1}}{(1+2a)^{N-1}}, L_{k'-1}\right] \\ ||G||||X_{k'-1}|| + ||\omega_k||, & ||X_{k'-1}|| \in \left[0, \frac{L_{k'-1}}{(1+2a)^{N-1}}\right] \end{cases} \\ &< \begin{cases} \mu L_{k'-1} + ||\omega_k||, & ||X_{k'-1}|| \in \left(\frac{L_{k'-1}}{(1+2a)^{N-1}}, L_{k'-1}\right] \\ ||G||\frac{L_{k'-1}}{(1+2a)^{N-1}} + ||\omega_k||, & ||X_{k'-1}|| \in \left[0, \frac{L_{k'-1}}{(1+2a)^{N-1}}\right] \end{cases} \\ &\leq L_{k'-1}\max\left\{\mu, \frac{||G||}{(1+2a)^{N-1}}\right\} + ||\omega||_{[0,\infty)} \\ &\overset{(36),(42)}{\leq} L_{k_T}\Omega_{in}^{\delta+1} + ||\omega||_{[0,\infty)} \end{aligned}$$

and

$$\begin{aligned} ||X_{k'+1}|| &\overset{(38)}{<} ||G||||X_{k'}|| + ||\omega_{k'}|| \\ &< ||G||(L_{k_T}\Omega_{in}^{\delta+1} + ||\omega||_{[0,\infty)}) + ||\omega||_{[0,\infty)} \end{aligned}$$

From (39), we have $L_{k_T} > (1 + ||G||)||\omega||_{[0,\infty)}(1 - \Omega_{in}^{\delta+1}||G||)^{-1}$, so $||X_{k'+1}|| < L_{k_T}$. By the encoding or decoding procedure, update $L_{k'+1} = L_{\max} = L_{k_T}$, so

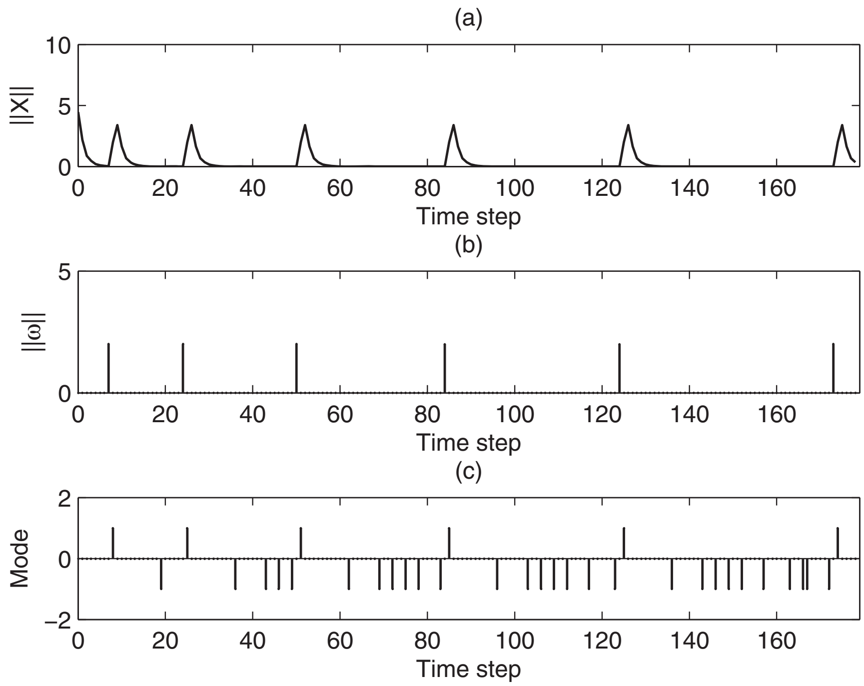

Figure 4. Simulation results for Example 3.

$||X_{k'+1}|| < L_{k_T} = L_{k'+1}$ and the system enters into measurement-update or measurement-wait mode again. Repeat the above procedure, the proof is completed. ■

Lemma 5.2: *Consider the system consisting of plant* (8) *and controller* (37). *Under the coding scheme in Section* 5.1, $||X_k|| \leq L_{k_T}$ *and* $||L_k|| \leq L_{k_T}$ *for* $k \geq 0$.

Proof: It is similar to the proof of Lemma 4.2 and omitted. ■

Theorem 5.1: *Consider the system consisting of plant* (8) *and controller* (37). *Under the coding scheme in Section* 5.1, *the system is input-to-state stable with respect to unknown disturbances.*

Proof: Since the coding scheme guarantees that capture time is uniformly bounded, by Theorem 4.1, the proof is completed. ■

Example 5.1: To illustrate Theorem 5.1, we apply the coding scheme in Section 5 to the same system as in Example 1. The controller (37) has the same feedback gain matrix and the disturbance behaves in the same way as in Example 4.1. $\Omega_{in} = 0.62$, $\Omega_{\text{out}} = 17.1$ and the parameters of the quantiser in Definition 5.1 $M = 175$, $N = 75$, $\alpha = 0.023$ and $L_0 = 6$ satisfy (36). By (34), the data rate R is 30. The simulation result shows that the system exhibits non-increasing overshoots (Figure 4 (a)) in response to the bounded disturbance (Figure 4(b)) with $\hat{\varepsilon} = 2$ in (28) and is input-to-state stable. Figure 4(c) shows that three modes are switched among one another, where

$$\text{Mode}(k) = \begin{cases} 0, & \frac{L_k}{(1+2a)^{N-1}} < ||X_k|| \leq L_k \\ -1, & 0 < ||X_k|| \leq \frac{L_k}{(1+2a)^{N-1}} \\ 1, & ||X_k|| > L_k \end{cases}$$

6. Conclusion

This paper shows that a necessary and sufficient condition for ISS of the quantised feedback system with respect to external disturbances is that capture time under the coding scheme is uniformly bounded. The coding scheme under which capture time is only bounded cannot guarantee ISS of the system. Therefore, the presented necessary and sufficient condition is instructive to the design of coding scheme for ISS of the system. A coding scheme of finite data rate is designed for uniformly bounded capture time and therefore achieves ISS for the system.

Acknowledgments

The authors would like to thank the editor and anonymous reviewers whose detailed comments and suggestions have helped to improve the manuscript.

Disclosure statement

No potential conflict of interest was reported by the author.

Funding

This work was supported by the National Natural Science Foundation of China [grant number 21506014], [grant number 61304002], [grant number 61573072], [grant number 61572082]; the Programme of Science and Technology Department of Liaoning Province [grant number 2015020038]; and the Programme of Bohai University [grant number 0515bs040-1].

References

Brockett, R.W., & Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control, 45*(7), 1279–1289.

Ceragioli, F., De Persis, C., & Frasca, P. (2011). Discontinuities and hysteresis in quantized average consensus. *Automatica, 47*(7), 1919–1928.

Corradinia, M.L., & Orlandob, G. (2008). Robust quantized feedback stabilization of linear systems. *Automatica, 44*(10), 2458–2462.

Delchamps, D.F. (1990). Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control, 35*(8), 916–924.

Elia, N., & Mitter, S.K. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control, 46*(9), 1384–1400.

Fu, M., & Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic Control, 50*(11), 1698–1711.

Gurt, A., & Nair, G.N. (2007). *Performance analysis of bit-rate-limited stochastic control systems*. Mediterranean conference on control & automation. Divani Caravel Hotel, Athens, Greece.

Hespanha, J., Ortega, A., & Vasudevan, L. (2002, August). Towards the control of linear systems with minimum bit-rate. In *Proceedings of 15th international symposium mathematical theory of networks*. Notre Dame, IN: University of Notre Dame.

Ishii, H., & Francis, B.A. (2002). Stabilizing a linear system by switching control with dwell time. *IEEE Transactions on Automatic Control, 47*(12), 1962–1973.

Kameneva, T., & Nesic, D. (2008). On l_2 stabilization of linear systems with quantized control. *IEEE Transactions on Automatic Control, 53*(3), 399–405.

Kang, X., & Ishii, H. (2015). Coarsest quantization for networked control of uncertain linear systems. *Automatica, 51*, 1–8.

Liberzon, D. (2014). Finite data-rate feedback stabilization of switched and hybrid linear systems. *Automatica, 50*, 409–420

Liberzon, D., & Nesic, D. (2007). Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Transactions on Automatic Control, 52*(5), 767–781.

Martins, N.C., Dahleh, A., & Elia, N. (2006). Feedback stabilization of uncertain systems in the presence of a direct link. *IEEE Transactions on Automatic Control, 51*(3), 438–447.

Matveev, A.S., & Savkin, A.V. (2004). *Stabilization of stochastic linear plants via limited capacity stochastic communication channel*. 43rd IEEE conference on decision and control (pp. 484–489). Atlantis, Paradise Island, Bahamas.

Nair, G.N., & Evans, R.J. (2004). Stabilization of stochastic linear systems with finite feedback data rates. *SIAM Journal on Control and Optimization, 43*(2), 413–436.

Nesic, D., Teel, A.R., & Sontag, E.D. (1999). Formulas relating KL stability estimates of discrete-time and sampled-data nonlinear systems. *Systems & Control Letters, 38*(1), 49–60.

Niu, Y., & Ho, D.W.C. (2014). Control strategy with adaptive quantizer's parameters under digital communication channels. *Automatica, 50*, 2665–2671.

Picasso, B., & Colaneri, P. (2008, December). Stabilization of discrete-time quantized linear systems: An H_∞/l_1 approach. Proceedings of 47th IEEE conference on decision and control (pp. 2868–2873), Cancun, Mexico.

Saldi, N., Linder, T., & Yüksel, S. (2015). Asymptotic optimality and rates of convergence of quantized stationary policies in stochastic control. *IEEE Transactions on Automatic Control, 60*(2), 553–558.

Savkin, A.V., & Cheng T., (2007). Detectability and output feedback stabilizability of nonlinear networked control systems. *IEEE Transactions on Automatic Control, 52*(4), 730–735.

Sharon, Y., & Liberzon, D. (2012). Input to state stabilizing controller for systems with coarse quantization. *IEEE Transactions on Automatic Control, 57*(4), 830–844.

Sontag, E.D., & Wang, Y. (1996). New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control, 41*(9), 1283–1294.

Tatikonda, S., & Mitter, S. (2004). Control under communication constraints. *IEEE Transactions on Automatic Control, 49*(7), 1056–1068.

Wang, J., & Yan, Z. (2014). Coding scheme based on spherical polar coordinate for control over packet erasure channel. *International Journal of Robust and Nonlinear Control, 24*, 1159–1176.

Wong, W.S., & Brockett, R.W. (1999). Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback. *IEEE Transactions on Automatic Control, 44*(5), 1049–1053.