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A note on establishing convergence in adaptive systems[☆]

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ABSTRACT

We present a new formulation of a convergence result for Lyapunov function candidates satisfying a differential inequality with integrable coefficients that often appears in adaptive control problems. Usually, Barbalat's Lemma is invoked, requiring boundedness of the time derivative of the Lyapunov function candidate which can sometimes be hard to establish. By connecting results from the literature, an alternative route avoiding Barbalat's Lemma is suggested.

1. Introduction

Consider the adaptive control problem of regulating the scalar state x of the system

$$\dot{x} = ax + u \tag{1}$$

to zero, where a is an unknown constant and u is the control input. Following a standard identifier-based approach to design u, we select the identifier

$$\dot{\hat{x}} = -\gamma_0(\hat{x} - x) + \hat{a}x + u + k_0(x - \hat{x})x^2 \tag{2}$$

where γ_0 and k_0 are positive design gains. The error $e = x - \hat{x}$ satisfies

$$\dot{e} = -\gamma_0 e + \tilde{a}x - k_0 e x^2 \tag{3}$$

where the parameter estimation error $\tilde{a} = a - \hat{a}$ has been defined. Consider the Lyapunov function candidate V_1, defined as

$$V_1 = \frac{1}{2}e^2 + \frac{1}{2\gamma_1}\tilde{a}^2 \tag{4}$$

for some design scalar $\gamma_1 > 0$. Differentiating (4) with respect to time and inserting the dynamics (3), we obtain

$$\dot{V}_1 = -\gamma_0 e^2 - k_0 e^2 x^2 \tag{5}$$

where we have chosen the adaptive law

$$\dot{\hat{a}} = \gamma_1 e x. \tag{6}$$

From (5) it is clear that V_1 is non-increasing, and therefore

$$e, \tilde{a} \in \mathcal{L}_\infty \quad \text{(bounded)}. \tag{7}$$

Since V_1 is non-increasing and bounded from below, V_1 has a limit as $t \to \infty$, and so (5) can be integrated from $t = 0$ to infinity to obtain

$$e, ex \in \mathcal{L}_2 \quad \text{(square-integrable)}. \tag{8}$$

Now, choosing the control law

$$u = -\hat{a}x - \gamma_2\hat{x} \tag{9}$$

for a design gain $\gamma_2 > 0$, and substituting into (2), we get

$$\dot{\hat{x}} = -\gamma_2\hat{x} + \gamma_0 e + k_0 e x^2. \tag{10}$$

Consider the Lyapunov function candidate

$$V_2 = \frac{1}{2}\hat{x}^2 + \frac{1}{2}e^2. \tag{11}$$

Differentiating (11) with respect to time and inserting the dynamics (3) and (10), and using Young's inequality, yield

$$\begin{aligned} \dot{V}_2 &= -\gamma_2\hat{x}^2 + \hat{x}\gamma_0 e + k_0\hat{x}ex^2 - \gamma_0 e^2 + e\tilde{a}x - k_0e^2x^2 \\ &\leq -\gamma_2\hat{x}^2 + \frac{\rho_1\gamma_0\hat{x}^2}{2} + \frac{\gamma_0 e^2}{2\rho_1} + \frac{k_0\rho_2\hat{x}^2e^2x^2}{2} + \frac{k_0\hat{x}^2}{\rho_2} \\ &\quad + \frac{k_0e^2}{\rho_2} - \gamma_0e^2 + \frac{\rho_3 e^2}{2} + \frac{\tilde{a}^2\hat{x}^2}{\rho_3} + \frac{\tilde{a}^2e^2}{\rho_3} - k_0e^2x^2 \end{aligned} \tag{12}$$

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for arbitrary positive constants ρ_1, ρ_2, ρ_3. Choosing $\rho_1 = \frac{\gamma_2}{3\gamma_0}$, $\rho_2 = \frac{6}{\gamma_2 k_0}$, $\rho_3 = \frac{6a_0^2}{\gamma_2}$, where a_0 upper bounds $|\hat{a}|$, and recalling that $e, ex \in \mathcal{L}_2$, we obtain

$$\dot{V}_2 \leq -cV_2 + l_1 V_2 + l_2 \tag{13}$$

where $c = \min\{\gamma_2, 2\gamma_0\}$ is a positive constant and

$$l_1 = \frac{6}{\gamma_2} e^2 x^2 \tag{14a}$$

$$l_2 = \left(\frac{3}{2}\frac{\gamma_0^2}{\gamma_2} + \frac{\gamma_2 k_0^2}{6} + \frac{3a_0^2}{\gamma_2} + \frac{\gamma_2}{6}\right) e^2 \tag{14b}$$

are integrable functions (i.e. $l_1, l_2 \in \mathcal{L}_1$).

At this point it is customary to set the stage for applying Barbalat's Lemma by invoking the following result:

Lemma 1 (*Lemma B.6 from Krstić, Kanellakopoulos, & Kokotović, 1995*). *Let $v(t), l_1(t), l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose,*[1]

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{15a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{15b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{15c}$$

where c is a positive constant. Then

$$v \in \mathcal{L}_1 \cap \mathcal{L}_\infty. \tag{16}$$

To apply Barbalat's Lemma (Lemma 4 or Corollary 5 in the Appendix) for concluding $V_2 \to 0$, one must in addition to (16), establish that $\dot{V}_2 \in \mathcal{L}_\infty$, which happens to be the case in this example. Another option is to use Lemma 3.1 from Liu and Krstić (2001) (Lemma 6 in the Appendix), which requires $\dot{V}_2$ to be bounded from above and not necessarily from below.

It turns out, however, that the conditions of Lemma 1 are sufficient to obtain convergence without requiring any form of boundedness on $\dot{V}_2$, a fact that follows trivially from combining Lemma 1 and the following Lemma.

Lemma 2 (*Lemma 2.17 from Tao, 2003*). *Consider a signal g satisfying*

$$\dot{g}(t) = -ag(t) + bh(t) \tag{17}$$

for a signal $h \in \mathcal{L}_1$ and some constants $a > 0, b > 0$. Then

$$g \in \mathcal{L}_\infty \tag{18}$$

and

$$\lim_{t\to\infty} g(t) = 0. \tag{19}$$

2. Extension of Lemma 1

We will here state the main point of this note, which is an extension of Lemma 1.

Lemma 3. *Let $v(t), l_1(t), l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose*

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{20a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{20b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{20c}$$

where c is a positive constant. Then

$$v \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{21}$$

and

$$\lim_{t\to\infty} v(t) = 0. \tag{22}$$

Proof. Property (21) follows from Lemma 1. Writing (20c) as

$$\dot{v}(t) \leq -cv(t) + f(t) \tag{23}$$

where

$$f(t) = l_1(t)v(t) + l_2(t) \tag{24}$$

satisfies $f \in \mathcal{L}_1$ and $f(t) \geq 0, \ \forall t \geq 0$ since $l_1, l_2 \in \mathcal{L}_1, l_1(t), l_2(t) \geq 0, \ \forall t \geq 0$ and $v \in \mathcal{L}_\infty$. Lemma 2 can be invoked for (23) with equality. The result (22) then follows from the comparison lemma.

An alternative, direct proof of (22) goes as follows. For (22) to hold, we must show that for every $\epsilon_1 > 0$, there exists $T_1 > 0$ such that

$$v(t) < \epsilon_1 \tag{25}$$

for all $t > T_1$. We will prove that such a T_1 exists by constructing it. Since $f \in \mathcal{L}_1$, there exists $T_0 > 0$ such that

$$\int_{T_0}^{\infty} f(s)ds < \epsilon_0 \tag{26}$$

for any $\epsilon_0 > 0$. Solving

$$\dot{w}(t) = -cw(t) + f(t), \tag{27}$$

and applying the comparison principle, gives the following bound for $v(t)$

$$v(t) \leq v(0)e^{-ct} + \int_0^t e^{-c(t-\tau)} f(\tau)d\tau. \tag{28}$$

Splitting the integral at $\tau = T_0$ gives

$$\begin{aligned} v(t) &\leq v(0)e^{-ct} + e^{-c(t-T_0)} \int_0^{T_0} e^{-c(T_0-\tau)} f(\tau)d\tau \\ &\quad + \int_{T_0}^t e^{-c(t-\tau)} f(\tau)d\tau \\ &\leq Me^{-ct} + \int_{T_0}^t f(\tau)d\tau \end{aligned} \tag{29}$$

for $t > T_0$, where

$$\begin{aligned} M &= v(0) + e^{cT_0} \int_0^{T_0} f(\tau)d\tau \\ &\leq v(0) + e^{cT_0} \|f\|_1 \end{aligned} \tag{30}$$

is a finite, positive constant. Using (26) with

$$\epsilon_0 = \frac{1}{2}\epsilon_1, \tag{31}$$

we have

$$\begin{aligned} v(t) &\leq Me^{-ct} + \int_{T_0}^t f(\tau)d\tau < Me^{-ct} + \epsilon_0 \\ &< Me^{-ct} + \frac{1}{2}\epsilon_1. \end{aligned} \tag{32}$$

Now, choosing T_1 as

$$T_1 = \max\left\{T_0, \frac{1}{c}\log\left(\frac{2M}{\epsilon_1}\right)\right\} \tag{33}$$

we obtain

$$v(t) < \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_1 = \epsilon_1 \tag{34}$$

for all $t > T_1$, which proves (22).

[1] In Krstić et al. (1995) $v(0) \geq 0$ is assumed rather than $v(t) \geq 0$.

3. Application to a PDE with uncertain boundary condition

While boundedness of $\dot{V}_1$ is easily established for the example in Section 1 rendering the advantage of Lemma 3 over the combination of Lemma 1 and Barbalat's Lemma marginal, this section offers an example for which Lemma 3 is crucial. Consider the linear hyperbolic partial differential equation (PDE)

$$v_t(x,t) - v_x(x,t) = 0, \; v(1,t) = \theta v(0,t) + U(t), \tag{35}$$

for an uncertain constant θ, with initial data $v(x,0) = v_0(x)$ satisfying $v_0 \in L_2([0,1])$, that is $\|v_0\| = \sqrt{\int_0^1 |v(x,0)|^2 dx} < \infty$. This is a pure transport delay, which, for the uncontrolled case, $U \equiv 0$, has an equilibrium at the origin which is unstable for $|\theta| > 1$, stable for $|\theta| = 1$ and asymptotically stable for $|\theta| < 1$. We will use swapping to design a controller $U(t)$ so that $\|v(t)\|$ is bounded, square integrable and converges asymptotically to zero as $t \to \infty$ using the measurement

$$y(t) = v(0,t) \tag{36}$$

only.

3.1. Filter design

Consider the filters

$$\psi_t(x,t) - \psi_x(x,t) = 0, \qquad \psi(1,t) = U(t) \tag{37a}$$

$$\phi_t(x,t) - \phi_x(x,t) = 0, \qquad \phi(1,t) = v(0,t) \tag{37b}$$

with initial conditions $\psi(x,0) = \psi_0(x)$, $\phi(x,0) = \phi_0(x)$ satisfying $\psi_0, \phi_0 \in L_2([0,1])$.

A non-adaptive state estimate $\bar{v}(x)$ can be generated from

$$\bar{v}(x,t) = \psi(x,t) + \theta\phi(x,t). \tag{38}$$

The non-adaptive state estimation error

$$\epsilon(x,t) = v(x,t) - \bar{v}(x,t) \tag{39}$$

satisfies the dynamics

$$\epsilon_t(x,t) - \epsilon_x(x,t) = 0, \qquad \epsilon(1,t) = 0 \tag{40}$$

with initial condition $\epsilon(x,0) = \epsilon_0(x)$ satisfying $\epsilon_0 \in L_2([0,1])$. Eq. (40) can explicitly be solved to yield

$$\epsilon(x,t) = \begin{cases} \epsilon_0(x+t) & \text{for } t < 1-x \\ \epsilon(1, t-1+x) & \text{for } t \geq 1-x. \end{cases} \tag{41}$$

Since $\epsilon(1,t) = 0$ for all $t \geq 0$, it is evident that $\epsilon \equiv 0$ for $t \geq 1$. In other words, we have

$$y(t) = v(0,t) = \psi(0,t) + \theta\phi(0,t) + \epsilon(0,t) \tag{42}$$

with $\epsilon(0,t) \equiv 0$ for $t \geq 1$, which provides a linear parametric model for designing parameter estimation schemes.

3.2. Adaptive law

Motivated by the parametrization (42), we propose the gradient law with normalization

$$\dot{\hat{\theta}}(t) = \begin{cases} \gamma \dfrac{\hat{\epsilon}(0,t)\phi(0,t)}{1+\phi^2(0,t)} & \text{for } t > 1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

for any positive design gain γ, with

$$\hat{\epsilon}(0,t) = v(0,t) - \hat{v}(0,t) \tag{44}$$

where $\hat{v}(x,t)$ is the adaptive estimate of the state $v(x,t)$, generated from

$$\hat{v}(x,t) = \psi(x,t) + \hat{\theta}(t)\phi(x,t). \tag{45}$$

The adaptive law (43) has the properties

$$\tilde{\theta} \in \mathcal{L}_\infty, \qquad \sigma \in \mathcal{L}_2 \cap \mathcal{L}_\infty([1,\infty)) \tag{46a}$$

$$\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{46b}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and

$$\sigma(t) = \frac{\hat{\epsilon}(0,t)}{\sqrt{1+\phi^2(0,t)}} \tag{47}$$

have been defined. This can be shown using the Lyapunov function candidate

$$V_3(t) = \frac{1}{2\gamma}\tilde{\theta}^2(t). \tag{48}$$

For details, see Appendix B.1.

3.3. Lyapunov analysis

The dynamics of (45) can be shown to satisfy

$$\hat{v}_t(x,t) - \hat{v}_x(x,t) = \dot{\hat{\theta}}(t)\phi(x,t), \; \hat{v}(1,t) = 0 \tag{49}$$

with initial condition $\hat{v}(x,0) = \hat{v}_0(x)$ satisfying $\hat{v}_0 \in L_2([0,1])$, where the controller has been selected as

$$U(t) = -\hat{\theta}(t)y(t). \tag{50}$$

Consider the Lyapunov function candidate V_4, defined as

$$V_4(t) = \int_0^1 (1+x)(4\hat{v}^2(x,t) + \phi^2(x,t))dx. \tag{51}$$

It can be shown (see Appendix B.2 for details) that its time derivative satisfies

$$\dot{V}_4 \leq -cV_4(t) + l_1(t)V_4(t) + l_3(t) \tag{52}$$

where $c = \frac{1}{4}$, and

$$l_1(t) = 16\dot{\hat{\theta}}^2, \; l_3(t) = 4\sigma^2(t) + \sigma^2(t)\phi^2(0,t) \tag{53}$$

are integrable functions. Lemma 1 gives $V_4 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$ and hence

$$\|\hat{v}\|, \|\phi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{54}$$

but neither Barbalat's lemma, Corollary 5 nor Lemma 6 can be used to prove convergence to zero, since boundedness of l_3 cannot be guaranteed. However, Lemma 3 gives $V_4 \to 0$ and thus

$$\|\hat{v}\|, \|\phi\| \to 0. \tag{55}$$

From (45) it then follows that $\|\psi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\|\psi\| \to 0$, while from (38) with $\epsilon \equiv 0$ in finite time, we have

$$\|v\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \; \|v\| \to 0. \tag{56}$$

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The authors would like to thank the anonymous reviewer for bringing (Tao, 2003, Lemma 2.17) to our attention, the power of which seems to have been overlooked in many other texts on adaptive control.

Appendix A. Previous convergence results

Lemma 4 (*Barbalat's Lemma*). *Let f be a real-valued function defined for $t \geq 0$. Suppose*

(1) *f has a finite limit as $t \to \infty$,*
(2) *$\dot{f}$ is uniformly continuous.*

Then

$$\lim_{t\to\infty} \dot{f}(t) = 0. \tag{A.1}$$

Proof of Barbalat's lemma can be found in many sources, e.g. Lemma A.6 in Krstić et al. (1995).

An immediate result of Barbalat's lemma is the following corollary, which is on a form that facilitates for use on signals satisfying (5).

Corollary 5 (*Corollary A.7 in Krstić et al., 1995*). *Let ϕ be a real-valued function defined for $t \geq 0$. Suppose*

(1) *$\phi \in \mathcal{L}_p \cap \mathcal{L}_\infty$ for some $p \in [1, \infty)$*
(2) *$\dot{\phi} \in \mathcal{L}_\infty$.*

Then

$$\lim_{t\to\infty} \phi(t) = 0. \tag{A.2}$$

An alternative to Barbalat's lemma (not a corollary) particularly suited for proving convergence to zero of Lyapunov functions, was presented in Liu and Krstić (2001). The lemma goes as follows:

Lemma 6 (*Lemma 3.1 from Liu & Krstić, 2001*). *Let g be a real valued function defined for $t \geq 0$. Suppose:*

(1) *$g(t) \geq 0$ for all $t \in [0, \infty)$,*
(2) *$g(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that $g'(t) \leq M$, for all $t \geq 0$,*
(3) *$g \in \mathcal{L}_1$.*

Then

$$\lim_{t\to\infty} g(t) = 0. \tag{A.3}$$

The requirements of Lemma 6 are less restrictive than those of Corollary 5 in the sense that $\dot{g}$ is only required to be bounded from above, as opposed to Corollary 5, where dual-sided boundedness is assumed.

Appendix B. Some further details

B.1. Proof of (46)

The following steps are standard, see for instance (Ioannou & Sun, 1995). Differentiating (48) and inserting the adaptive law (43), we find

$$\dot{V}_3(t) = \begin{cases} -\tilde{\theta}(t)\dfrac{\hat{\epsilon}(0,t)\phi(0,t)}{1+\phi^2(0,t)} & \text{for } t \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{B.1}$$

We note that

$$\hat{\epsilon}(0,t) = \epsilon(0,t) + \tilde{\theta}(t)\phi(0,t) \tag{B.2}$$

with $\epsilon(0,t) = 0$ for $t \geq 1$, hence

$$\dot{V}_3(t) = \begin{cases} -\sigma^2(t) & \text{for } t > 1 \\ 0 & \text{otherwise} \end{cases} \tag{B.3}$$

for σ defined in (47), which proves that $V_3(t)$ is nonincreasing, bounded and hence has a limit as $t \to \infty$. This gives $\tilde{\theta} \in \mathcal{L}_\infty$. Integrating (B.3) from zero to infinity gives $\sigma \in \mathcal{L}_2([1,\infty))$, while $\epsilon(x,0) \in L_2([0,1])$ ensures that $\sigma(t)$ is square integrable on $0 \leq t \leq 1$ as well. Thus, $\sigma \in \mathcal{L}_2$. Moreover, for $t \geq 1$, we have

$$|\sigma(t)| = \frac{|\tilde{\theta}(t)\phi(0,t)|}{\sqrt{1+\phi^2(0,t)}} \leq |\tilde{\theta}(t)| \tag{B.4}$$

and hence $\sigma \in \mathcal{L}_\infty([1,\infty))$. From the adaptive law, we have $\dot{\hat{\theta}} = 0$ for $t < 1$, while for $t \geq 1$

$$|\dot{\hat{\theta}}(t)| \leq \gamma|\sigma(t)|\frac{|\phi(0,t)|}{\sqrt{1+\phi^2(0,t)}} \leq \gamma|\sigma(t)| \tag{B.5}$$

which proves $\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

B.2. Details of the Lyapunov analysis in Section 3.3

Consider (51), and notice that $V_4(t) \leq 8\|\hat{v}(0)\|^2 + 2\|v(0)\|^2 + 2\|\phi(0)\|^2$ for $t \leq 1$ due to (37b), (43) and (49). To analyze the case $t \geq 1$, we differentiate (51) with respect to time, and insert the dynamics (49) and (37b) to obtain

$$\begin{aligned}\dot{V}_4 &= 8\int_0^1 (1+x)\hat{v}(x,t)\left[\hat{v}_x(x,t) + \dot{\hat{\theta}}(t)\phi(x,t)\right]dx \\ &\quad + 2\int_0^1 (1+x)\phi(x,t)\phi_x(x,t)dx.\end{aligned} \tag{B.6}$$

Integration by parts and using Young's inequality on the cross term yield

$$\begin{aligned}\dot{V}_4 &= 8\hat{v}^2(1,t) - 4\hat{v}^2(0,t) - 4\int_0^1 \hat{v}^2(x,t)dx \\ &\quad + 4\int_0^1 (1+x)\left[\rho_1\hat{v}^2(x,t) + \frac{1}{\rho_1}\dot{\hat{\theta}}^2(t)\phi^2(x,t)\right]dx \\ &\quad + 2\phi^2(1,t) - \phi^2(0,t) - \int_0^1 \phi^2(x,t)dx\end{aligned} \tag{B.7}$$

for some arbitrary positive constant ρ_1. Inserting the boundary conditions and choosing $\rho_1 = \frac{1}{4}$ yield

$$\begin{aligned}\dot{V}_4(t) &\leq -\int_0^1 (1+x)\left(\hat{v}^2(x,t) + \frac{1}{2}\phi^2(x,t)\right)dx \\ &\quad + 16\dot{\hat{\theta}}^2(t)\int_0^1 (1+x)\phi^2(x,t)dx \\ &\quad - \left[1 - 4\sigma^2(t)\right]\phi^2(0,t) + 4\sigma^2(t)\end{aligned} \tag{B.8}$$

where we have used $\phi(1,t) = v(0,t) = \hat{v}(0,t) + \hat{\epsilon}(0,t)$, and the relationship $\hat{\epsilon}^2(0,t) = \sigma^2(t)(1+\phi^2(0,t))$. Inequality (B.8) can be written as

$$\begin{aligned}\dot{V}_4(t) &\leq -\frac{1}{4}V_4(t) + l_1(t)V_4(t) + l_2(t) \\ &\quad - \left[1 - 4\sigma^2(t)\right]\phi^2(0,t)\end{aligned} \tag{B.9}$$

where

$$l_1(t) = 16\dot{\hat{\theta}}^2, \qquad l_2(t) = 4\sigma^2(t) \tag{B.10}$$

are integrable functions. We already know from Lemma 1 that $V_4 \in \mathcal{L}_\infty$ if the term in the brackets is nonnegative. If $V_4 \notin \mathcal{L}_\infty$ then $\sigma^2(t)$ must be positive on a set whose measure increases unboundedly as $t \to \infty$. Supposing this is the case, there must exist constants $T_1 > 0, T_0 > 0$ and $\rho > 0$ so that $\int_t^{t+T_0} \sigma^2(\tau)d\tau \geq \rho$ for $t > T_1$. This is the requirement for persistence of excitation in (B.3), meaning that V_3 converges to zero and can be made as small as desired.

However, for $t > 1$, we have $\hat{\epsilon}(0, t) = \tilde{\theta}(t)\phi(0, t)$ and

$$\sigma^2(t) = \frac{\tilde{\theta}^2(t)\phi^2(0, t)}{1 + \phi^2(0, t)} \leq \tilde{\theta}^2(t) = 2\gamma V_3(t), \tag{B.11}$$

and hence, $\sigma^2(t)$ can also be made as small as desired. In particular, there must exist a time $T_2 > 0$ after which $\sigma^2(t) < \frac{1}{4}$ for all $t > T_2$, resulting in the expression in the brackets in (B.9) being positive for all $t > T_2$, contradicting the initial assumption. Hence $V_4 \in \mathcal{L}_\infty$, and

$$\|\hat{v}\|, \|\phi\| \in \mathcal{L}_\infty. \tag{B.12}$$

Since $\|\phi\|$ is bounded, $\phi^2(0)$ must be bounded for almost all $t \geq 0$. And hence $\sigma^2\phi^2(0) \in \mathcal{L}_1$ since $\sigma^2 \in \mathcal{L}_1$. Thus, we may write (B.9) as

$$\dot{V}_4(t) \leq -cV_4(t) + l_1(t)V_4(t) + l_3(t) \tag{B.13}$$

where

$$l_3(t) = l_2(t) + \sigma^2(t)\phi^2(0, t) \tag{B.14}$$

is integrable, but not necessarily bounded, and $c = \frac{1}{4}$.

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