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Brief Paper

Input-to-state stability for a class of Lurie systems☆

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Abstract

We analyze input-to-state stability (ISS) for the feedback interconnection of a linear block and a nonlinear element. This study is of importance for establishing robustness against actuator nonlinearities and disturbances. In the absolute stability framework, we prove ISS from a positive real property of the linear block, by restricting the sector nonlinearity to grow unbounded as its argument tends to infinity. When this growth condition is violated, examples show that the ISS property is lost. The result is used to give a simple proof of boundedness for negative resistance oscillators, such as the van der Pol oscillator. In a separate application, we relax the minimum phase assumption of an earlier boundedness result for systems with nonlinearities that grow faster than linear.

Keywords: Nonlinear systems; Absolute stability; Robustness to disturbances

1. Introduction

The absolute stability problem, formulated by Lurie and coworkers in the 1940s, has been a well-studied and fruitful area of research, as presented in Aizerman and Gantmacher (1964), and Narendra and Taylor (1973). Several stability criteria have been developed for the system in Fig. 1 which consists of the linear block $H(s)=C(sI-A)^{-1}B$ in feedback with the nonlinearity $\phi(\cdot)$. These criteria make use of the input–output properties of the linear block $H(s)$, and characterize classes of nonlinearities which ensure stability. The most well known among them is the *circle criterion* of Sandberg (1964a, b) and Zames (1966), which proves linear-gain $\mathscr{L}_p$-stability ($p\in[1,\infty]$) from the disturbance d to the output y, when the nonlinearity $\phi(\cdot)$ satisfies the *sector property*

$$y\phi(y)>0 \quad \forall y\neq 0 \tag{1}$$

and $H(s)$ is *strictly positive real* (SPR). However, the SPR property requires that $H(s)$ be asymptotically stable and minimum phase, and disallows a large class of systems with poles or zeros on the imaginary axis. With a less restrictive *positive real* (PR) condition Teel (2002) showed that linear-gain stability is lost, and instead, derived nonlinear gain functions for $\mathscr{L}_p$-stability with $p<\infty$.

In this paper we study $\mathscr{L}_\infty$-stability of the feedback system in Fig. 1, within the *input-to-state stability* (ISS) framework of Sontag (1989). The main result (Theorem 1) establishes ISS when $H(s)$ is PR and the sector nonlinearity $\phi(\cdot)$ grows unbounded as its argument goes to infinity

$$|y|\to\infty \Rightarrow |\phi(y)|\to\infty. \tag{2}$$

This growth condition is crucial for the ISS property because, with the saturation function $\mathrm{sat}(y):=\mathrm{sgn}(y)\min\{1,|y|\}$, which violates (2), the solutions of the system

$$\dot y=-\mathrm{sat}(y)+d \tag{3}$$

are driven to infinity by a constant disturbance $d>1$. This means that system (3), which is as in Fig. 1 with $H(s)=1/s$, is not ISS because the saturation function is not powerful enough to counteract large disturbances (see Liu, Chitour, & Sontag (1996) for a detailed discussion on saturation functions, and an ISS property with respect to small disturbances).

Our ISS result, presented in Section 2, is used in Section 3 to prove boundedness for a class of systems that encompasses *negative resistance oscillators* (Khalil, 2002,

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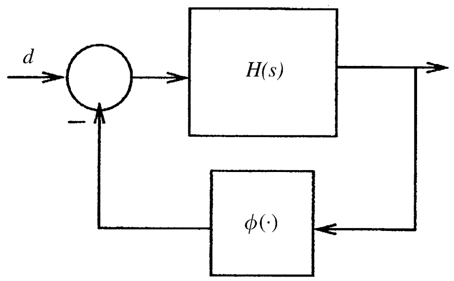

Fig. 1. The Lurie system (4)–(5), represented as the feedback interconnection of the linear block $H(s)$ and the nonlinearity $\phi(\cdot)$.

Example 2.9). In Section 4, we study relative degree one systems in feedback with a nonlinearity that grows faster than linear. For such systems we prove boundedness with a PR condition on the zero dynamics, which relaxes an earlier minimum phase assumption in Arcak, Larsen, and Kokotović (2002). Conclusions are given in Section 5. To ensure existence of solutions, henceforth the nonlinearity $\phi(\cdot)$ is assumed to be continuous, and the disturbance d is assumed to be a measurable function of time.

2. Main result

We now study the system

$$\dot{x} = Ax + B[-\phi(y) + d], \tag{4}$$

$$y = Cx, \tag{5}$$

which is as in Fig. 1 with $H(s) = C(sI - A)^{-1}B$, and prove ISS with respect to d for a class of multivariable nonlinearities $\phi(\cdot): \mathbb{R}^m \to \mathbb{R}^m$ defined by inequalities (7) and (8) below.

Theorem 1. *Consider system (4)–(5), where $x \in \mathbb{R}^n$, $\phi(\cdot): \mathbb{R}^m \to \mathbb{R}^m$, and (C, A) is detectable. If there exist a matrix $P = P^T \geqslant 0$ satisfying*

$$A^TP + PA \leqslant 0, \quad PB = C^T \tag{6}$$

a constant $\mu > 0$, and a class-$\mathscr{K}_\infty$ function $\phi_l(\cdot)$, such that

$$|y|\phi_l(|y|) \leqslant y^T\phi(y) \quad \text{for all } y \in \mathbb{R}^m, \tag{7}$$

$$|\phi(y)| \leqslant y^T\phi(y) \quad \text{when } |y| \geqslant \mu \tag{8}$$

then the system is ISS with respect to d.

Remark 1. When (A, B, C) is a minimal realization a straightforward modification of the Positive Real Lemma (Khalil, 2002, Lemma 6.2) for $P \geqslant 0$ shows that our assumption (6) is equivalent to the PR property of $H(s) = C(sI - A)^{-1}B$. For a more general result, in Theorem 1 we allow nonminimal realizations and only restrict (C, A) to be detectable.

Remark 2. For scalar nonlinearities $\phi(\cdot): \mathbb{R} \to \mathbb{R}$ condition (7) is equivalent to the sector property (1) and the growth condition (2). For scalar nonlinearities condition (8) is redundant because (7) implies $y^T\phi(y) = |y|\,|\phi(y)|$ and, thus, (8) holds with $\mu = 1$. For multivariable nonlinearities, (7) does not imply (8). A counterexample is

$$\phi(y) := y + |y|^3 Jy, \tag{9}$$

where J satisfies $J + J^T = 0$ and $J^TJ = I$. In this example, $y^T\phi(y) = |y|^2$ and $|\phi(y)| = \sqrt{|y|^2 + |y|^8}$, which mean that (7) is satisfied with $\phi_l(|y|) = |y|$, but (8) is violated.

Proof of Theorem 1. Before proceeding with the proof of ISS, we claim from (7) and (8) that there exist a constant $\varepsilon > 0$ and a class-$\mathscr{K}$ function $\eta(\cdot)$, satisfying

$$\varepsilon(|\phi(y)| + |y|) \leqslant y^T\phi(y) \quad \text{when } |y| \geqslant \mu, \tag{10}$$

$$\eta(|y|)|y|^2 + \eta(|\phi(y)|)|\phi(y)|^2 \leqslant y^T\phi(y) \quad \text{when } |y| \leqslant \mu. \tag{11}$$

Inequality (10) holds with $\varepsilon \leqslant \frac{1}{2}\min\{1, \phi_l(\mu)\}$, because, from (8) and (7), $|y| \geqslant \mu$ implies

$$\varepsilon|\phi(y)| \leqslant \tfrac{1}{2}y^T\phi(y), \tag{12}$$

$$\varepsilon|y| \leqslant \tfrac{1}{2}\phi_l(\mu)|y| \leqslant \tfrac{1}{2}\phi_l(|y|)|y| \leqslant \tfrac{1}{2}y^T\phi(y). \tag{13}$$

To prove (11), we let $\phi_u(\cdot)$ be a class-$\mathscr{K}_\infty$ function such that $|\phi(y)| \leqslant \phi_u(|y|)$ for all $y \in \mathbb{R}^m$, and show that we can select a class-$\mathscr{K}$ function $\eta(\cdot)$ to satisfy

$$\eta(|y|)|y|^2 + \eta(\phi_u(|y|))\phi_u(|y|)^2 \leqslant |y|\phi_l(|y|) \quad \text{when } |y| \leqslant \mu. \tag{14}$$

To this end, we note that the left-hand side of (14) satisfies

$$\begin{aligned} &\eta(|y|)|y|^2 + \eta(\phi_u(|y|))\phi_u(|y|)^2 \\ &\leqslant (\eta(|y|) + \eta(\phi_u(|y|)))(|y|^2 + \phi_u(|y|)^2) \\ &\leqslant 2\eta(p(|y|))q(|y|), \end{aligned} \tag{15}$$

where $p(|y|) := |y| + \phi_u(|y|)$ and $q(|y|) := |y|^2 + \phi_u(|y|)^2$. Thus, by selecting $\eta(\cdot)$ such that, when $|y| \leqslant \mu$,

$$2\eta(p(|y|))q(|y|) \leq |y|\phi_l(|y|) \tag{16}$$

we ensure from (15) that (14) holds. Finally, (14) implies (11) because of (7).

To proceed with the proof of ISS, we let $f(x, d)$ denote the right-hand side of (4); that is,

$$\dot{x} = f(x, d) := Ax - B[\phi(B^TPx) + d] \tag{17}$$

and construct a positive definite, radially unbounded *ISS-Lyapunov function* $V(x)$ satisfying

$$\langle \nabla V(x), f(x, d) \rangle \leqslant -\alpha(|x|) + \beta(|d|) \tag{18}$$

for some class-$\mathscr{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$. The first component of our ISS-Lyapunov function is

$$V_0(x) := x^TPx, \tag{19}$$

which, from (6), satisfies

$$\langle \nabla V_0(x), f(x, d) \rangle \leqslant -2y^T\phi(y) + 2y^Td \tag{20}$$

with

$$y := B^{\mathrm{T}} P x = C x. \tag{21}$$

Considering the two cases $|d| \leqslant \frac{1}{2}\phi_l(|y|)$ and $|d| \geqslant \frac{1}{2}\phi_l(|y|)$, and using (7), we obtain the inequality

$$2y^{\mathrm{T}} d \leqslant 2|y||d| \leqslant |y|\phi_l(|y|) + 2\phi_l^{-1}(2|d|)|d| \leqslant y^{\mathrm{T}}\phi(y) + 2\phi_l^{-1}(2|d|)|d|, \tag{22}$$

which results in

$$\langle \nabla V_0(x), f(x,d)\rangle \leqslant -\, y^{\mathrm{T}}\phi(y) + 2\phi_l^{-1}(2|d|)|d|. \tag{23}$$

Because $-y^{\mathrm{T}}\phi(y)$ is only a semidefinite function of x in general, we will add to $V_0(x)$ another term $V_1(x)$, which renders the right-hand side negative definite in x. To this end, we first rewrite (17) as

$$\dot{x} = (A - LC)x + Ly + B[-\phi(y) + d], \tag{24}$$

where L is chosen so that $A - LC$ is Hurwitz. Such a matrix exists since the pair (C, A) is detectable. Next, we let

$$V_1(x) := \rho(x^{\mathrm{T}} P_1 x), \tag{25}$$

where $P_1 = P_1^{\mathrm{T}} > 0$ is such that

$$(A - LC)^{\mathrm{T}} P_1 + P_1(A - LC) \leqslant -I \tag{26}$$

and

$$\rho(s) := \varepsilon_1 \int_0^s \min\left\{1, \frac{1}{\sqrt{\tau}}, \pi(\tau)\right\} \mathrm{d}\tau, \tag{27}$$

where the constant $\varepsilon_1 > 0$ and the class-$\mathscr{K}$ function $\pi(\cdot)$ are to be specified. By construction, $V_1(\cdot)$ is positive definite and radially unbounded. To prove that

$$V(x) := V_0(x) + V_1(x) \tag{28}$$

is an ISS-Lyapunov function as in (18), we let $k > 0$ be such that $2\max\{|B^{\mathrm{T}} P_1 x|, |L^{\mathrm{T}} P_1 x|\} \leqslant k|x|$, and note from (24) that

$$\langle \nabla V_1(x), f(x,d)\rangle \leqslant \rho'(x^{\mathrm{T}} P_1 x)[-|x|^2 + k|x|(|\phi(y)| + |y| + |d|)]. \tag{29}$$

Because $\rho'(\tau) \leqslant \varepsilon_1/\sqrt{\tau}$, we can find a constant $c > 0$, independent of ε_1, such that

$$\rho'(x^{\mathrm{T}} P_1 x)k|x| \leqslant \varepsilon_1 c, \quad \forall x \in \mathbb{R}^n \tag{30}$$

and obtain

$$\langle \nabla V_1(x), f(x,d)\rangle \leqslant -\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + \varepsilon_1 c(|\phi(y)| + |y| + |d|). \tag{31}$$

Choosing $\varepsilon_1 \leqslant \varepsilon/c$ in (27), and using (10), we ensure that

$$|y| \geqslant \mu \Rightarrow \langle \nabla V_1(x), f(x,d)\rangle \leqslant -\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + y^{\mathrm{T}}\phi(y) + \varepsilon|d|. \tag{32}$$

To show that a similar estimate holds when $|y| \leqslant \mu$, we denote by $\bar{\lambda}$ the maximum eigenvalue of P_1, and note that

$$\rho'(x^{\mathrm{T}} P_1 x)k|x|\,|\phi(y)| \leqslant \tfrac{1}{4}\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + 4k^2|\phi(y)|^2\pi(16\bar{\lambda}k^2|\phi(y)|^2) \tag{33}$$

(consider the two cases $|\phi(y)| \leqslant 1/4k|x|$ and $|x| \leqslant 4k|\phi(y)|$, and use $\rho'(\tau) \leqslant \pi(\tau)$ from (27)), and

$$\rho'(x^{\mathrm{T}} P_1 x)k|x|\,|y| \leqslant \tfrac{1}{4}\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + 4k^2|y|^2\pi(16\bar{\lambda}k^2|y|^2) \tag{34}$$

(consider the two cases $|y| \leqslant 1/4k|x|$ and $|x| \leqslant 4k|y|$). Thus, with the choice

$$\pi(\tau) := \frac{1}{4k^2}\,\eta\left(\sqrt{\frac{\tau}{16\bar{\lambda}k^2}}\right) \tag{35}$$

substitution of (33) and (34) in (29) results in

$$\langle \nabla V_1(x), f(x,d)\rangle \leqslant -\tfrac{1}{2}\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + \eta(|y|)|y|^2 + \eta(|\phi(y)|)|\phi(y)|^2 + \varepsilon|d|, \tag{36}$$

which, from (11), implies

$$|y| \leqslant \mu \Rightarrow \langle \nabla V_1(x), f(x,d)\rangle \leqslant -\tfrac{1}{2}\rho'(x^{\mathrm{T}} P_1 x)|x|^2 + y^{\mathrm{T}}\phi(y) + \varepsilon|d|. \tag{37}$$

Finally, noting from (27) that $\rho'(\tau) = \varepsilon_1/\sqrt{\tau}$ for sufficiently large τ, we can find a class-$\mathscr{K}_\infty$ function $\alpha(\cdot)$ such that

$$\tfrac{1}{2}\rho'(x^{\mathrm{T}} P_1 x)|x|^2 \geqslant \alpha(|x|) \quad \forall x \in \mathbb{R}^n. \tag{38}$$

Thus, it follows from (32) and (37) that, for all values of x and d,

$$\langle \nabla V_1(x), f(x,d)\rangle \leqslant -\,\alpha(|x|) + y^{\mathrm{T}}\phi(y) + \varepsilon|d|. \tag{39}$$

Adding inequalities (23) and (39), we conclude that $V(x) = V_0(x) + V_1(x)$ is an ISS-Lyapunov function, because it satisfies (18) with $\beta(|d|) := \varepsilon|d| + 2\phi_l^{-1}(2|d|)|d|$. □

In the absence of the disturbance d, when global asymptotic stability of the origin of (4)–(5) is of interest, the proof technique used above can be viewed as an alternative to using LaSalle's invariance principle. The proof of Theorem 1 exploits the detectability property that would be used in LaSalle's invariance theorem, instead to prove ISS in the presence of d. Indeed, the second component $V_1(x)$ of our ISS-Lyapunov function translates the detectability property into the Lyapunov inequality (39), which renders $\dot{V}$ negative definite in x. (See Angeli (1999) for a related approach to proving ISS from detectability.)

3. Boundedness of negative resistance oscillators

We now use Theorem 1 to give a simple proof of boundedness for the negative resistance oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - \phi(x_2) \tag{40}$$

(Khalil, 2002, Example 2.9), where the nonlinearity $\phi(\cdot)$ satisfies $\phi(0) = 0$, $\phi'(0) < 0$,

$$\phi(y) \to \infty \text{ as } y \to \infty \text{ and } \phi(y) \to -\infty \text{ as } y \to -\infty. \tag{41}$$

This class of nonlinearities differs from that considered in Theorem 1, because the sector condition (1) is violated around the origin due to $\phi'(0) < 0$. Although the origin is unstable, (41) ensures boundedness of trajectories, as proved by Levinson and Smith (1942), Cartwright (1950), LaSalle and Lefschetz (1961), Miller and Michel (1982), and Khalil (2002). We now use the ISS result of Theorem 1 to generalize this boundedness result to the larger class (4)–(5), which includes higher order systems and bounded disturbances:

Theorem 2. *Consider system* (4)–(5), *where* $x \in \mathbb{R}^n$, $\phi(\cdot): \mathbb{R} \to \mathbb{R}$, (C, A) *is detectable, and d is a bounded disturbance. If there exists a matrix* $P = P^{\mathrm{T}} \geqslant 0$ *satisfying* (6), *and if the nonlinearity* $\phi(\cdot)$ *satisfies* (41), *then the trajectories are bounded.*

Proof. We first note from (41) that we can find a constant $a > 0$ such that

$$|y| > a \Rightarrow y\phi(y) > 0. \tag{42}$$

Then, we let $\tilde{\phi}(y)$ be a continuous function such that

$$\tilde{\phi}(y) = \phi(y) \quad \text{when } |y| > a \tag{43}$$

and $y\tilde{\phi}(y) > 0$ for all $y \neq 0$. It follows that (1) and (2) hold for $\tilde{\phi}$, the latter because of (41), so that, according to Remark 2, $\tilde{\phi}(\cdot)$ satisfies the assumptions of Theorem 1. Rewriting (4)–(5) as

$$\dot{x} = Ax - B[\tilde{\phi}(y) + \tilde{d}(y) + d], \tag{44}$$

where $\tilde{d}(y) := \phi(y) - \tilde{\phi}(y)$ is bounded because $\tilde{d}(y) = 0$ when $|y| > a$, we conclude from Theorem 1 that (44) is ISS with respect to the bounded disturbance $\tilde{d}(y) + d$ and, hence, the trajectories are bounded. □

Theorem 2 encompasses the negative resistance oscillator (40), which is of the form (4)–(5) and (6), with $d = 0$,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1] \quad \text{and } P = I. \tag{45}$$

It is important to note that, by computing the ISS gain function $\gamma(\cdot)$ from the proof of Theorem 1, we can also obtain an asymptotic upper bound on $|x(t)|$

$$\limsup_{t\to\infty} |x(t)| \leqslant \gamma \left(d_0 + \limsup_{t\to\infty} |d(t)| \right),$$

$$d_0 := \max_{|y| \leqslant a} \{|\tilde{d}(y)|\}. \tag{46}$$

Thus, our ISS approach not only proves boundedness, but also gives an estimate for the size of the limit cycle.

4. Boundedness in systems with stiffening nonlinearities

In this section we use Theorem 2 to extend an earlier result, (Arcak et al., 2002, Theorem 1), which establishes boundedness of trajectories for a relative degree one, minimum phase, linear block, in feedback with a "stiffening" nonlinearity $\phi(\cdot)$, defined by the property

$$\lim_{|y|\to\infty} \frac{\phi(y)}{y} \to +\infty. \tag{47}$$

Using the Isidori normal form Isidori (1995) for relative degree one systems, this feedback interconnection is expressed as

$$\dot{z} = A_0 z + B_0 y, \tag{48}$$

$$\dot{y} = -C_0 z - ay - \phi(y) + d, \tag{49}$$

where the z-subsystem represents the zero dynamics of the linear block. The proof in Arcak et al. (2002) relies on the minimum phase property; that is, the eigenvalues of A_0 are assumed to be in the open left half-plane. We now prove boundedness for a class of systems with imaginary axis zeros:

Theorem 3. *Consider system* (48)–(49), *where d is a bounded disturbance*, (C_0, A_0) *is a detectable pair, and there exists a matrix* $P_0 = P_0^{\mathrm{T}} \geqslant 0$ *such that*

$$A_0^{\mathrm{T}} P_0 + P_0 A_0 \leqslant 0, \quad P_0 B_0 = C_0^{\mathrm{T}}. \tag{50}$$

If the nonlinearity $\phi(\cdot): \mathbb{R} \to \mathbb{R}$ *satisfies the stiffening property* (47), *then the trajectories are bounded.*

Proof. To apply Theorem 2, we rewrite (48)–(49) as in (4)–(5), with

$$A = \begin{bmatrix} A_0 & B_0 \\ -C_0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1] \tag{51}$$

and

$$\bar{\phi}(y) := \phi(y) + ay. \tag{52}$$

We note that

$$P = \begin{bmatrix} P_0 & 0 \\ 0 & 1 \end{bmatrix} \tag{53}$$

satisfies (6) because of (50), and (C, A) is detectable because (C_0, A_0) is detectable. Likewise, $\bar{\phi}(y)$ satisfies (41) because, from (47)

$$\lim_{|y|\to\infty} \frac{\bar{\phi}(y)}{y} \to +\infty. \tag{54}$$

The conditions of Theorem 2 being satisfied, we conclude that the trajectories of system (48)–(49) are bounded. □

Our positive real assumption (50) for the zero dynamics is less restrictive than the minimum phase assumption of Arcak et al. (2002), as illustrated in the following example:

Example 1. Consider the feedback interconnection in Fig. 1 with

$$H(s) = \frac{s^2 + 1}{s^3 - s^2 + 2s - 1} \tag{55}$$

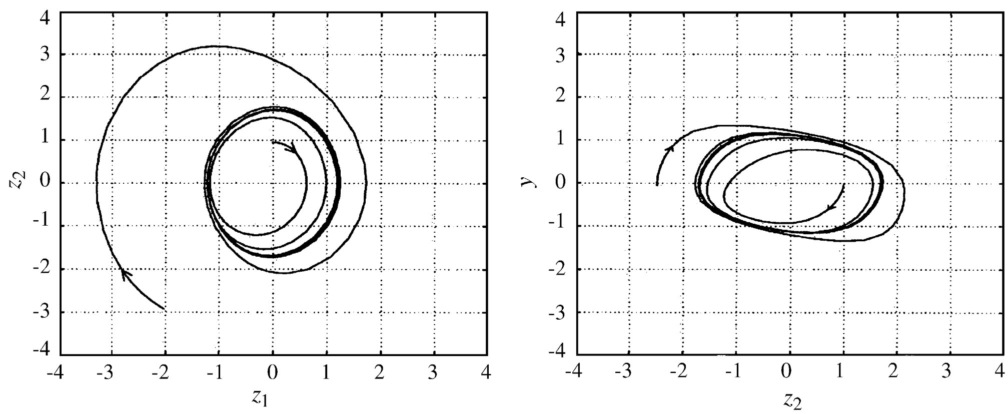

Fig. 2. Limit cycle in system (55)–(56), projected onto the (z_1, z_2)-plane (left), and the (z_2, y)-plane (right).

and the stiffening nonlinearity

$$\phi(y) = y^3. \tag{56}$$

The boundedness result of Arcak et al. (2002) is not applicable because $H(s)$ has a pair of zeros on the imaginary axis. To apply Theorem 3, we note that $H(s)$ is relative degree one, and rewrite system (55)–(56) as in (48)–(49), with $d = 0$ and

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = [0 \quad 1], \quad a = -1. \tag{57}$$

The origin is unstable from the Jacobian linearization. However, because (50) holds with $P_0 = I$, Theorem 3 ensures boundedness. Numerical simulations indicate that the trajectories converge to one of the two stable equilibria $(z_1, z_2, y) = \mp(1, 0, 1)$, or the limit cycle in Fig. 2, projected onto the (z_1, z_2)-plane (left), and the (z_2, y)-plane (right).

5. Conclusion

We have established ISS for the Lurie system in Fig. 1, by restricting the linear block to be PR, and the sector nonlinearity in the feedback loop to grow unbounded. A useful extension of our result would be to further relax the PR assumption using multiplier methods, such as the Popov and Zames–Falb multipliers. These extensions would also enlarge the classes of systems for our boundedness results, Theorems 2 and 3.

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