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On stabilization of switching linear systems[☆]

Giorgio Battistelli[1]

Dipartimento di Ingegneria dell'Informazione - Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy

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ABSTRACT

This paper addresses the problem of controlling a continuous-time linear system that may switch among different modes taken from a finite set. The current mode of the system is supposed to be unknown. Moreover, unknown but bounded disturbances are assumed to affect the dynamics as well as the measurements. The proposed methodology is based on a minimum-distance mode estimator which orchestrates controller switching according to a dwell-time switching logic. Provided that the controllers are designed so as to ensure a certain mode observability condition and that the plant switching signal is slow on the average, the resulting control system turns out to be exponentially input-to-state stable.

1. Introduction

Over the last decade, a lot of attention has been devoted to switching systems from both research and industrial areas, as they allow one to represent and investigate the properties of a large class of plants in numerous applications resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions (see Liberzon, 2003). In a switching system, the system dynamics as well as the measurement equations may switch, at each time instant, among different configurations (system modes) taken from a finite set. Under the assumption that an exact knowledge of the current system configuration is available online without delay, the stabilization of a switching system is now a well understood problem. In fact, necessary and sufficient conditions for the existence of a switching controller that stabilizes the plant under arbitrary switching have been derived both in the continuous-time (Blanchini, Miani, & Mesquine, 2009) and in the discrete-time case (Lee & Dullerud, 2006). Similar methodologies can also be exploited in order to address the case of delayed information on the plant configuration (Xie & Wu, 2009).

On the other hand, the case in which the knowledge of the plant configuration is not available, neither in real time nor with delay, still poses many challenges. In this framework, the approaches proposed in the literature are based on the idea of estimating the current plant mode on the grounds of the plant state (Caravani & De Santis, 2009; Cheng, Guo, Lin, & Wang, 2005) or output (Li, Wen, & Soh, 2003; Vale & Miller, 2011). However, such results deal with particular settings (e.g., accessible state, absence of disturbances, etc.) and/or consider very specific class of controllers (Vale & Miller, 2011). Indeed, to the best of our knowledge, the problem of orchestrating the controller switching so as to ensure exponential stability for arbitrary initial conditions, arbitrary noise amplitudes, and generic classes of controllers is still an open issue. Further, even if in recent years extensive theoretical studies have been carried out on mode observability and mode estimation (see, for instance, Babaali & Pappas, 2005; Baglietto, Battistelli, & Scardovi, 2007, 2009; De Santis, Di Benedetto, & Pola, 2009; Di Benedetto, Di Gennaro, & D'innocenzo, 2009; Vidal, Chiuso, Soatto, & Sastry, 2003, and the references therein), such results have yet to be fully exploited in the context of adaptive stabilization of switching linear systems (with the exception of Caravani & De Santis, 2009). In this regard, a still open problem concerns the possibility of determining control law ensuring mode observability as well as stabilization of the considered switching system.

Motivated by this, a method is proposed to estimate the plant mode that naturally arises from mode observability considerations. Such a technique is based on the idea of evaluating the distance of the plant input/output data collected over a moving horizon from the set of behaviors associated to each possible mode. The minimum-distance mode estimator is then embedded in a supervisory unit that orchestrates the switching between the candidate controllers according to a *dwell-time switching logic* (DTSL) (Morse, 1995). Provided that all the candidate controllers are designed so as to satisfy a certain closed-loop mode observability condition, it

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E-mail address: giorgio.battistelli@unifi.it.

[1] Tel.: +39 0554796574; fax: +39 0554796363.

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is shown that the proposed minimum-distance criterion provides a reliable estimation of the current plant mode even in the presence of disturbances. Moreover, the exponential input-to-state stability of the resulting control system can be proved under the additional assumption that the plant switching signal is sufficiently slow on the average. Finally, it is proved that under mild assumptions on the plant, the set of controllers ensuring mode observability is generic.

The paper is organized as follows. Section 2 analyzes the mode observability of feedback control systems and provides guidelines for designing stabilizing controllers that also satisfy such a property. In Section 3, the adaptive stabilization scheme based on minimum-distance mode estimation is described and its stability properties analyzed in the noise-free case. Section 4 deals with the effects of persistent disturbances on the stability of the proposed control scheme. In Section 5, conclusions are drawn. For the reader's convenience, all the proofs are given in Appendix A. Finally, in Appendix B some possible relaxations of the derived results are discussed.

Before concluding this section, let us introduce some notations and basic definitions. Given a vector $v \in \mathbb{R}^n$, $|v|$ denotes its Euclidean norm. Given a symmetric, positive semi-definite matrix P, we denote by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimum and maximum eigenvalues of P, respectively. Given a matrix M, $M^\top$ is its transpose and $|M| = \left[\lambda_{\max}(M^\top M)\right]^{1/2}$ its spectral norm. Given a measurable time function $v : \mathbb{R}^+ \to \mathbb{R}^n$ and a time interval $\mathcal{I} \subseteq \mathbb{R}^+$, we denote the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of $v(\cdot)$ on $\mathcal{I}$ as $\|v\|_{2,\mathcal{I}} = \sqrt{\int_{\mathcal{I}} |v(t)|^2 dt}$ and $\|v\|_{\infty,\mathcal{I}} = \operatorname{ess\,sup}_{t\in\mathcal{I}} |v(t)|$ respectively. When $\mathcal{I} = R^+$, we simply write $\|v\|_2$ and $\|v\|_\infty$. Further, $\mathcal{L}_2(\mathcal{I})$ and $\mathcal{L}_\infty(\mathcal{I})$ denote the sets of square integrable and, respectively, (essentially) bounded time functions on $\mathcal{I}$.

2. Mode-observability of feedback linear switching systems

Consider a plant $\mathcal{P}_{\sigma(t)}$ described by a continuous-time switching linear system of the form

$$\mathcal{P}_{\sigma(t)} : \begin{cases} \dot{x}(t) = A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t) \\ y(t) = C_{\sigma(t)}\, x(t) \end{cases} \tag{1}$$

where $t \in \Re_+$ is the time instant, $x(t) \in \Re^{n_x}$ is the plant state vector, $\sigma(t) \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}$ is the plant mode, $u(t) \in \Re^{n_u}$ is the control input, $y(t) \in \Re^{n_y}$ is the vector of the measurements. A_i, B_i, and C_i, $i \in \mathcal{N}$, are constant matrices of appropriate dimensions. In order to ensure well-posedness of the solution of (1), It is supposed that the unknown and unobserved switching signal $\sigma : \Re^+ \to \mathcal{N}$ belongs to the class Σ of all the functions that are piecewise constant, right continuous, and admit no Zeno behavior (i.e., the number of switching instants is finite on every finite interval).

For the plant $\mathcal{P}_{\sigma(t)}$, we consider a linear switching controller $\mathcal{C}_{\hat{\sigma}(t)}$ of the form

$$\mathcal{C}_{\hat{\sigma}(t)} : \begin{cases} \dot{q}(t) = F_{\hat{\sigma}(t)}\, q(t) + G_{\hat{\sigma}(t)}\, y(t) \\ u(t) = H_{\hat{\sigma}(t)}\, q(t) + K_{\hat{\sigma}(t)}\, y(t) \end{cases} \tag{2}$$

where $q(t) \in \Re^{n_q}$ is the controller state vector and $\hat{\sigma}(t) \in \mathcal{N}$ is the controller mode. F_i, G_i, H_i, and K_i, $i \in \mathcal{N}$, are constant matrices of appropriate dimensions. The switching signal $\hat{\sigma} : \Re^+ \to \mathcal{N}$ is supposed to be known and belonging to Σ.

In this section, attention will be devoted to the problem of inferring the plant mode $\sigma(t)$ from observation of the plant input/output data. To this end, it is convenient to consider the following state space realization for the closed loop system $(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)})$ resulting from the feedback interconnection of (1) with (2)

$$(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)}) : \begin{cases} \dot{w}(t) = A^{cl}_{\sigma(t)/\hat{\sigma}(t)}\, w(t) \\ z(t) = C^{cl}_{\sigma(t)/\hat{\sigma}(t)}\, w(t) \end{cases} \tag{3}$$

where

$$w(t) \triangleq \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}, \qquad z(t) \triangleq \begin{bmatrix} u(t) \\ y(t) \end{bmatrix},$$

$$A^{cl}_{i/j} \triangleq \begin{bmatrix} A_i + B_i K_j C_i & B_i H_j \\ G_j C_i & F_j \end{bmatrix}, \qquad C^{cl}_{i/j} \triangleq \begin{bmatrix} K_j C_i & H_j \\ C_i & 0 \end{bmatrix},$$

for each $i, j \in \mathcal{N}$.

Further, let us denote by

$$z_{i/j}(t, t_0, w_0) \triangleq C^{cl}_{i/j} e^{A^{cl}_{i/j}(t-t_0)} w_0$$

the output of (3) at time $t > t_0$ when the initial state at time t_0 is w_0, the controller switching signal is $\hat{\sigma}(\tau) = j$ for any $\tau \in [t_0, t]$, and the plant switching signal is $\sigma(\tau) = i$ for any $\tau \in [t_0, t]$. The following notion of *distinguishability* between two plant modes can now be introduced.

Definition 1. For system (3), two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are said to be *distinguishable* if

$$z_{i/j}(\cdot, t_0, w_0) \neq z_{i'/j}(\cdot, t_0, w'_0) \quad \text{a.e. on } [t_0, t]$$

for any t_0, t with $t > t_0, j \in \mathcal{N}$, and $w_0, w'_0 \in \Re^{n_x+n_q}$ with $w_0 \neq 0$ or $w'_0 \neq 0$. □

The rationale behind Definition 1 is as follows. At a given time t_0, supposing that a certain controller mode $C_j, j \in \mathcal{N}$ is inserted in the feedback loop, the active plant mode can be any $\mathcal{P}_i, i \in \mathcal{N}$. Our aim is therefore to address the problem of discerning which plant modes could have produced the collected input/output data z. Thus, roughly speaking, we say that two plant modes are distinguishable when, over any finite interval, they always lead to different input/output data provided that their state trajectories are not jointly null. To elaborate more on this issue, consider the two feedback loops (P_i/C_j) and $(P_{i'}/C_j)$ with initial states w_0 and w'_0, respectively. In order to have distinguishability, we require that the only case in which the two feedback loops exhibit the same input/output behavior is when their initial states are zero, i.e., $w_0 = w'_0 = 0$. This amounts to requiring that the behavioral data associated with such two feedback loops, i.e., $z_{i/j}(\cdot, t_0, w_0)$ and $z_{i'/j}(\cdot, t_0, w'_0)$, are different whenever at least one between w_0 and w'_0 is different from zero. Further, we require that such a condition holds irrespective of which controller mode is active. Notice that the adopted distinguishably notion is instrumental to the developments of the following sections, however other distinguishability notions are possible based on different perspectives (see De Santis, 2011, for an overview of the topic).

Definition 2. The feedback system (3) is said to be *mode-observable* if any two different plant modes $i, i' \in \mathcal{N}$ are distinguishable. □

Mode observability corresponds to the invertibility of the mapping from plant input/output data $z(\cdot)$ to the plant switching signal $\sigma(\cdot)$. In fact, it is an easy matter to see that, under mode observability, it is possible, at least in principles, to reconstruct the unknown switching signal $\sigma(\cdot)$ from observation of $z(\cdot)$, provided that the initial state $w(0)$ is not null. In what follows, necessary and sufficient conditions for mode observability of (3) will be derived.

To this end, some preliminary definitions are needed. Let $O^{(k)}_{i/j}$ be the observability matrix of order k of the feedback system $(\mathcal{P}_i/\mathcal{C}_j)$

$$O^{(k)}_{i/j} \triangleq \begin{bmatrix} C^{cl}_{i/j} \\ C^{cl}_{i/j} A^{cl}_{i/j} \\ \vdots \\ C^{cl}_{i/j} \left(A^{cl}_{i/j}\right)^{k-1} \end{bmatrix}.$$

Further, let

$$\Phi^{cl}_{i/j}(t) = C^{cl}_{i/j} e^{A^{cl}_{i/j} t}$$

be the state-to-output transition matrix of the feedback system $(\mathcal{P}_i/\mathcal{C}_j)$ and

$$W_{i/j}(t) \triangleq \int_0^t \left(\Phi^{cl}_{i/j}(\xi)\right)^\top \Phi^{cl}_{i/j}(\xi)\, d\xi$$

its observability Gramian.

The next lemma unveils that the *joint observability matrix*

$$\left[O^{(2n_x+2n_q)}_{i/j}\ O^{(2n_x+2n_q)}_{i'/j}\right]$$

plays a key role in determining the distinguishability of two plant modes i and i'.

Lemma 1. *Two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are distinguishable if and only if their joint observability matrix is full-rank, i.e.,*

$$\operatorname{rank}\left[O^{(2n_x+2n_q)}_{i/j}\ O^{(2n_x+2n_q)}_{i'/j}\right] = 2n_x + 2n_q, \quad \forall j \in \mathcal{N}. \tag{4}$$

As a consequence, the feedback system (3) is mode observable if and only if condition (4) holds for any pair of different plant modes $i, i' \in \mathcal{N}$. □

The proof of Lemma 1 is omitted since it is just an adaptation of well-known results about observability of switching linear systems (Babaali & Pappas, 2005; Vidal et al., 2003). Notice that, in view of Lemma 1, a necessary condition for mode observability is that all the pairs $(A^{cl}_{i/j}, C^{cl}_{i/j})$ be observable so that each $O^{(2n_x+2n_q)}_{i/j}$ have rank $n_x + n_q$. This, in turn, amounts to requiring that both the plant $\mathcal{P}_i$ and the controller $\mathcal{C}_j$ be observable for any fixed indices i and j. It is worth noting that such a requirement depends on the adopted distinguishability notion (see Definition 1). In fact, the presence of unobservable dynamics would entail the existence of non-zero trajectories of the state w corresponding to zero trajectories of the input/output data z and, hence, for which it would be impossible to infer the plant mode. The possibility of relaxing the observability requirement will be discussed later on.

Consider now a left polynomial matrix fraction description (LPMFD) of the plant

$$\mathcal{P}_{\sigma(t)} : P_{\sigma(t)}(D)\, y(t) = Q_{\sigma(t)}(D)\, u(t) \tag{5}$$

where for each $i \in \mathcal{N}$, $P_i(D)$ and $Q_i(D)$ are matrices of appropriate dimensions whose entries are polynomials in the differential operator $D \triangleq d/dt$ and such that $\det P_i(s) = \det(sI - A_i)$. Here, Eq. (5) is intended as a shorthand notation to mean that over each interval of time where $\sigma(t) = i$ is constant, $y(t)$ is the output of a LTI system described by the input/output relation $P_i(D)y(t) = Q_i(D)u(t)$ with the state at the beginning of this interval being initialized according to (1).

Similarly, consider a LPMFD of the controller

$$\mathcal{C}_{\hat\sigma(t)} : R_{\hat\sigma(t)}(D)\, u(t) = S_{\hat\sigma(t)}(D)\, y(t) \tag{6}$$

where, for each $j \in \mathcal{N}$, $R_j(D)$ and $S_j(D)$ are polynomial matrices of appropriate dimensions with

$$\det R_j(s) = \det\left(sI - F_j\right).$$

Notice that such LPMFDs always exist when the pairs (A_i, C_i) and (F_j, H_j) are observable (Antsaklis & Michel, 2006). Then, the following lemma can be stated which provides an alternative condition for studying mode observability of the feedback system (3).

Lemma 2. *Let the pairs (A_i, C_i) and (F_j, H_j) be observable for any $i, j \in \mathcal{N}$. Then, two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are distinguishable if and only if, for any $j \in \mathcal{N}$,*

$$\operatorname{rank}\begin{bmatrix} P_i(s) & -Q_i(s) \\ P_{i'}(s) & -Q_{i'}(s) \\ -S_j(s) & R_j(s) \end{bmatrix} = n_u + n_y, \quad \forall s \in \mathbb{C}. \quad \square \tag{7}$$

Recalling that the characteristic polynomial $\varphi_{i/j}(s)$ of the feedback system $(\mathcal{P}_i/\mathcal{C}_j)$ is

$$\varphi_{i/j}(s) = \det(sI - A^{cl}_{i/j}) = \det\begin{bmatrix} P_i(s) & -Q_i(s) \\ -S_j(s) & R_j(s) \end{bmatrix}$$

the following result can be readily established.

Proposition 1. *Let the pairs (A_i, C_i) and (F_j, H_j) be observable for any $i, j \in \mathcal{N}$ and consider two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$. If, for any $j \in \mathcal{N}$, the closed loop characteristic polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime, then i, i' are distinguishable.* □

In general, Proposition 1 provides only a sufficient condition for distinguishability, however in the case of a single-input single-output (SISO) also necessity holds in the following sense.

Proposition 2. *Let the plant be SISO, i.e., $n_u = n_y = 1$. Further, let the pairs (A_i, C_i) and (F_j, H_j) be observable for any $i, j \in \mathcal{N}$ and the pairs (F_j, G_j) be controllable for any $j \in \mathcal{N}$. Then, two plant modes $i, i' \in \mathcal{N}$ with $i \neq i'$ are distinguishable if and only if, for any $j \in \mathcal{N}$, the closed loop characteristic polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime.* □

Propositions 1 and 2 suggest that mode-observability of the feedback system (3) can be guaranteed by designing each controller $\mathcal{C}_j$ so that, for any pair $i, i' \in \mathcal{N}$ with $i \neq i'$, the closed loop polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ have no common roots. We point out that, since the plant can take on only a finite number of possible configurations (i.e., the set $\mathcal{N}$ is finite), this condition can be easily satisfied in practice. More precisely, in the SISO case, the following result can be stated.

Lemma 3. *Let the plant be SISO, i.e., $n_u = n_y = 1$. Then a controller ensuring distinguishability of two plant modes $i, i' \in \mathcal{N}$ exists if and only if the following three conditions hold:*

(a) *the pairs (A_i, C_i) and $(A_{i'}, C_{i'})$ are observable;*
(b) *the input/output maps $C_i(sI - A_i)^{-1}B_i$ and $C_{i'}(sI - A_{i'})^{-1}B_{i'}$ are different;*
(c) *the characteristic polynomials of the uncontrollable parts of $\mathcal{P}_i$ and $\mathcal{P}_{i'}$ are coprime.*

Further, when conditions (a)–(c) *hold, for any given controller order n_q the set of controller matrices (F_j, G_j, H_j, K_j) ensuring distinguishability is generic.*[2]

Notice that condition (b) is the necessary and sufficient condition for the distinguishability of modes i and i', starting from any initial state, by applying a generic input function (see De Santis, 2011). The additional conditions (a) and (c) depend on the fact that the control input u is supposed to be generated by a linear controller $\mathcal{C}_j$ and that distinguishability is required for any non-zero initial state of the feedback loop. Recalling that the countable intersection of generic sets is also generic, we can conclude that, under conditions (a)–(c), almost all the choices of the controller matrices (F_j, G_j, H_j, K_j) lead to mode observability of the feedback system. Building on the above lemma, it is possible to prove that the last claim holds also in the general MIMO case.

[2] Recall that a subset $\mathcal{X}$ of a topological space is generic when the following two conditions hold: for any $x \in \mathcal{X}$, then there exists a neighborhood of x contained in $\mathcal{X}$; for any $x \notin \mathcal{X}$, then every neighborhood of x contains an element of $\mathcal{X}$.

Lemma 4. *Consider two plant modes $i, i' \in \mathcal{N}$ and suppose that conditions* (a)–(c) *of Lemma 3 hold. Then, for any given controller order n_q, the set of controller matrices (F_j, G_j, H_j, K_j) ensuring distinguishability is generic.*

As it can be seen from the statements of Lemmas 3 and 4, there are no particular restrictions concerning the order of each controller mode $\mathcal{C}_j$. Clearly, in case he adopted design procedure leads to controller transfer functions having the same order for each mode $j \in \mathcal{N}$, one can simply let the state dimension n_q coincide with such an order. In the negative, the simplest way to realize the switching controller consists of: setting the state dimension n_q as the maximum among the orders of all controller modes; for each controller mode having order less than n_q, adopt a non-minimal but stabilizable and observable realization of order n_q.

We conclude this section with a comment on the role that controller realization plays on mode observability. The first important observation is that, as it can be easily checked, mode observability is a structural property in the sense that it is invariant upon change of coordinates. In particular, all minimal realizations of the controller share the same mode observability properties. On the other hand, the adoption of non-minimal realizations will in general affect mode observability. This is a relevant issue because, when dealing with switching controllers, non-minimal realization can come into play for several reasons (for instance, in shared-state architecture for switching between controllers of different orders as discussed above or for improving transient performance as will be discussed at the end of Section 3.2). As already pointed out after Lemma 1, the presence in the controller of unobservable dynamics is not compatible with mode observability. On the contrary, the effect of uncontrollable dynamics is less extreme. For instance, in the SISO case, it is possible to see that distinguishability of two plant modes $i, i' \in \mathcal{N}$ under controller $\mathcal{C}_j$ is preserved provided that, for any uncontrollable eigenvalue s_0 of $\mathcal{C}_j$, one has

$$\operatorname{rank}\begin{bmatrix} P_i(s_0) & -Q_i(s_0) \\ P_{i'}(s_0) & -Q_{i'}(s_0) \end{bmatrix} = 2$$

so that the rank condition (7) is satisfied. A generalization of such a result to the general MIMO case is possible, albeit non-trivial, and would require an analysis similar to the one adopted in Baglietto, Battistelli, and Tesi (2013) for mode observability in the presence of exogenous signals (the uncontrollable part of the controller playing the role of an exosystem acting on the controllable part of the feedback loop).

2.1. An example

In order to illustrate how to design controllers ensuring mode observability, let us consider the two-tank system of Blanchini et al. (2009) which can be modeled as in (1) with $N = 2$ and

$$A_1 = A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$C_1 = [0\,1], \qquad C_2 = [1\,0]. \tag{8}$$

The two plant modes have transfer functions from u to y equal to $\mathfrak{P}_1(s) = 1/(s^2 + 2s)$ and $\mathfrak{P}_2(s) = -\mathfrak{P}_1(s)$, respectively. For each plant mode a proportional–integral controller is considered with transfer function

$$\mathfrak{C}_i(s) = \frac{K_{P,i}\, s + K_{I,i}}{s}, \quad i = 1, 2. \tag{9}$$

When controller $\mathcal{C}_1$ is active, the resulting closed-loop characteristic polynomials are

$$\varphi_{1/1}(s) = s^3 + 2s^2 - K_{P,1}\, s - K_{I,1}$$
$$\varphi_{2/1}(s) = s^3 + 2s^2 + K_{P,1}\, s + K_{I,1}.$$

In view of Proposition 2, in order to ensure distinguishability the two gains $K_{P,1}$ and $K_{I,1}$ have to be chosen so that $\varphi_{1/1}(s)$ and $\varphi_{2/1}(s)$ are coprime. As well known, coprimeness of two polynomials is equivalent to the fact that their Sylvester resultant (i.e., the determinant of the Sylvester matrix associated with the two polynomials) is different from 0. Via elementary calculations, it can be seen that the Sylvester resultant of $\varphi_{1/1}(s)$ and $\varphi_{2/1}(s)$ is $8\,K_{I,1}\,(K_{I,1} - 2\,K_{P,1})$ so that distinguishability holds when $K_{I,1} \neq 0$ and $K_{I,1} \neq 2K_{P,1}$. Analogous considerations hold when $\mathcal{C}_2$ is active.

3. Constructing a stabilizing controller

In this section, it will be shown that stability of the feedback system (3) can be achieved by means of a suitable choice of the controller switching signal $\hat{\sigma}(t)$. To this end, it is supposed that the controllers $\mathcal{C}_i, i \in \mathcal{N}$ are designed so as to satisfy the following basic assumptions.

A1. For each index $i \in \mathcal{N}$, the i-th tuned loop $(\mathcal{P}_i/\mathcal{C}_i)$ is asymptotically stable.

A2. The feedback system (3) is mode-observable.

The choice of the control action to use, among all the available candidate controllers $\mathcal{C}_i, i \in \mathcal{N}$, is carried out in real-time by a data-driven high-level unit called *mode estimator*. At each time $t \in \mathbb{R}_+$, the mode estimator generates an estimate $\hat{\sigma}(t, z(\cdot)) \in \mathcal{N}$ of the current plant mode on the basis of the plant input/output data $z(\cdot)$ up to the current time t. Such an estimate is then used as the controller switching signal, i.e.,

$$\hat{\sigma}(t) = \hat{\sigma}(t, z(\cdot)).$$

As to the generation of the estimates, it is supposed that the mode estimator updates its estimate $\hat{\sigma}(t, z(\cdot))$ of the plant mode $\sigma(t)$ at discrete-time instants of the type kT where $k \in \mathbb{Z}_+$ and T, a positive real, is the so called *dwell time*. This amounts to assuming the controller switching signal $\hat{\sigma}(t)$ to be constant over each interval $\mathcal{I}_k \triangleq [kT, (k+1)T)$, i.e.,

$$\hat{\sigma}(t) = \hat{\sigma}_k, \quad \forall t \in \mathcal{I}_k.$$

In other words, the switching between controllers is orchestrated according to a DTSL.

3.1. A minimum-distance mode estimator

Thanks to the adoption of the DTSL, a simple criterion for the determination of the estimate $\hat{\sigma}_k$ can be devised. To this end, notice first that, whenever also the plant mode takes on a constant value, say i, over $\mathcal{I}_k$, the evolution of the plant input/output data on $\mathcal{I}_k$ can be written as

$$z(t) = z_{i/\hat{\sigma}_k}(t, kT, w(kT)), \quad t \in \mathcal{I}_k.$$

Thus the set $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$ of all the possible plant input/output data over $\mathcal{I}_k$ associated with a plant mode i and a controller mode $\hat{\sigma}_k$ corresponds to the linear subspace

$$\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k) \triangleq \{\hat{z} \in \mathcal{L}_2(\mathcal{I}_k) : \hat{z}(\cdot) = z_{i/\hat{\sigma}_k}(\cdot, kT, \hat{w}) \text{ on } \mathcal{I}_k, \text{ for some } \hat{w} \in \Re^{n_x+n_q}\}.$$

It is an easy matter to see that under mode observability the following result holds.

Proposition 3. *Under assumption* A2, *for any two different plant modes $i, i' \in \mathcal{N}$ and any controller mode $\hat{\sigma}_k \in \mathcal{N}$, one has $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k) \cap \mathcal{S}_{i'/\hat{\sigma}_k}(\mathcal{I}_k) = \{0\}$.* $\square$

For the reader's convenience, a graphical representation of the condition of Proposition 3 is provided in Fig. 1. In view of the

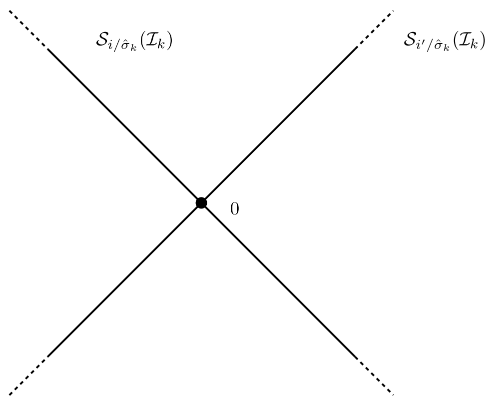

Fig. 1. Graphical representation of the condition of Proposition 3. Under mode observability, the intersection between the linear subspaces associated with two different plant modes i and i' is always trivial. Thus, for any non-zero trajectory, the plant mode can be reconstructed by looking to which one of the sets $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$, $i \in \mathcal{N}$ the plant input/output data belong.

above considerations, one can see that a particularly convenient approach for estimating the plant mode $\sigma(\cdot)$ on $\mathcal{I}_k$ consists in choosing the index i for which the distance between the observed plant input/output data $z(\cdot)$ on $\mathcal{I}_k$ and the linear subspace $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$ is minimal. Accordingly, at the generic time $(k+1)T$ the estimate $\hat{\sigma}_{k+1}$ can be obtained according to a minimum-distance criterion. To this end, let

$$\delta_{i/j}(z(\cdot), \mathcal{I}_k) \triangleq \min_{\hat{w} \in \Re^{n_x+n_q}} \left\| z(\cdot) - z_{i/j}(\cdot, kT, \hat{w}) \right\|_{2,\mathcal{I}_k}. \tag{10}$$

Then, the controller mode can be updated according to the following logic: at any time instant kT one first selects, arbitrarily, a value $i_* \in \mathcal{N}$ among those which achieve the minimum of $\delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$, i.e.,

$$i_* \in \arg\min_{i \in \mathcal{N}} \delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k); \tag{11}$$

if $\delta_{i_*/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$ is smaller than $\delta_{\hat{\sigma}_k/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$, then $\hat{\sigma}_{k+1}$ is set equal to i_*, otherwise the controller mode is left unchanged. Notice that, being the pair $(A^{cl}_{i/j}, C^{cl}_{i/j})$ completely observable by hypothesis, the minimization in (10) yields

$$\delta_{i/j}(z(\cdot), \mathcal{I}_k) = \left(\int_{\mathcal{I}_k} \left| z(t) - \Phi^{cl}_{i/j}(t-kT) \left(W_{i/j}(kT)\right)^{-1} \times \int_{\mathcal{I}_k} \left(\Phi^{cl}_{i/j}(\xi - kT)\right)^{\top} z(\xi)\, d\xi \right|^2 dt \right)^{1/2}.$$

The next lemma points out an important property of the minimum-distance criterion (11).

Lemma 5. *Suppose that assumption* A2 *holds, that* $w(kT) \neq 0$ *and the plant mode is constant on* $\mathcal{I}_k$, *i.e.,*

$$\sigma(t) = \sigma_k, \quad \forall t \in \mathcal{I}_k.$$

Then, if the minimum distance criterion (11) *is used, one has* $\hat{\sigma}_{k+1} = \sigma_k$. □

Lemma 5 indicates that, under the stated assumptions, the proposed minimum distance criterion always leads to the exact reconstruction of the plant mode provided that no switch occurs in the observation interval. In other words, this implies that an identification error can be incurred only in those intervals characterized by at least one plant mode variation.

Before proceeding on discussing such stability results, some comments on the practical applicability of (11) are in order. First of all, it is important to note that, instead of computing the integral in (10) on the whole interval $\mathcal{I}_k$ (which would require real-time operations), one could restrict its attention to a subinterval of the type $[kT, kT+\tau)$ with $\tau \in (0, T)$, reserving the remaining subinterval $[kT+\tau, (k+1)T)$ for all the necessary computations. It is immediate to verify that Lemma 5 holds also in this more general setting. Further, when continuous-time integration is not feasible, one could replace the integral in (10) with a finite summation without compromising the validity of Lemma (11), provided that the sampling instants are adequately chosen. To see this, consider a positive real Δ and a positive integer M such that $\Delta M < T$ and suppose that the distance $\delta_{i/j}(z(\cdot), \mathcal{I}_k)$ is replaced by

$$\tilde{\delta}_{i/j}(z(\cdot), \mathcal{I}_k) \triangleq \min_{\hat{w} \in \Re^{n_x+n_q}} \left(\sum_{h=0}^{M} \left| z(kT + h\Delta) - \Phi^{cl}_{i/j}(h\Delta)\hat{w} \right|^2 \right)^{1/2}.$$

Further, let $\Lambda_{i/j}$ be the spectrum of $A^{cl}_{i/j}$, then the following result can be stated that descends from the well-known Kalman–Bertram criterion for the observability of sampled-data systems (Antsaklis & Michel, 2006).

Corollary 1. *Suppose that assumption* A2 *holds, that* $w(kT) \neq 0$ *and the plant mode is constant on* $\mathcal{I}_k$, *i.e.,*

$$\sigma(t) = i, \quad t \in \mathcal{I}_k.$$

Further, let the minimum distance criterion (11) *be used with* $\delta_{i/j}(z(\cdot), \mathcal{I}_k)$ *replaced by* $\tilde{\delta}_{i/j}(z(\cdot), \mathcal{I}_k)$. *Then,*

$$\hat{\sigma}_{k+1} = i$$

provided that

$$M - 1 \geq n_x + n_q$$

and the sampling period Δ *is such that, for any two different plant modes* $i, i' \in \mathcal{N}$ *and any controller mode* $j \in \mathcal{N}$,

$$\mathrm{Im}(\lambda - \lambda') \neq \frac{2\pi h}{\Delta} \quad \text{for } h \in \mathbb{Z} \setminus \{0\} \text{ whenever } \mathrm{Re}(\lambda - \lambda') = 0$$

for any $\lambda, \lambda' \in \Lambda_{i/j} \cup \Lambda_{i'/j}$. □

As a final remark, notice that it is always possible to find suitable values of Δ satisfying the requirements of Corollary 1.

3.2. Stability under an average dwell-time

Thanks to Lemma 5, it is possible to show that the proposed control system with mode estimator yields an exponentially stable closed-loop system provided that the plant switching signal $\sigma(t)$ is slow on the average, i.e., the number of switches in any finite interval grows linearly with the length of the interval, with sufficiently small growth rate. In this respect, let $N_\sigma(t, t_0)$ be the number of discontinuities of σ in the interval (t_0, t), then the following assumption is needed (Hespanha, 2004; Hespanha & Morse, 1999).

A3. There exist a positive real τ_D, called *average dwell-time*, and a positive integer N_0, called *chatter bound*, such that

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_D}$$

for any $t, t_0 \in \mathbb{R}_+$ with $t > t_0$.

In order to study the stability of the overall control scheme, we consider a generic vector norm $\|\cdot\|$ on $\mathbb{R}^{n_x+n_q}$ and the corresponding induced matrix norm. Such a norm can be, for example, a weighted Euclidean norm or a polyhedral norm (see, for instance, Blanchini, 1995, and the references therein). These are, in fact, the two most typical choices when dealing with switching systems. Note now that assumption A1 ensures that, for any given vector norm $\|\cdot\|$, there exist two positive reals μ and λ such that

$$\|e^{A^{cl}_{i/i}t}\| \le \mu e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+, \forall i \in \mathcal{N}. \tag{12}$$

Further, since the set $\mathcal{N}$ is finite, one has

$$\|e^{A^{cl}_{i/j}t}\| \le e^{\rho t}, \quad \forall t \in \mathbb{R}_+, \forall i, j \in \mathcal{N} \tag{13}$$

for some positive real ρ. Of course, the numerical values of the constants μ, λ, and ρ will depend on the particular choice of the vector norm $\|\cdot\|$. For instance, whenever possible a particularly convenient choice would be to choose $\|\cdot\|$ so that $\|w\|$ be a common Lyapunov function for the $\mathcal{N}$ tuned loops $(\mathcal{P}_i/\mathcal{C}_i)$. In fact, in this case, the constant μ would be equal to 1. With this respect, the use of polyhedral norms may prove relevant since they are known to be a generic class of Lyapunov functions for switching systems, in the sense that stability of a continuous-time switching system is equivalent to the existence of a polyhedral Lyapunov function.

The main stability result of this section can now be stated.

Theorem 1. *Suppose that assumptions* A1–A3 *holds, that* $w(0) \neq 0$ *and let the minimum distance criterion* (11) *be used. Then, the state transition matrix* $\Phi(t, t_0)$ *of the closed-loop system* $(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)})$ *can be upper bounded as*

$$\|\Phi(t, t_0)\| \le \beta\, e^{-\alpha\,(t-t_0)} \tag{14}$$

where

$$\alpha = \lambda - [\log \mu + 2(\lambda + \rho)T]/\tau_D$$
$$\beta = \left(\mu e^{2(\lambda+\rho)T}\right)^{N_0+1}. \quad \square$$

The following corollary follows at once.

Corollary 2. *Suppose that assumptions* A1–A3 *holds and let the minimum distance criterion* (11) *be used. If the average dwell-time* τ_D *is such that*

$$\tau_D > [\log \mu + 2(\lambda + \rho)T]/\lambda, \tag{15}$$

then the closed-loop system $(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)})$ *is exponentially stable for any plant switching signal* $\sigma(\cdot)$. $\square$

Clearly, the right-hand side of (15) represents an upper bound on the minimum plant average-dwell time compatible with the stability of the closed-loop system. As it can be seen from (15), such an upper bound can be reduced by decreasing the switching logic dwell-time T or by making the tuned loops $(\mathcal{P}_i/\mathcal{C}_i)$, $i \in \mathcal{N}$ "more stable" by increasing the convergence rate λ. Notice also that when the constant μ is equal to 1, i.e., $\|w\|$ is a common Lyapunov function for the $\mathcal{N}$ tuned loops $(\mathcal{P}_i/\mathcal{C}_i)$, inequality (15) simplifies to $\tau_D > 2(\lambda + \rho)T/\lambda$. Thus, in this case, it would be theoretically possible to achieve stability for any, arbitrarily small, plant dwell time τ_D by making the controller dwell time T suitably small. It is nevertheless important to point out that, in general, making the controller dwell time too small can lead to unacceptable performance in the presence of noises and/or disturbances. This issue will be discussed in some detail in the next section (see Remark 1).

A final issue is worth mentioning. As well known, under switching the controller realization plays a fundamental role. In fact, depending on the adopted controller realization, a common Lyapunov function for the tuned loops $(\mathcal{P}_i/\mathcal{C}_i)$ may exist or not (Hespanha & Morse, 2002). As pointed out at the end of Section 2, all controller realizations which are minimal are equivalent as far as mode observability is concerned. Further, it is possible to introduce uncontrollable dynamics without compromising mode-observability of the feedback loop. On the contrary, the introduction of unobservable dynamics in the controller makes mode-observability as defined in Section 2 impossible to achieve. This is a relevant problem because the techniques proposed in the literature for constructing realizations which are stable under switching rely on unobservable and uncontrollable dynamics (Blanchini et al., 2009). Interestingly enough, it is possible to show that the presence of unobservable but stable dynamics in the controller does not destroy stability. The proof of this fact is sketched in Appendix B. Similarly, in Appendix B, it is also shown that the requirement of observability of each plant mode $\mathcal{P}_i$ can be relaxed to detectability without compromising stability.

3.3. An example (continued)

In order to illustrate the effectiveness of the proposed approach for adaptive stabilization of switching plants, let us consider again the two-tank system of Section 2.1. For each plant mode a stabilizing PI controller is available of the form (9) with gains $K_{P,1} = -1$, $K_{I,1} = -0.1$ and $K_{P,2} = 1$, $K_{I,2} = 0.1$, respectively. The resulting closed-loop characteristic polynomials are

$$\varphi_{1/1}(s) = \varphi_{2/2}(s) = s^3 + 2\,s^2 + s + 0.1$$
$$\varphi_{1/2}(s) = \varphi_{2/1}(s) = s^3 + 2\,s^2 - s - 0.1.$$

Since $\varphi_{1/1}(s)$ and $\varphi_{1/2}(s)$ are coprime, mode-observability of the feedback system holds (see Proposition 2). Thus, provided that the plant variations are sufficiently slow on the average, Corollary 2 ensures exponential stability when the controller switching is orchestrated according to the minimum distance criterion (11). This was also confirmed by means of numerical simulations. With this respect, Fig. 2 shows the time behavior of the norm of the feedback system state w (top) and of the output y (middle) when the plant mode switches between 2 and 1 every 10 s (the controller dwell-time was set equal to 1 s; the controller state was initialized to $q(0) = 0$ whereas the plant initial state was set to $x(0) = [80 \quad 5]^\top$). As it can be seen from Fig. 2 (bottom), the estimate $\hat{\sigma}$ follows the true plant switching signal σ with a delay equal to the controller dwell time.

4. Stability under persistent disturbances

In this section, the effects of persistent disturbances on the stability of the proposed control scheme are analyzed. To this end, suppose that the plant state and measurement equations be affected by additive disturbances $d(\cdot)$ and $n(\cdot)$, respectively, i.e.,

$$\mathcal{P}_{\sigma(t)} : \begin{cases} \dot{x}(t) = A_{\sigma(t)}\,x(t) + B_{\sigma(t)}\,u(t) + d(t) \\ y(t) = C_{\sigma(t)}\,x(t) + n(t) \end{cases} \tag{16}$$

with $d(t) \in \mathbb{R}^{n_x}$ and $n(t) \in \mathbb{R}^{n_y}$. Then, it is an easy matter to verify that a state space representation of the closed-loop system takes the form

$$(\mathcal{P}_{\sigma(t)}/\mathcal{C}_{\hat{\sigma}(t)}) : \begin{cases} \dot{w}(t) = A^{cl}_{\sigma(t)/\hat{\sigma}(t)}\,w(t) + B^{cl}_{\sigma(t)/\hat{\sigma}(t)}\,v(t) \\ z(t) = C^{cl}_{\sigma(t)/\hat{\sigma}(t)}\,w(t) + D^{cl}_{\sigma(t)/\hat{\sigma}(t)}\,v(t) \end{cases} \tag{17}$$

where $v(t) \triangleq [d(t)^\top \quad n(t)^\top]^\top$ and

$$B^{cl}_{i/j} \triangleq \begin{bmatrix} I & B_i K_j \\ 0 & G_j \end{bmatrix}, \qquad D^{cl}_{i/j} \triangleq \begin{bmatrix} 0 & K_j \\ 0 & I \end{bmatrix}, \quad i, j \in \mathcal{N}.$$

The main complication arising in this case concerns the effects of the disturbances on the quality of the estimate computed via the minimum distance criterion (11). In fact, even supposing that the plant mode $\sigma(t)$ takes a constant value, say σ_k, on a certain interval $\mathcal{I}_k$, the presence of the disturbance $v(t)$ prevents one from applying Lemma 5 given that neither the plant input/output data $z(\cdot)$ need

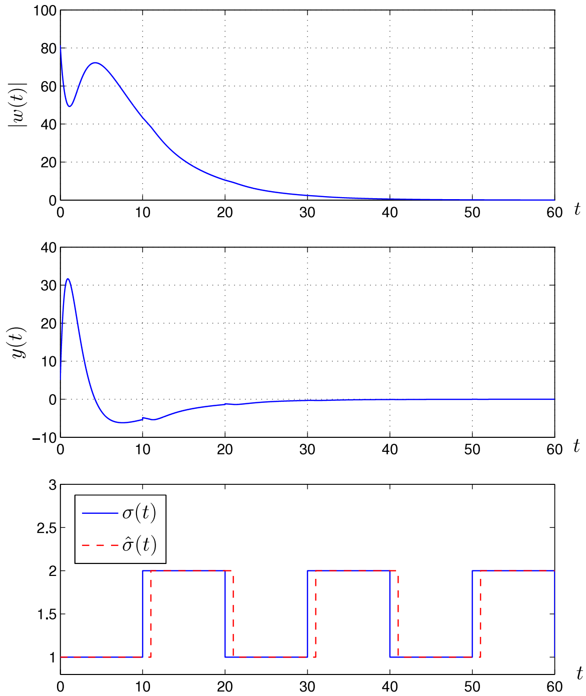

Fig. 2. Time behaviors of $|w(t)|$ (top), $y(t)$ (middle), $\sigma(t)$ and its estimate $\hat{\sigma}(t)$ (bottom).

to belong to $\mathcal{S}_{\sigma_k/\hat{\sigma}_k}(\mathcal{I}_k)$ nor the distance $\delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k)$ needs to take its minimum value for $i = \sigma_k$.

Nevertheless, the mode observability property ensures that an estimate $\hat{\sigma}_{k+1}$ computed as in (11) becomes reliable provided that the input/output data $z(\cdot)$ contain a sufficient level of excitement. To see this, some further considerations concerning mode observability are in order. While Lemma 1 provides an answer on the binary question of whether or not the feedback system (3) is mode observable, a measure of the degree of mode observability of (3) can be obtained by analyzing the *joint observability Gramian*

$$W_{i,i'/j}(t) \triangleq \int_0^t \begin{bmatrix} \left(\Phi^{cl}_{i/j}(\xi)\right)^\top \\ \left(\Phi^{cl}_{i'/j}(\xi)\right)^\top \end{bmatrix} \begin{bmatrix} \Phi^{cl}_{i/j}(\xi) & \Phi^{cl}_{i'/j}(\xi) \end{bmatrix} d\xi. \tag{18}$$

In fact, since

$$\begin{aligned} &\|z_{i/j}(\cdot, t_0, w_0) - z_{i'/j}(\cdot, t_0, w'_0)\|^2_{2,[t_0,t]} \\ &\quad = \begin{bmatrix} w_0^\top & -w_0'^\top \end{bmatrix} W_{i,i'/j}(t-t_0) \begin{bmatrix} w_0 \\ -w'_0 \end{bmatrix} \\ &\quad \geq \lambda_{\min}\{W_{i,i'/j}(t-t_0)\} \left(|w_0|^2 + |w'_0|^2\right), \end{aligned}$$

it can be seen that the greater is $\lambda_{\min}\{W_{i,i'/j}(t-t_0)\}$ the more distinguishable are the two plant modes i and i' under controller $\mathcal{C}_j$. Then, a measure of the degree of mode observability of (3) over an interval of length T is provided by

$$\omega_{\min}(T) = \min_{i,i',j\in\mathcal{N};i\neq i'} \lambda_{\min}\{W_{i,i'/j}(T)\}. \tag{19}$$

As will be clear in the following, such a *mode-observability index* plays a crucial role in the presence of disturbances. We notice that $\omega_{\min}(T)$ tends to zero as the observation interval tends to zero, which is coherent with the fact that the shorter is the observation interval the more difficult it is to distinguish between two different input–output behaviors.

Let us now introduce the following definitions

$$\kappa_A \triangleq \max_{i,j\in\mathcal{N}} |A^{cl}_{i/j}|, \qquad \kappa_B \triangleq \max_{i,j\in\mathcal{N}} |B^{cl}_{i/j}|,$$

$$\kappa_C \triangleq \max_{i,j\in\mathcal{N}} |C^{cl}_{i/j}|, \qquad \kappa_D \triangleq \max_{i,j\in\mathcal{N}} |D^{cl}_{i/j}|.$$

Further, let θ be a positive scalar such that, for any matrix M belonging to $\mathbb{R}^{(n_x+n_q)\times(n_x+n_q)}$, one has $|M| \leq \theta\,\|M\|$ (notice that such a θ does exist since we are dealing with finite-dimensional normed-vector spaces). Then, Lemma 5 can be replaced by the following.

Lemma 6. *Suppose that assumption* A2 *holds and that the plant mode is constant on $\mathcal{I}_k$, i.e.,*

$$\sigma(t) = \sigma_k, \quad \forall t \in \mathcal{I}_k. \tag{20}$$

Then, if the minimum distance criterion (11) is applied to the noisy feedback system (17), one has

$$\hat{\sigma}_{k+1} = \sigma_k$$

provided that

$$|w(kT)| \geq \frac{2\,\psi(T)\,\|v(\cdot)\|_{\infty,\mathcal{I}_k}}{\sqrt{\omega_{\min}(T)}} \tag{21}$$

where $\omega_{\min}(T)$ is the mode-observability index (19) and

$$\psi(T) \triangleq \sqrt{T}\left(\kappa_B\,\kappa_C\,\theta\,\frac{e^{\rho T}-1}{\rho} + \kappa_D\right). \quad \square \tag{22}$$

Thus, under the stated assumptions and provided that the initial state at the beginning of the observation interval $\mathcal{I}_k$ is "far enough" from the origin, the minimum-distance criterion (11) leads to the exact identification of the plant mode even in the presence of disturbances. Further, it can be seen that condition (21) becomes less stringent the smaller are the disturbances and the greater is the mode-observability index. This state of affairs can be understood by recalling that under assumption (20), for any $t \in \mathcal{I}_k$, the input–output data $z(t)$ can be decomposed as

$$z(t) = z^{(n)}(t) + z^{(f)}(t) \tag{23}$$

where $z^{(n)}(t)$ is the natural response

$$z^{(n)}(t) = \Phi^{cl}_{\sigma_k/\hat{\sigma}_k}(t-KT)\,w(kT)$$

and $z^{(f)}(t)$ is the forced response

$$z^{(f)}(t) = \int_{kT}^t \Phi^{cl}_{\sigma_k/\hat{\sigma}_k}(t-\tau)B^{cl}_{\sigma_k/\hat{\sigma}_k}v(\tau)\,d\tau + D^{cl}_{\sigma_k/\hat{\sigma}_k}v(t).$$

As pointed out in the previous sections, when mode observability holds the plant mode can be reconstructed by observing the natural response (in fact, when there are no disturbances, $z(t) = z^{(n)}(t)$). Then the main idea behind Lemma 6 is that, when the forced response due to the disturbances becomes "negligible" with respect to the natural response (in the sense of condition (21)), everything goes as in the noise-free case and the plant mode can be uniquely determined. This situation is depicted in Fig. 3. A formal proof of Lemma 6 is given in the Appendix.

An important consequence of Lemma 6 is that, when the disturbances are bounded in the $\mathcal{L}_\infty$ sense, their effect on the mode estimator disappears as soon as the system state exceeds a certain threshold. In view of this result, it is possible to show that the feedback control system (17) is exponentially input-to-state stable. More specifically, the following theorem can be stated.

Theorem 2. *Suppose that assumptions* A1–A3 *holds and let the minimum distance criterion (11) be used. If the average dwell-time τ_D satisfies inequality (15), then the noisy closed-loop system (17) is exponentially input-to-state stable in that, for any $t_0, t \in \mathbb{R}^+$ with $t \geq t_0$ and for any plant switching signal $\sigma(\cdot)$,*

$$|w(t)| \leq \beta\,\theta\,e^{-\alpha\,(t-t_0)}\,|w(t_0)| + \gamma\,\|v(\cdot)\|_{\infty,[t_0,t]} \tag{24}$$

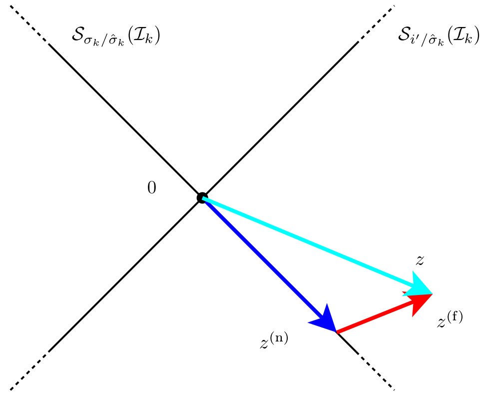

Fig. 3. Decomposition of the input/output data $z(t)$ on the interval $\mathcal{I}_k$. When the plant mode is constant on $\mathcal{I}_k$ and equal to σ_k, then the natural response $z^{(n)}(t)$, $t \in \mathcal{I}_k$ (depicted in blue) belongs to the linear subspace $\mathcal{S}_{\sigma_k/\hat{\sigma}_k}(\mathcal{I}_k)$. Due to the forced response $z^{(f)}(t)$, $t \in \mathcal{I}_k$ (depicted in red), the total response z will in general lie outside such a set. However, when condition (21) holds, the natural response becomes dominant and the plant mode can still be reconstructed by means of the minimum distance criterion. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

with α and β the same as in Theorem 1 and

$$\gamma = \frac{\beta\,\theta}{\alpha}\left[\kappa_B + \frac{4\,\theta\,\psi(T)\,\kappa_A}{\sqrt{\omega_{\min}(T)}}e^{2\,\rho\,T} + \frac{2\,\theta\,\kappa_A\,\kappa_B}{\rho}(e^{2\,\rho\,T}-1)\right]. \quad \square$$

It is worth pointing out that Theorem 2 provides a quite strong stability result in that inequality (24) holds regardless of the magnitude of the disturbances $v(\cdot)$ and of the initial state $w(t_0)$. Some remarks on such a result are in order.

Remark 1. It is an interesting feature of the proposed control scheme that the upper bound on the average-dwell time remains unchanged in the presence of persistent disturbances, so that all the considerations made at the end of Section 3 are still valid. In particular, it can be seen from (24) that the mode observability index $\omega_{\min}(T)$ does not affect the decay rate α and only acts on the disturbance-to-state gain γ. Clearly, in the presence of noises, there is still a trade-off between allowable plant dwell-time and acceptable performance, since making the controller dwell-time T too small can lead to a very high gain γ; thus compromising the performance (albeit not the stability) of the overall control system.

Remark 2. As well known (Sontag & Wang, 1995), input-to-state stability implies also some level of robustness with respect to model uncertainties. In fact, by resorting to standard small-gain arguments, it could be proved that the stability of the proposed control scheme is preserved also when there is some uncertainty on the knowledge of the system matrices (A_i, B_i, C_i), $i \in \mathcal{N}$.

Remark 3. The proposed mode estimation methodology follows a finite memory paradigm in that, in order to estimate the plant mode at time $(k+1)T$, only the data collected in the interval $\mathcal{I}_k$ are used. In the presence of disturbances, the performance of the proposed control scheme could be improved by modifying the mode estimation criterion so as to account also for the previously available data. For example, a possibility consists in computing some estimate $\bar{w}(kT)$ of the feedback system state $w(kT)$ on the basis of the data collected up to time kT and, then, considering the alternative distance measure

$$\min_{\hat{w}\in\Re^{n_x+n_q}}\left\{\omega\,|\hat{w}-\bar{w}(kT)|^2 + (1-\omega)\,\left\|z(\cdot)-z_{i/j}(\cdot,kT,\hat{w})\right\|^2_{2,\mathcal{I}_k}\right\}$$

where $\omega \in (0,1)$ is some suitable weight. In fact, it is natural to expect that, when the estimate $\bar{w}(kT)$ is close to the true state, the adoption of such a distance measure not only would not destroy stability but also would lead to a more reliable mode estimate. Guidelines on how to estimate the state of a switching linear system with guaranteed bound on the estimation error can be found, for example, in Alessandri, Baglietto, and Battistelli (2005, 2010).

Remark 4. As established by Theorem 2 the considered mode estimation technique, which is based on computing the distances of the plant input/output data from the linear subspaces associated with the possible plant modes, turns out to be quite effective in the considered setting. However, several other techniques for mode inference can be conceived, for example, based on prediction errors (Hespanha, Liberzon, & Morse, 2003) or on virtual experiments (Baldi, Battistelli, Mosca, & Tesi, 2010). With this respect, a interesting, yet unanswered, question is whether, under the stated assumptions, similar stability results hold also for other mode estimation approaches.

4.1. An example (continued)

Let us consider again the switching system of Section 3.3 and suppose that both the system and measurement equations are affected by persistent disturbances d and n. More specifically, let d and n be white noises with zero mean and covariance matrices diag(0.1, 0.1) and 0.1, respectively. The plant state was initialized to $x(0) = [20 \quad -90]^\top$. All the other simulation parameters are the same as in Section 3.3. As it can be seen from Fig. 4 (top) where the time behavior of the norm of the feedback system state w for a random simulation is shown, despite the facts that the plant switching signal σ is unknown and that the data are corrupted by noises, the state w is exponentially driven to a neighborhood of the origin. Some insights on this behavior can be gained from observing the middle and bottom plots of Fig. 4. In particular, it can be seen that initially the estimate $\hat{\sigma}$ follows the true switching signal σ with an unavoidable delay equal to the controller dwell time. In fact, in the first part of the simulation run, the state is far from the origin and therefore the natural response dominates the forced one. Afterwards, some errors in the estimation of the plant mode occur due to the presence of the disturbances. However, stability is not destroyed since such errors occur only when the plant output and state are in a neighborhood of the origin (recall Lemma 6).

5. Conclusions

The problem of stabilizing a switching linear plant has been addressed under the assumption that the plant switching signal is not available, nor in real-time neither with delay. The proposed methodology is based on a supervisory unit that periodically switches the controller operating mode. The controller switching signal is generated by resorting to a minimum distance criterion, for the estimation of the plant mode, that naturally arises from mode observability considerations. It has been shown that, even in the presence of persistent disturbances, the proposed control scheme yields a stable feedback system provided that the plant switching signal is sufficiently slow on the average.

Acknowledgments

The author would like to thank Pietro Tesi and Marco Baglietto for their precious help and valuable comments on the paper.

Appendix A. Proofs

Proof of Lemma 2. Given the LPMFDs (5) and (6), then a polynomial matrix description (PMD) (Antsaklis & Michel, 2006) of the

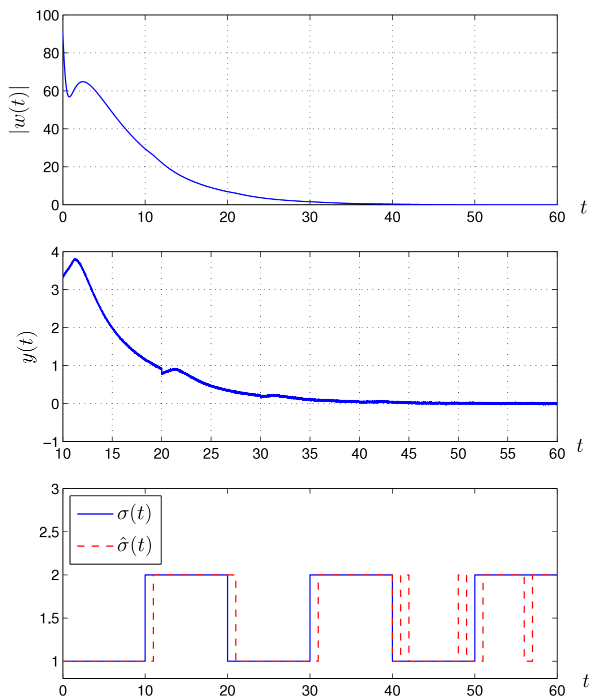

Fig. 4. Time behaviors of $|w(t)|$ (top), $y(t)$ (middle), $\sigma(t)$ and its estimate $\hat{\sigma}(t)$ (bottom) in the presence of disturbances.

feedback system $(\mathscr{P}_i/\mathscr{C}_j)$ is

$$\Gamma_{i/j}(D)\,z(t) = 0 \tag{25}$$

where

$$\Gamma_{i/j}(s) \triangleq \begin{bmatrix} -Q_i(s) & P_i(s) \\ R_j(s) & -S_j(s) \end{bmatrix}.$$

Note now that the joint observability matrix

$$\left[O_{i/j}^{(2n_x+2n_q)} \; O_{i'/j}^{(2n_x+2n_q)} \right]$$

coincides with the observability matrix of the $2n_x + 2n_q$-dimensional system

$$\begin{aligned} \dot{\chi}(t) &= \begin{bmatrix} A_{i/j}^{cl} & 0 \\ 0 & A_{i'/j}^{cl} \end{bmatrix} \chi(t) \\ \zeta(t) &= \begin{bmatrix} C_{i/j}^{cl} & C_{i'/j}^{cl} \end{bmatrix} \chi(t) \end{aligned} \tag{26}$$

obtained from the parallel connection of $(\mathscr{P}_i/\mathscr{C}_j)$ with $(\mathscr{P}_{i'}/\mathscr{C}_j)$. Then, the joint observability condition (4) is algebraically equivalent to the observability of the state $\chi(t)$ of system (26) from its output $\zeta(t)$.

In view of (25), system (26) admits the PMD

$$\begin{bmatrix} \Gamma_{i/j}(D) & 0 \\ 0 & \Gamma_{i'/j}(D) \end{bmatrix} \xi(t) = 0$$
$$\zeta(t) = \begin{bmatrix} I & I \end{bmatrix} \xi(t).$$

Then, system (26) is completely observable, i.e., the joint observability condition (4) holds, if and only if

$$\mathrm{rank} \begin{bmatrix} \Gamma_{i/j}(s) & 0 \\ 0 & \Gamma_{i'/j}(s) \\ I & I \end{bmatrix} = \dim(\xi(t)) = 2n_u + 2n_y, \tag{27}$$

for any $s \in \mathbb{C}$. By standard manipulation, condition (27) turns out to be equivalent to

$$\mathrm{rank} \begin{bmatrix} \Gamma_{i/j}(s) \\ \Gamma_{i'/j}(s) \end{bmatrix} = n_u + n_y, \quad \forall s \in \mathbb{C}.$$

Then the proof is concluded by noting that the latter can be rewritten as in (7). □

Proof of Proposition 1. This is adapted from a classic result in linear system theory. It is a direct consequence of the fact that for any $s \in \mathbb{C}$ such that $\varphi_{i/j}(s) \neq 0$

$$\mathrm{rank} \begin{bmatrix} P_i(s) & -Q_i(s) \\ -S_j(s) & R_j(s) \end{bmatrix} = n_u + n_y. \tag{28}$$

Then, when $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime, for any $s \in \mathbb{C}$ either (28) or

$$\mathrm{rank} \begin{bmatrix} P_{i'}(s) & -Q_{i'}(s) \\ -S_j(s) & R_j(s) \end{bmatrix} = n_u + n_y$$

holds. □

Proof of Proposition 2. (Only if) Let s_0 be a common root of $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$. By hypothesis, $R_j(s)$ and $S_j(s)$ are coprime for any $j \in \mathscr{N}$. Then, there must exist two scalars $\nu_i, \nu_{i'} \in \mathbb{C}$ such that

$$\begin{bmatrix} P_i(s_0) & -Q_i(s_0) \end{bmatrix} = \nu_i \begin{bmatrix} -S_j(s_0) & R_j(s_0) \end{bmatrix}$$
$$\begin{bmatrix} P_{i'}(s_0) & -Q_{i'}(s_0) \end{bmatrix} = \nu_{i'} \begin{bmatrix} -S_j(s_0) & R_j(s_0) \end{bmatrix}.$$

This in turn implies that

$$\begin{bmatrix} P_i(s_0) & -Q_i(s_0) \\ P_{i'}(s_0) & -Q_{i'}(s_0) \\ -S_j(s_0) & R_j(s_0) \end{bmatrix} = \begin{bmatrix} \nu_i \\ \nu_{i'} \\ 1 \end{bmatrix} \begin{bmatrix} -S_j(s_0) & R_j(s_0) \end{bmatrix}$$

whose rank is always $1 < n_u + n_y = 2$. □

Proof of Lemma 3. The necessity of (a) has been discussed after Lemma 1. The necessity of (b) is obvious from the fact that, when the two transfer functions are equal, the two modes $\mathscr{P}_i$ and $\mathscr{P}_{i'}$ always exhibit the same response for zero initial conditions. Finally, the necessity of (c) stems from the observation that, for a common uncontrollable eigenvalue s_0, one would have $P_i(s_0) = Q_i(s_0) = P_{i'}(s_0) = Q_{i'}(s_0) = 0$ which would clearly imply that the rank condition (7) cannot hold.

Consider now the problem of proving the genericity of the set of controllers ensuring distinguishability when conditions (a)–(c) hold. Recalling that controllability and observability are generic properties, then without loss of generality we can restrict our attention to controllable and observable controllers and invoke Proposition 2. Accordingly, we are interested in characterizing the set of controllers $\mathscr{C}_j$ for which the closed loop polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime. To this end, recall that two polynomials have a common root if only if their Sylvester resultant (i.e., the determinant of the Sylvester matrix associated with the two polynomials) is 0. Notice now that the entries of the Sylvester matrix depend on the coefficients of the two polynomials which, in turn, are polynomial functions of the entries of the matrices (F_j, G_j, H_j, K_j). Hence those (controllable and observable) controllers for which distinguishability does not hold are given by a polynomial equation in the elements of (F_j, G_j, H_j, K_j) and, thus, define an algebraic set. Since the complement of a proper algebraic set is always a generic set, then the proof can be concluded simply by showing that, for any given order n_q there exists at least one controller for which the closed-loop polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime. To see this, let $\tilde{S}_j(s)$ and $\tilde{R}_j(s)$ be any two coprime polynomials of degree n_q and consider a controller of the form $C_j(s) = \kappa\,\tilde{S}_j(s)/\tilde{R}_j(s)$ with $\kappa \in \mathbb{R}$. Then the two closed-loop polynomials take the form

$$\varphi_{i/j}(s) = P_i(s)\tilde{R}_j(s) + \kappa\, Q_i(s)\tilde{S}_j(s),$$
$$\varphi_{i'/j}(s) = P_{i'}(s)\tilde{R}_j(s) + \kappa\, Q_{i'}(s)\tilde{S}_j(s).$$

Under conditions (a) and (c), we have that the greatest common divisor of the four polynomials $P_i(s)\tilde{R}_j(s)$, $Q_i(s)\tilde{S}_j(s)$, $P_{i'}(s)\tilde{R}_j(s)$, $Q_{i'}(s)\tilde{S}_j(s)$ is 1. Further, condition (b) ensures that $P_i(s)\tilde{R}_j(s)\, Q_{i'}(s)\, \tilde{S}_j(s) \neq P_{i'}(s)\tilde{R}_j(s)Q_i(s)\tilde{S}_j(s)$. Then, we can invoke Lemma 3 of Vidyasagar, Levy, and Viswanadham (1986) and conclude that $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime for all but a finite number of values of $\kappa \in \mathbb{R}$. □

Proof of Lemma 4. Like in the proof of Lemma 3, we restrict our attention to observable controllers and focus on the sufficient condition of Proposition 1. Again, since the set of (observable) controllers for which the closed loop polynomials $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime define an algebraic set, its genericity can be proved simply by showing that one such controller exists. To this end, the basic idea is that of transforming the original MIMO plant into a SISO one that still satisfies conditions (a)–(c). This can be done by considering a novel scalar input $\tilde{u}$ and a novel scalar output $\tilde{y}$ related to the original input/output pair (u, y) by the equations

$$u = \tilde{K} y + b\, \tilde{u}$$
$$\tilde{y} = c^\top y \tag{29}$$

where b, c, and $\tilde{K}$ are constant vectors and, respectively, matrix of appropriate size. If we consider now the fictitious SISO plant $\tilde{\mathcal{P}}_{\sigma(t)}$ with input $\tilde{u}$ and output $\tilde{y}$, the state-space descriptions of the two modes $\mathcal{P}_i$ and $\mathcal{P}_{i'}$ are characterized by the triples $(A_i + B_i \tilde{K} C_i, B_i\, b, c^\top C_i)$ and $(A_{i'} + B_{i'} \tilde{K} C_{i'}, B_{i'}\, b, c^\top C_{i'})$, respectively. By invoking Theorem 4 of Davison and Wang (1973), we can claim that generically, i.e., for almost every choice of the triple $(b, c, \tilde{K})$, the two SISO systems $\tilde{\mathcal{P}}_i$ and $\tilde{\mathcal{P}}_{i'}$ will have the same controllability and observability properties of the two original MIMO systems P_i and, respectively, $P_{i'}$. Hence, under the stated hypotheses, conditions (a) and (c) generically hold when considering $\tilde{\mathcal{P}}_i$ and $\tilde{\mathcal{P}}_{i'}$. The same is true also for condition (b) since, when the two MIMO transfer functions $C_i(sI - A_i)^{-1}B_i$ and $C_{i'}(sI - A_{i'})^{-1}B_{i'}$ are different, then the set of triples $(b, c, \tilde{K})$ for which the two SISO transfer functions $c^\top C_i(sI - A_i - B_i \tilde{K} C_i)^{-1}B_i\, b$ and $c^\top C_{i'}(sI - A_{i'} - B_{i'} \tilde{K} C_{i'})^{-1}B_{i'}\, b$ are different is a proper algebraic set (as it can be easily verified by noting that the coefficients of such transfer functions are polynomial functions of the triple $(b, c, \tilde{K})$). As shown in the proof of Lemma 3, we can thus conclude that there exists a SISO controller, say $(\tilde{F}, \tilde{g}, \tilde{h}^\top, \tilde{k})$, that when applied to the two SISO systems $\tilde{\mathcal{P}}_i$ and $\tilde{\mathcal{P}}_{i'}$ yield two coprime closed-loop polynomials. Recalling (29), this in turn implies that the controller (F_j, G_j, H_j, K_j) with the choice

$$F_j = \tilde{F}, \qquad G_j = \tilde{g}\, c^\top, \qquad H_j = b\, \tilde{h}^\top, \qquad K_j = b\, \tilde{k}\, c^\top + \tilde{K}$$

is such that $\varphi_{i/j}(s)$ and $\varphi_{i'/j}(s)$ are coprime, which concludes the proof. □

Proof of Lemma 5. When $\sigma(t)$ is constant in the interval $\mathcal{I}_k$, one has

$$z(t) = \Phi^{cl}_{\sigma_k/\hat{\sigma}_k}(t - kT)\, w(kT) \quad t \in \mathcal{I}_k$$

and, consequently,

$$\delta_{\sigma_k/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k) = 0.$$

On the other hand, in view of Proposition 3, when $w(kT) \neq 0$ one has that, for any $i \neq \sigma_k$, the input/output data $z(\cdot)$ on the interval $\mathcal{I}_k$ do not belong to the set $\mathcal{S}_{i/\hat{\sigma}_k}(\mathcal{I}_k)$ so that

$$\delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k) > 0$$

which concludes the proof. □

Proof of Theorem 1. Consider a generic interval $[t_0, t]$ and let t_h be the h-th discontinuity of σ in such an interval. Further, let us denote by $\sigma_h \in \mathcal{N}$ the value taken on by σ on the interval $[t_h, t_{h+1})$. Then, the closed-loop state transition matrix $\Phi(t, t_0)$ can be decomposed as

$$\Phi(t, t_0) = \prod_{h=0}^{N_\sigma(t_0,t)} \Phi(t_{h+1}, t_h) \tag{30}$$

where, for the sake of compactness, $t_{N_\sigma(t_0,t)+1} \triangleq t$. Notice now that each interval $[t_h, t_{h+1})$ with $t_{h+1} - t_h \geq 2T$ can be further decomposed into three subintervals $[t_h, k_hT)$, $[k_hT, (k_h+1)T)$, and $[(k_h + 1)T, t_{h+1})$ where k_h is the least integer such that $k_hT > t_h$. Further, by virtue Lemma 5, one has that $\hat{\sigma}(t) = \sigma_h$ for any $t \in [(k_h + 1)T, t_{h+1})$. Then, taking into account (12) and (13), one obtains the upper bound

$$\begin{aligned}\|\Phi(t_{h+1}, t_h)\| &\leq \|\Phi(t_h, k_hT)\|\, \|\Phi(k_hT, (k_h + 1)T)\| \\ &\quad \times \|\Phi((k_h + 1)T, t_{h+1})\| \\ &\leq e^{2T\rho}\, \mu\, e^{-\lambda(t_{h+1}-t_h-2T)}\end{aligned} \tag{31}$$

where the latter inequality can be derived by noting that $k_hT - t_h \leq T$ and $t_{h+1} - (k_h + 1)T \geq t_{h+1} - t_h - 2T$. It is an easy matter to see that (31) holds also when $t_{h+1} - t_h < 2T$. As a consequence, (31) together with (30) yields the upper bound

$$\begin{aligned}\|\Phi(t, t_0)\| &\leq \prod_{h=0}^{N_\sigma(t_0,t)} \|\Phi(t_{h+1}, t_h)\| \\ &\leq \prod_{h=0}^{N_\sigma(t_0,t)} \left(e^{2T\rho}\, \mu\, e^{-\lambda(t_{h+1}-t_h-2T)}\right) \\ &= \left(\mu\, e^{2T(\rho+\lambda)}\right)^{N_\sigma(t_0,t)+1} e^{-\lambda \sum_{h=0}^{N_\sigma(t_0,t)} (t_{h+1}-t_h)} \\ &= \left(\mu\, e^{2T(\rho+\lambda)}\right)^{N_\sigma(t_0,t)+1} e^{-\lambda(t-t_0)}.\end{aligned}$$

The proof is concluded by noting that, under assumption A3, the latter implies

$$\|\Phi(t, t_0)\|\, \left(\mu\, e^{2T(\rho+\lambda)}\right)^{N_0+1} \left(\mu\, e^{2T(\rho+\lambda)}\right)^{(t-t_0)/\tau} e^{-\lambda(t-t_0)}$$

which can be rewritten as (14). □

Proof of Lemma 6. By exploiting the decomposition (23) and invoking the triangular inequality, it is immediate to verify that, for any $i \in \mathcal{N}$,

$$\begin{aligned}&\delta_{i/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) - \|z^{(f)}(\cdot)\|_{2,\mathcal{I}_k} \leq \delta_{i/\hat{\sigma}_k}(z(\cdot), \mathcal{I}_k) \\ &\quad \leq \delta_{i/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) + \|z^{(f)}(\cdot)\|_{2,\mathcal{I}_k}.\end{aligned}$$

Then, a sufficient condition for $\hat{\sigma}_{k+1}$ to be equal to σ_k is that

$$\begin{aligned}&\delta_{i/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) - \|z^{(f)}(\cdot)\|_{2,\mathcal{I}_k} \\ &\quad \geq \delta_{\sigma_k/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) + \|z^{(f)}(\cdot)\|_{2,\mathcal{I}_k}\end{aligned} \tag{32}$$

for any $i \neq \sigma_k$.

Proceeding as in the proof of Lemma 5, we easily find that

$$\delta_{\sigma_k/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) = 0.$$

As for the distance $\delta_{i/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k)$, one can write

$$\begin{aligned}\delta^2_{i/\hat{\sigma}_k}(z^{(n)}(\cdot), \mathcal{I}_k) &= \int_{\mathcal{I}_k} \Big| \Phi^{cl}_{\sigma_k/\hat{\sigma}_k}(t - kT)\, w(kT) \\ &\quad - \Phi^{cl}_{i/\hat{\sigma}_k}(t - kT)\, \Psi_{i,\sigma_k/\hat{\sigma}_k}\, w(kT)\Big|^2 dt\end{aligned} \tag{33}$$

with

$$\Psi_{i,i'/j} = \left(W_{i/j}(kT)\right)^{-1} \int_{\mathcal{I}_k} \left(\Phi^{cl}_{i/j}(\xi - kT)\right)^\top \Phi^{cl}_{i'/j}(\xi - kT)\, d\xi.$$

Further, recalling (18), one can write

$$\delta^2_{i/\hat{\sigma}_k}(z^{(n)}(\cdot),\mathcal{I}_k) = \begin{bmatrix} w(kT) \\ -\Psi_{i,\sigma_k/\hat{\sigma}_k}\, w(kT)\end{bmatrix}^\top \times W_{\sigma_k,i/\sigma_k}(t-t_0)\begin{bmatrix} w(kT) \\ -\Psi_{i,\sigma_k/\hat{\sigma}_k}\, w(kT)\end{bmatrix}.$$

This, in turn, implies that for $i \neq \sigma_k$

$$\delta^2_{i/\hat{\sigma}_k}(z^{(n)}(\cdot),\mathcal{I}_k) \geq \omega_{\min}(T)\,|w(kT)|^2.$$

Note that, under mode observability, the index $\omega_{\min}(T)$ is always strictly positive for any $T > 0$.

Moreover, inequality (13) implies that, for any $t \in \mathcal{I}_k$,

$$|z^{(f)}(t)| \leq \kappa_C\,\kappa_B\,\theta \int_{KT}^{t} e^{\rho(t-\tau)}\,|v(\tau)|\,d\tau + \kappa_D\,|v(t)|$$

which, in turn, yields

$$\|z^{(f)}(\cdot)\|_{2,\mathcal{I}_k} \leq \psi(T)\|v(\cdot)\|_{\infty,\mathcal{I}_k}$$

with $\psi(T)$ defined as in (22). As a consequence, a sufficient condition for (32) to hold is that

$$\sqrt{\omega_{\min}(T)}\,|w(kT)| - \psi(T)\|v(\cdot)\|_{\infty,\mathcal{I}_k} \geq \psi(T)\|v(\cdot)\|_{\infty,\mathcal{I}_k}$$

which can be rewritten as (21). $\square$

Proof of Theorem 2. Let t_h, σ_h, and k_h be defined as in the proof of Theorem 1. In each interval $[t_h,(k_h+1)T)$, the estimate of the system mode is not reliable and the closed-loop dynamics takes the form

$$\dot{w}(t) = A^{cl}_{\sigma_h/\hat{\sigma}(t)}\,w(t) + \tilde{v}(t) \tag{34}$$

with $\tilde{v}(t) = B^{cl}_{\sigma_h/\hat{\sigma}(t)}\,v(t)$ and $A^{cl}_{\sigma_h/\hat{\sigma}(t)}$ possibly unstable due to the fact that $\hat{\sigma}(t)$ is in general different from σ_h. On the other hand, for $t \in [(k_h+1)T, t_{h+1})$, the estimate $\hat{\sigma}(t)$ becomes reliable in the sense of Lemma 6. In order to exploit this property, it is convenient to rewrite the closed loop dynamics in the interval $[(k_h+1)T, t_{h+1})$ as

$$\dot{w}(t) = A^{cl}_{\sigma_h/\sigma_h}\,w(t) + \tilde{v}(t) \tag{35}$$

where the signal

$$\tilde{v}(t) = \left(A^{cl}_{\sigma_h/\hat{\sigma}(t)} - A^{cl}_{\sigma_h/\sigma_h}\right)\,w(t) + B^{cl}_{\sigma_h/\hat{\sigma}(t)}\,v(t)$$

accounts for both the exogenous disturbances and the possible mismatch between $\hat{\sigma}(t)$ and σ_h.

Notice now that the two Eqs. (34) and (35) describe a switched system with exogenous input $\tilde{v}(t)$ and state transition matrix $\tilde{\Phi}(\tau,t_0)$ that, for any $\tau \in [t_0,t)$, can be upper bounded as

$$\|\tilde{\Phi}(\tau,t_0)\| \leq \beta\,e^{-\alpha\,(\tau-t_0)}$$

with α and β the same as in Theorem 1 (this can be shown along the lines of the proof of Theorem 1).

Given that $w(t)$ can be written as

$$w(t) = \tilde{\Phi}(t,t_0)w(t_0) + \int_{t_0}^{t}\tilde{\Phi}(t,\tau)\tilde{v}(\tau)d\tau,$$

when inequality (15) holds it is immediate to derive the upper bound

$$|w(t)| \leq \beta\,\theta\,e^{-\alpha\,(\tau-t_0)}|w(t_0)| + \frac{\beta\,\theta}{\alpha}\,\|\tilde{v}(\cdot)\|_{\infty,[t_0,t]}.$$

Then, in order to complete the proof, it is sufficient to derive an upper bound on $\|\tilde{v}(\cdot)\|_{\infty,[t_0,t]}$.

To this end, notice that $\tilde{v}(t)$ can differ from $B^{cl}_{\sigma_h/\hat{\sigma}(t)}v(t)$ only in all those intervals $\mathcal{I}_k$ with $k = k_h+1,\ldots,k_{h+1}$ for which $\hat{\sigma}_k \neq \sigma_h$. However, in view of Lemma 6, this can happen only when

$$|w((k-1)T)| \leq \frac{2\,\psi(T)\,\|v(\cdot)\|_{\infty,[t_0,t]}}{\sqrt{\omega_{\min}(T)}}. \tag{36}$$

Thus, in such intervals, $\hat{\sigma}_k \neq \sigma_h$ implies that the state $w((k-1)T)$ can be upper bounded as in (36) and, therefore, that in this case for the state $w(t)$, $t \in \mathcal{I}_k$ the following inequality holds

$$\begin{aligned}|w(t)| &\leq \theta e^{2\,\rho\,T}|w((k-1)T)| + \frac{\theta\,\kappa_B}{\rho}\,(e^{2\,\rho\,T}-1)\,\|v(\cdot)\|_{\infty,[t_0,t]} \\ &\leq \left(\frac{2\,\theta\,\psi(T)}{\sqrt{\omega_{\min}(T)}}e^{2\,\rho\,T} + \frac{\theta\,\kappa_B}{\rho}\,(e^{2\,\rho\,T}-1)\right)\|v(\cdot)\|_{\infty,[t_0,t]}.\end{aligned}$$

This, in turn, implies

$$\|\tilde{v}(\cdot)\|_{\infty,[t_0,t]} \leq \left(\kappa_B + \frac{4\,\theta\,\psi(T)\,\kappa_A}{\sqrt{\omega_{\min}(T)}}e^{2\,\rho\,T} + \frac{2\,\theta\,\kappa_A\,\kappa_B}{\rho}\,(e^{2\,\rho\,T}-1)\right)\|v(\cdot)\|_{\infty,[t_0,t]}. \quad \square$$

Appendix B. Relaxations

In this section, it is discussed how the observability requirement on both plant and controller can be relaxed. Notice that the stability analysis for detectable plants can be carried over to the noisy-case with straightforward reasonings. On the other hand, the effect of controller unobservable dynamics in the presence of disturbances deserves further investigations. In order to avoid cumbersome notations and without loss of generality, the two relaxations are treated separately.

Stability for detectable plants. Let us first address the case where some of the plant modes $\mathcal{P}_i$ are not completely observable but detectable. Then, as well known, through a suitable change of variables it is possible to decompose the dynamics of $\mathcal{P}_i$ with respect to its observability from y by considering a nonsingular matrix T_i such that

$$T_i^{-1}A_iT^i = \begin{bmatrix} A_{i,o} & 0 \\ A_{i,m} & A_{i,\bar{o}}\end{bmatrix}, \qquad C_iT_i = \begin{bmatrix} C_{i,o}\,0\end{bmatrix},$$

$$(T^i)^{-1}B_i = \begin{bmatrix} B_{i,o} \\ B_{i,\bar{o}}\end{bmatrix}$$

with $(A_{i,o},C_{i,o})$ completely observable and $A_{i,\bar{o}}$ stable. In the novel coordinates, the state vector can be written as $T_i^{-1}x = \text{col}\left(x_{i,o},x_{i,\bar{o}}\right)$ where $x_{i,o}$ is the state of the observable subsystem

$$\dot{x}_{i,o} = A_{i,o}\,x_{i,o} + B_{i,o}\,u \tag{37}$$

$$y = C_{i,o}\,x_{i,o}. \tag{38}$$

Notice that, in this case, the input–output representation (5) accounts only for the observable subsystem (37)–(38). As already pointed out, since the plant is not completely observable, it is not possible to achieve mode observability of the feedback system. However, in place of assumption A2, one can consider the following weaker assumption.

A2′. The controllers are designed so that the rank condition (7) of Lemma 2 is satisfied for any pair $i,i' \in \mathcal{N}$ and any $j \in \mathcal{N}$.

This amounts to assuming that distinguishability of two plant modes $i,i' \in \mathcal{N}$ holds at least for the observable part of the feedback system.

Now, exploiting the above-defined change of variables, it is possible to redefine the feedback system (3) as

$$w(t) \triangleq \begin{bmatrix} x_{i,\bar{o}}(t) \\ x_{i,o}(t) \\ q(t)\end{bmatrix},$$

$$A^{cl}_{i/j} \triangleq \begin{bmatrix} A_{i,\bar{o}} & A_{i,m} & B_{i,\bar{o}}H_j \\ 0 & A_{i,o}+B_{i,o}K_jC_{i,o} & B_{i,o}H_j \\ 0 & G_jC_{i,o} & F_j \end{bmatrix},$$

$$C^{cl}_{i/j} \triangleq \begin{bmatrix} 0 & K_jC_{i,o} & H_j \\ 0 & C_{i,o} & 0 \end{bmatrix}, \quad i,j \in \mathcal{N}.$$

While Lemma 5 is no longer valid, it is not difficult to show that, under assumption A2′, when the plant mode is constant on $\mathcal{I}_k$, i.e.,

$$\sigma(t) = \sigma_k, \quad \forall t \in \mathcal{I}_k$$

the minimum distance criterion (11) leads to a correct estimation of the plant mode, i.e., $\hat{\sigma}_{k+1} = \sigma_k$, provided that the observable part of the state $\mathrm{col}\left(x_{\sigma_k,o}(kT), q(kT)\right)$ is different from 0. Then, in such intervals, an error in the estimation of the plant mode can occur only when the state at time kT takes the form $w(kT) = \mathrm{col}\left(x_{\sigma_k,\bar{o}}(kT), 0, 0\right)$ for some $x_{\sigma_k,\bar{o}}(kT)$. Supposing now that the plant mode is equal to σ_k also in the interval $\mathcal{I}_{k+1}$, it turns out that $\hat{\sigma}_{k+1} \neq \sigma_k$ implies that $w(t) = \mathrm{col}\left(x_{\sigma_k,\bar{o}}(t), 0, 0\right), t \in \mathcal{I}_k$. Since, for such states, we have

$$A^{cl}_{\sigma_k/j} w(t) = \mathrm{col}\left(A_{i,\bar{o}}, x_{\sigma_k,\bar{o}}(t), 0, 0\right), \quad \forall j \in \mathcal{N},$$

we can conclude that, irrespective of which controller $\mathcal{C}_j$ is inserted in the loop, the state trajectory on $\mathcal{I}_{k+1}$ would be the very same as if the correct controller $\mathcal{C}_{\sigma_k}$ were active. This, in turn, implies that the stability result of Corollary 2 can still be derived.

Stability for detectable controllers. Consider now the case where some of the controller modes $\mathcal{C}_j$ are not completely observable but detectable. Again, one can decompose the dynamics of $\mathcal{C}_j$ with respect to its observability from u through a suitable nonsingular matrix T_j, so that the controller state can be written as $T_j^{-1}q = \mathrm{col}\left(q_{j,o}, q_{j,\bar{o}}\right)$ where $q_{j,o}$ is the state of the observable subsystem. As before, we consider an input–output representation (6) accounting for the observable dynamics of the controller and suppose that assumption A2′ holds in place of A2.

Then, it is not difficult to show that, under assumption A2′, when the plant mode is constant on $\mathcal{I}_k$, i.e.,

$$\sigma(t) = \sigma_h, \quad \forall t \in \mathcal{I}_k$$

the minimum distance criterion (11) leads to a correct estimation of the plant mode, i.e., $\hat{\sigma}_{k+1} = \sigma_h$, provided that the observable part of the state $\mathrm{col}\left(x(kT), q_{\hat{\sigma}_k,o}(kT)\right)$ is different from 0. Then, in such intervals, an error in the estimation of the plant mode can occur only when $x(kT) = 0$ and $q_{\hat{\sigma}_k,o}(kT) = 0$. This, in turn, implies that in such intervals the feedback loop $(\mathcal{P}_{\sigma_h}/\mathcal{C}_{\hat{\sigma}_k})$ behaves like $(\mathcal{P}_{\hat{\sigma}_k}/\mathcal{C}_{\hat{\sigma}_k})$ and, thus, exhibits an exponential decrease.

Suppose now that the plant mode is constant and equal to σ_h over an interval $[t_h, t_H)$ and let $t_h - t_H \geq 2T$ (as discussed in the proof of Theorem 1 this is the only relevant case for stability). Further, let k_h denote the least integer such that $k_hT \geq t_h$. Clearly, at time k_hT there are two possibilities: (i) $x(k_hT)$ and $q_{\hat{\sigma}_{k_h},o}(k_hT)$ are both null; (ii) at least one between $x(k_hT)$ and $q_{\hat{\sigma}_{k_h},o}(k_hT)$ is different from 0. In the latter case, the correct controller $\mathcal{C}_{\sigma_h}$ is switched on at time $(k_h+1)T$ and never switched off until k_HT. Then everything goes as in the proof of Theorem 1. In case (i), in view of the foregoing considerations and of the adopted switching logic, we have that $\mathcal{C}_{\hat{\sigma}_{k_h}}$ is never switched off until k_HT, with $(\mathcal{P}_{\sigma_h}/\mathcal{C}_{\hat{\sigma}_{k_h}})$ behaving like $(\mathcal{P}_{\hat{\sigma}_{k_h}}/\mathcal{C}_{\hat{\sigma}_{k_h}})$ so that

$$\|\Phi(t_H, k_hT)w(k_hT)\| \leq \mu\, e^{-\lambda(t_H - k_hT)}\|w(k_hT)\|.$$

The latter inequality implies that the stability result of Corollary 2 can still be derived.

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Giorgio Battistelli received the Laurea degree in electronic engineering and the Ph.D. degree in robotics, both from the University of Genoa, in 2000 and 2004, respectively. From 2004 to 2006 he was a research associate at the Dipartimento di Informatica, Sistemistica e Telematica of the University of Genoa. He is currently an Assistant Professor of Automatic Control at the Dipartimento di Ingegneria dell'Informazione of the University of Florence. His research interests include adaptive switching control, linear and nonlinear estimation, hybrid systems, sensor networks and data fusion, neural networks and learning applications in automatic control. Dr. Battistelli is currently an editor of the IFAC Journal Engineering Applications of Artificial Intelligence, an associate editor of the IEEE Transactions on Neural Networks, and a member of the IEEE Control Systems Society Conference Editorial Board.