Sum of Squares Certificates for Stability of Planar, Homogeneous, and Switched Systems

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***Abstract*—We show that existence of a global polynomial Lyapunov function for a homogeneous polynomial vector field or a planar polynomial vector field (under a mild condition) implies existence of a polynomial Lyapunov function that is a sum of squares (sos) and that the negative of its derivative is also a sum of squares. This result is extended to show that such sos-based certificates of stability are guaranteed to exist for all stable switched linear systems. For this class of systems, we further show that if the derivative inequality of the Lyapunov function has an sos certificate, then the Lyapunov function itself is automatically a sum of squares. These converse results establish cases where semidefinite programming is guaranteed to succeed in finding proofs of Lyapunov inequalities. Finally, we demonstrate some merits of replacing the sos requirement on a polynomial Lyapunov function with an sos requirement on its top homogeneous component. In particular, we show that this is a weaker algebraic requirement in addition to being cheaper to impose computationally.**

***Index Terms*—Asymptotic stability, polynomial vector fields, semidefinite programming, sum of squares lyapunov functions.**

I. Introduction

Consider a continuous time dynamical system

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial and has an equilibrium at the origin, i.e., $f(0) = 0$. When a polynomial function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is used as a candidate Lyapunov function for stability analysis of system (1), conditions of Lyapunov's theorem reduce to a set of polynomial inequalities. For instance, if establishing global asymptotic stability of the origin is desired (see, e.g., [14, Chap. 4] for a formal definition), one would require a radially unbounded polynomial Lyapunov function candidate to satisfy:

$$V(x) > 0 \quad \forall x \neq 0 \tag{2}$$

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle < 0 \quad \forall x \neq 0. \tag{3}$$

Here, $\dot{V}$ denotes the time derivative of V along the trajectories of (1), ∇V is the gradient vector of V, and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. In some other variants of the analysis problem, e.g. if LaSalle's invariance principle is to be used, or if the goal is to prove boundedness of trajectories of (1), then the inequality in (3) is replaced with

$$\dot{V}(x) \leq 0 \quad \forall x. \tag{4}$$

In either case, the problem arising from this analysis approach is that even though polynomials of a given degree are finitely parameterized, the computational problem of searching for a polynomial V satisfying inequalities of the type (2), (3), (4) is intractable. In fact, even deciding if a given polynomial V of degree four or larger satisfies (2) is NP-hard [17], [2, Prop. 1].

An approach pioneered in [19] and quite popular by now is to replace the positivity or nonnegativity conditions by the requirement of existence of a sum of squares (sos) decomposition (see Section II for a definition):

$$V \quad \text{sos} \tag{5}$$

$$-\dot{V} = -\langle \nabla V, f \rangle \quad \text{sos}. \tag{6}$$

Clearly, if a polynomial is a sum of squares of other polynomials, then it must be nonnegative. Moreover, it is well known that an sos decomposition constraint on a polynomial can be cast as a semidefinite programming (SDP) problem [20], which can be solved efficiently. [1]

Over the last decade, Lyapunov analysis with sum of squares techniques has become a relatively well-established approach for a variety of problems in controls. Examples include stability analysis of switched and hybrid systems, design of nonlinear controllers, and formal verification of safety-critical systems, just to name a few; see, e.g., [11], [8], [13], [24], [18], and references therein. Despite the wealth of research in this area, the literature by and large focuses on proposing the sum of squares constraints as a sufficient condition for the underlying Lyapunov inequalities, without studying their necessity. For example, even for the basic notion of global asymptotic stability (GAS), the following question is to the best of our knowledge open:

Problem 1: Does existence of a polynomial Lyapunov function satisfying (2)–(3) imply existence of a polynomial Lyapunov function (of possibly higher degree) satisfying (5)–(6)?

We consider this question for polynomial vector fields, as well as switched systems, and provide a positive answer in some special cases. (We also study other related questions.) Before we outline our contributions, some remarks about the statement of Problem 1 are in order.

First, we point out that between the two sos requirements in (5)–(6), the condition in (5) can simply be met by squaring a polynomial Lyapunov function (see Lemma III.1). By contrast, the condition in (6) is typically more challenging to ensure. Second, we remark that imposing the sos conditions in (5)–(6) is not the only way to use the sos relaxation for proving GAS of (1). For example, even when these polynomials are not sos, one can multiply them by a positive polynomial (e.g., a power of $\sum x_i^2$) and the result may become sos and

[1] If the SDP resulting from (5) and (6) is strictly feasible, then any interior solution automatically satisfies (2)–(3); see, e.g., [1, p. 41].

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then certify the desired inequalities. In this technical note, however, we are interested in knowing whether the sos conditions on a Lyapunov function and its derivative can be met just by increasing the degree of the Lyapunov function. This is a very basic question in our opinion and it is in fact how the sos relaxation is most commonly used in practice. Finally, since our interest is mainly in establishing GAS of (1), a natural question that comes before Problem 1 is the following: "Does global asymptotic stability of a polynomial vector field imply existence of a polynomial Lyapunov function satisfying (2)–(3)?" The answer to this question is in general negative. The interested reader can find explicit counterexamples in [6], [3].

A. Contributions

Many of the results in this technical note can be seen as establishing cases where semidefinite programming is guaranteed to succeed in finding proofs of Lyapunov inequalities. More precisely, in Section III, we give a positive answer to Problem 1 in the case where the vector field is homogeneous (Theorem III.3) or when it is planar and an additional mild assumption is met (Theorem III.5). The general case remains open. The proofs of these two theorems are quite simple and rely on powerful and relatively recent Positivstellensatz results due to Scheiderer (Theorems III.2 and III.4).

In Section III-A, we extend these results for stability analysis of switched linear systems. These are a widely-studied subclass of hybrid systems. Our result combined with a result of Mason *et al.* [15] shows that if such a system is asymptotically stable under arbitrary switching, then it admits a common polynomial Lyapunov function that is sos and that the negative of its derivative is also sos (Theorem III.6). We also show that for switched linear systems (both in discrete and continuous time), if the inequality on the decrease condition of a Lyapunov function is satisfied with a sum of squares certificate, then the Lyapunov function itself is automatically a sum of squares (Propositions III.9 and III.10). This statement, however, is shown to be false for nonlinear systems (Lemma III.11).

Finally, in Section IV, we demonstrate a related curious fact, that instead of requiring a candidate polynomial Lyapunov function to be sos, it is better to ask its top homogeneous component to be sos. We show that this still implies GAS (Proposition IV.1 and Lemma IV.2), is a weaker algebraic requirement (Lemma IV.2), introduces no conservative in dimension two, and is cheaper to impose.

B. Related Literature

In related work [21], [23], Peet and Papachristodoulou study similar questions for the notion of exponential stability on compact sets. In [21], Peet proves that exponentially stable polynomial systems have polynomial Lyapunov functions on bounded regions. In [22], [23], Peet and Papachristodoulou provide a degree bound for this Lyapunov function (depending on decay rate of trajectories) and show that it can be made sos. There is no guarantee in [22], [23], however, that the inequality on the derivative of the Lyapunov function has an sos certificate.

II. Preliminaries

Throughout the technical note, we will be concerned with Lyapunov functions that are (multivariate) polynomials. We say that a polynomial function $V : \mathbb{R}^n \to \mathbb{R}$ is *nonnegative* if $V(x) \geq 0$ for all x, *positive definite* if $V(x) > 0$ for all $x \neq 0$, *negative definite* if $-V$ is positive definite, and *radially unbounded* if $V(x) \to \infty$ as $||x|| \to \infty$. A polynomial V of degree d is said to be *homogeneous* if it satisfies $V(\lambda x) = \lambda^d V(x)$ for any scalar $\lambda \in \mathbb{R}$. This condition holds if and only if all monomials of V have degree d. A homogeneous polynomial is also called a *form*. The *top homogeneous component* of a polynomial p is the homogeneous polynomial formed by the collection of the highest order monomials of p.

We say that a polynomial V is a *sum of squares* (sos) if $V = \sum_i^m q_i^2$ for some polynomials q_i. We do not present here the SDP that decides if a given polynomial is sos since it has already appeared in several places. The unfamiliar reader is referred to [20]. If V is sos, then V is nonnegative, though the converse is in general not true [12], [25].

III. Converse SOS Lyapunov Theorems

We start by observing that existence of a polynomial Lyapunov function immediately implies existence of a Lyapunov function that is a sum of squares.

Lemma III.1: Given a polynomial vector field, suppose there exists a polynomial Lyapunov function V such that V and $-\dot{V}$ are positive definite. Then, there also exists a polynomial Lyapunov function W such that W and $-\dot{W}$ are positive definite and W is sos.

Proof: Take $W = V^2$. Then W and $-\dot{W} = -2V\dot{V}$ are positive definite and W is sos (though $-\dot{W}$ may not be sos). ∎

We will next prove a result that guarantees the derivative of the Lyapunov function will also satisfy the sos condition, though this result is restricted to homogeneous systems.

A polynomial vector field $\dot{x} = f(x)$ is *homogeneous* if all entries of f are homogeneous polynomials of the same degree, i.e., if all the monomials in all the entries of f have the same degree. Homogeneous systems are extensively studied in the control literature; see e.g. [28], [5], [10], [27], [16], [7], and references therein.

A basic fact about homogeneous vector fields is that for these systems the notions of local and global stability are equivalent. Indeed, a homogeneous vector field of degree d satisfies $f(\lambda x) = \lambda^d f(x)$ for any scalar λ, and therefore the value of f on the unit sphere determines its value everywhere. It is also well-known that an asymptotically stable homogeneous system admits a homogeneous Lyapunov function [27].

We will use the following Positivstellensatz result due to Scheiderer to prove our converse sos Lyapunov theorem.

Theorem III.2 (Scheiderer, [30]): Given any two positive definite homogeneous polynomials p and q, there exists an integer k such that pq^k is a sum of squares.

Theorem III.3: Given a homogeneous polynomial vector field, suppose there exists a homogeneous polynomial Lyapunov function V such that V and $-\dot{V}$ are positive definite. Then, there also exists a homogeneous polynomial Lyapunov function W such that W and $-\dot{W}$ are both sos (and positive definite).

Proof: Observe that V^2 and $-2V\dot{V}$ are both positive definite and homogeneous polynomials. (Homogeneous polynomials are closed under products and the gradient of a homogeneous polynomial has homogeneous entries.) Applying Theorem III.2 to these two polynomials, we conclude that there exists an integer k such that $(-2V\dot{V})(V^2)^k$ is sos. Let

$$W = V^{2k+2}.$$

Then, W is clearly sos since it is a perfect even power. Moreover,

$$-\dot{W} = -(2k+2)V^{2k+1}\dot{V} = -(k+1)2V^{2k}V\dot{V}$$

is also sos by the previous claim. Positive definiteness of W and $-\dot{W}$ is clear from the construction. ∎

The polynomial W constructed in the proof above has higher degree than the polynomial V, though it seems difficult to construct vector fields for which this degree increase is necessary. In [4], we give a GAS vector field that admits a quadratic Lyapunov function, but for which the minimum degree of a polynomial Lyapunov function satisfying the sos constraints in (5)–(6) is four. In [2], we show that unless $\mathrm{P} = \mathrm{NP}$, there cannot be a polynomial time algorithm to test whether a cubic homogeneous vector field is globally asymptotically stable, or even to test whether it admits a quadratic Lyapunov function. Hence, just from complexity considerations, we expect there to exist GAS cubic vector fields (of possibly high dimensions) that admit a degree-2 polynomial Lyapunov function satisfying (2)–(3), but for which the minimum degree of a polynomial satisfying the sos constraints in (5)–(6) is arbitrarily high. One difficulty with explicitly constructing such examples stems from non-uniqueness of Lyapunov functions. This makes it insufficient to simply engineer V or $-\dot{V}$ to be "positive but not sos"; one needs to show that *any* polynomial Lyapunov function of a given degree fails to satisfy the sos constraints in (5)–(6).

Next, we develop a theorem that removes the homogeneity assumption from the vector field in Theorem III.3, but instead is restricted to vector fields on the plane. For this, we need another result of Scheiderer.

Theorem III.4 (Scheiderer, [29, Cor. 3.12]): Let $p := p(x_1, x_2, x_3)$ and $q := q(x_1, x_2, x_3)$ be two homogeneous polynomials in three variables, with p positive semidefinite and q positive definite. Then, there exists an integer k such that pq^k is a sum of squares.

Theorem III.5: Given a (not necessarily homogeneous) polynomial vector field in two variables, suppose there exists a positive definite polynomial Lyapunov function V, with $-\dot{V}$ positive definite, and such that the top homogeneous component of V has no zeros[2]. Then, there also exists a polynomial Lyapunov function W such that W and $-\dot{W}$ are both sos (and positive definite).

Proof: Let $\tilde{V} = V + 1$. So, $\dot{\tilde{V}} = \dot{V}$. Let us denote the degrees of $\tilde{V}$ and $\dot{\tilde{V}}$ by d_1 and d_2 respectively. Consider the (non-homogeneous) polynomials $\tilde{V}^2$ and $-2\tilde{V}\dot{\tilde{V}}$ in the variables $x := (x_1, x_2)$. Note that $\tilde{V}^2$ is nowhere zero and $-2\tilde{V}\dot{\tilde{V}}$ is only zero at the origin. Our first step is to homogenize these polynomials by introducing a new variable y. Observing that the homogenization of products of polynomials equals the product of their homogenizations, we obtain the following two trivariate forms:

$$y^{2d_1}\tilde{V}^2(\tfrac{x}{y}), \tag{7}$$

$$-2y^{d_1}y^{d_2}\tilde{V}(\tfrac{x}{y})\dot{\tilde{V}}(\tfrac{x}{y}). \tag{8}$$

Since by assumption the highest order term of V has no zeros, the form in (7) is positive definite. The form in (8), however, is only positive semidefinite. In particular, since $\dot{\tilde{V}} = \dot{V}$ has to vanish at the origin, the form in (8) has a zero at the point $(x_1, x_2, y) = (0, 0, 1)$. Nevertheless, since Theorem III.4 allows for positive semidefiniteness of one of the two forms, by applying it to the forms in (7) and (8), we conclude that there exists an integer k such that

$$-2y^{d_1(2k+1)}y^{d_2}\tilde{V}(\tfrac{x}{y})\dot{\tilde{V}}(\tfrac{x}{y})\tilde{V}^{2k}(\tfrac{x}{y}) \tag{9}$$

is sos. Let $W = \tilde{V}^{2k+2}$. Then, W is clearly sos. Moreover,

$$-\dot{W} = -(2k+2)\tilde{V}^{2k+1}\dot{\tilde{V}} = -(k+1)2\tilde{V}^{2k}\tilde{V}\dot{\tilde{V}}$$

is also sos because this polynomial is obtained from (9) by setting $y = 1$. Positive definiteness of W and $-\dot{W}$ is again clear from the construction. Note that while W does not vanish at the origin, it achieves its minimum there, and hence provides a proof of asymptotic stability. ∎

[2] This requirement is only slightly stronger than the requirement of radial unboundedness, which is imposed on V by Lyapunov's theorem anyway. See Section IV.

A. SOS Certificates for Stability of Switched Linear Systems

The result of Theorem III.3 extends in a straightforward manner to Lyapunov analysis of switched systems. In particular, we are interested in the highly-studied problem of establishing stability of arbitrary switched linear systems:

$$\dot{x} = A_i x, \quad i \in \{1, \ldots, m\}, \tag{10}$$

$A_i \in \mathbb{R}^{n\times n}$. We assume the minimum dwell time of the system (i.e., the minimum time between two consecutive switches) is bounded away from zero. This guarantees that the solutions of (10) are well-defined. The (global) asymptotic stability under arbitrary switching (ASUAS) of system (10) is equivalent to asymptotic stability of the linear differential inclusion

$$\dot{x} \in co\{A_i\}x, \quad i \in \{1, \ldots, m\},$$

where co here denotes the convex hull. A common approach for analyzing stability of these systems is to use the sos technique to search for a common polynomial Lyapunov function [24], [9]. We will prove the following result.

Theorem III.6: The switched linear system in (10) is asymptotically stable under arbitrary switching if and only if there exists a common homogeneous polynomial Lyapunov function W such that

$$\begin{aligned} W &\ \text{sos} \\ -\dot{W}_i = -\langle \nabla W(x), A_i x\rangle &\ \text{sos}, \end{aligned}$$

for $i = 1, \ldots, m$, where the polynomials W and $-\dot{W}_i$ are all positive definite.

To prove this result, we make use of the following theorem of Mason *et al.*

Theorem III.7 (Mason et al., [15]): If the switched linear system in (10) is asymptotically stable under arbitrary switching, then there exists a common homogeneous polynomial Lyapunov function V such that

$$\begin{aligned} V &> 0 \quad \forall x \neq 0 \\ -\dot{V}_i(x) = -\langle \nabla V(x), A_i x\rangle &> 0 \quad \forall x \neq 0, \end{aligned}$$

for $i = 1, \ldots, m$.

The next proposition is an extension of Theorem III.3 to switched systems (not necessarily linear).

Proposition III.8: Consider an arbitrary switched dynamical system

$$\dot{x} = f_i(x), \quad i \in \{1, \ldots, m\},$$

where $f_i(x)$ is a homogeneous polynomial vector field of degree d_i (the degrees of the different vector fields can be different). Suppose there exists a common positive definite homogeneous polynomial Lyapunov function V such that

$$-\dot{V}_i(x) = -\langle \nabla V(x), f_i(x)\rangle$$

is positive definite for all $i \in \{1, \ldots, m\}$. Then there exists a common homogeneous polynomial Lyapunov function W such that W is sos (and positive definite) and the polynomials

$$-\dot{W}_i = -\langle \nabla W(x), f_i(x)\rangle,$$

for all $i \in \{1, \ldots, m\}$, are also sos (and positive definite).

Proof: Observe that for each i, the polynomials V^2 and $-2V\dot{V}_i$ are both positive definite and homogeneous. Applying Theorem III.2 m times to these pairs of polynomials, we conclude that there exist positive integers k_i such that

$$(-2V\dot{V}_i)(V^2)^{k_i} \text{ is sos,} \tag{11}$$

for $i = 1, \ldots, m$. Let

$$k = \max\{k_1, \ldots, k_m\},$$

and let $W = V^{2k+2}$. Then, W is clearly sos. Moreover, for each i, the polynomial

$$\begin{aligned} -\dot{W}_i &= -(2k+2)V^{2k+1}\dot{V}_i \\ &= -(k+1)2V\dot{V}_i V^{2k_i} V^{2(k-k_i)} \end{aligned}$$

is sos since $(-2V\dot{V}_i)(V^{2k_i})$ is sos by (11), $V^{2(k-k_i)}$ is sos as an even power, and products of sos polynomials are sos. ■

The proof of Theorem III.6 now simply follows from Theorem III.7 and Proposition III.8 in the special case where $d_i = 1$ for all i.

B. The "V sos" Requirement for Switched Linear Systems

Our next two propositions show that for switched linear systems, both in discrete time and in continuous time, the sos condition on the Lyapunov function itself is never conservative, in the sense that if one of the "decrease inequalities" has an sos certificate, then the Lyapunov function is automatically sos. These propositions are really statements about linear systems, so we will present them as such. However, since stable linear systems always admit (sos) quadratic Lyapunov functions, the propositions are only interesting in the context where a common polynomial Lyapunov function for a switched linear system is seeked.

Proposition III.9: Consider the linear dynamical system $x_{k+1} = Ax_k$ in discrete time. Suppose there exists a positive definite polynomial Lyapunov function V such that $V(x) - V(Ax)$ is positive definite and sos. Then, V is sos.

Proof: Consider the polynomial $V(x) - V(Ax)$ that is sos by assumption. If we replace x by Ax in this polynomial, we conclude that the polynomial $V(Ax) - V(A^2x)$ is also sos. Hence, by adding these two sos polynomials, we get that $V(x) - V(A^2x)$ is sos. This procedure can be repeated to infer that for any integer $k \geq 1$, the polynomial

$$V(x) - V(A^k x) \tag{12}$$

is sos. Since by assumption V and $V(x) - V(Ax)$ are positive definite, the linear system must be GAS, and hence A^k converges to the zero matrix as $k \to \infty$. Observe that for all k, the polynomials in (12) have degree equal to the degree of V, and that the coefficients of $V(x) - V(A^k x)$ converge to the coefficients of $V - V(0)$ as $k \to \infty$. Since for a fixed degree and dimension the cone of sos polynomials is closed [26], it follows that $V - V(0)$ is sos. Hence, V is sos. ■

Similarly, in continuous time, we have the following:

Proposition III.10: Consider the linear dynamical system $\dot{x} = Ax$ in continuous time. Suppose there exists a positive definite polynomial Lyapunov function V such that $-\dot{V} = -\langle \nabla V(x), Ax\rangle$ is positive definite and sos. Then, V is sos.

Proof: The value of the polynomial V along the trajectories of the dynamical system satisfies the relation

$$V(x(t)) = V(x(0)) + \int_o^t \dot{V}(x(\tau))d_\tau.$$

Since the assumptions imply that the system is GAS, $x(t) \to 0$ as t goes to infinity. By evaluating the above equation at $t = \infty$, rearranging terms, and substituting $e^{A\tau}x$ for the solution of the linear system at time τ starting at initial condition x, we obtain

$$V(x) = \int_0^\infty -\dot{V}(e^{A\tau}x)d_\tau + V(0).$$

By assumption, $-\dot{V}$ is sos and therefore for any value of τ, the integrand $-\dot{V}(e^{A\tau}x)$ is an sos polynomial. Since converging integrals of sos polynomials are sos, it follows that V is sos. ■

One may wonder if a similar statement holds for nonlinear vector fields? The answer is negative.

Lemma III.11: There exist a polynomial vector field $\dot{x} = f(x)$ and a polynomial Lyapunov function V, such that V and $-\dot{V}$ are positive definite, $-\dot{V}$ is sos, but V is not sos.

Proof: Consider any positive form V that is not a sum of squares. (An example is $x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2x_3^2 + x_3^6 + \frac{1}{250}(x_1^2 + x_2^2 + x_3^2)^3$.) Define a dynamical system by

$$\dot{x} = -\nabla V(x).$$

In this case, both V and $-\dot{V} = ||\nabla V(x)||^2$ are positive definite and $-\dot{V}$ is sos, though V is not sos. To see that $-\dot{V}$ is positive definite, note that a homogeneous function V of degree d satisfies the Euler identity: $V(x) = \frac{1}{d}x^T\nabla V(x)$. If we had $-\dot{V}(x) = 0$ for some $x \neq 0$, then we would have $\nabla V(x) = 0$ and hence also $V(x) = 0$, which is a contradiction. ■

IV. Working With the Top Homogeneous Component of V

We show in this final section that for global stability analysis with sos techniques, the requirement of the polynomial Lyapunov function being sos can be replaced with the requirement of its top homogeneous component being sos. We also show that doing so has a number of advantages. The point of departure is the following proposition, which states that in presence of radial unboundedness, the positivity requirement of the Lyapunov function is not needed.

Proposition IV.1: Consider the vector field (1). If there exists a continuously differentiable, radially unbounded Lyapunov function V that satisfies $\dot{V}(x) < 0, \forall x \neq 0$, then the origin is globally asymptotically stable.

Proof: We first observe that since V is radially unbounded and continuous, it must be lower bounded. In fact, radial unboundedness implies that the set $\{x|\ V(x) \leq V(1, 0, \ldots, 0)\}$ is compact, and continuity of V implies that the minimum of V on this set, which equals the minimum of V everywhere, is achieved. We further claim that this minimum can only be achieved at the origin. Suppose there was a point $\bar{x} \neq 0$ that was a global minimum for V. As a necessary condition of global optimality, we must have $\nabla V(\bar{x}) = 0$. But this implies that $\dot{V}(\bar{x}) = 0$, which is a contradiction. Now consider a new Lyapunov function W defined as $W(x) := V(x) - V(0)$. Then W is positive definite, radially unbounded, and has $\dot{W} = \dot{V} < 0$, for all $x \neq 0$. Hence, W satisfies all assumptions of Lyapunov's theorem (see, e.g., [14, Chap. 4]), therefore implying global asymptotic stability.[3] ■

A sufficient condition for a polynomial p to be radially unbounded is for its *top homogeneous component*, denoted by $t.h.c.(p)$, to form a positive definite polynomial. This condition is almost necessary: radial unboundedness of p implies that $t.h.c.(p)$ needs to be positive semidefinite. Since we seek for radially unbounded Lyapunov functions anyway, this suggests that in our SDP search for Lyapunov functions, we can replace the conditions "V sos and $-\dot{V}$ sos", with "$t.h.c.(V)$

[3]The conditions of Proposition IV.1 do not imply that $V(0) = 0$ as is customary for Lyapunov functions. However, what we really need is for V to attain its global minimum at the origin, which is the case here.

sos and $-\dot{V}$ sos". The following lemma tells us that this can only help us in terms of conservatism.

Lemma IV.2: Consider the vector field (1) and suppose there exists a polynomial function $V := V(x_1, \ldots, x_n)$ that makes both $t.h.c.(V)$ and $-\dot{V}$ sos (and positive definite), then the origin is GAS. Moreover the condition $t.h.c.(V)$ sos is never more conservative (and can potentially be less conservative) than the condition $t.h.c.(V)$ sos.

Proof: The claim of global asymptotic stability follows from Proposition IV.1. To prove the claim about conservatism, we show that if V is sos, then $t.h.c.(V)$ is sos, while the converse is not true (even if V is nonnegative). Let d be the degree of V and consider the standard homogenization V_h of V in $n+1$ variables given by $V_h(x_1, \ldots, x_1, y) = y^d V(\frac{x}{y})$. Since V is sos, V_h must be sos [25]. But this implies that $t.h.c.(V)$ must be sos since it is a restriction of V_h: $t.h.c.(V) = V_h(x_1, \ldots, x_n, 0)$. A counterexample to the converse is the Motzkin polynomial

$$x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1,$$

whose top homogeneous component $x_1^4 x_2^2 + x_1^2 x_2^4$ is sos, but the polynomial itself is not sos even though it is nonnegative [25]. ■

In the particular case of the plane ($n = 2$), the constraint "$t.h.c.(V)$ sos" in fact introduces no conservatism at all. This is because all nonnegative bivariate homogeneous polynomials (of any degree) are sums of squares [25]. By contrast, the alternative condition "V sos" can be conservative as there are (non-homogeneous) nonnegative bivariate polynomials that are not sums of squares. In addition to these features in terms of conservatism, the requirement "$t.h.c.(V)$ sos" is a cheaper constraint to impose than "V sos". Indeed, if V is an n-variate polynomial of degree $2d$, simple calculation shows that the SDP underlying the former condition has $\frac{2d(n+2d-1)!}{2d!n!}$ fewer equality constraints and $\frac{d(n+d-1)!((2n+d)(n+d-1)!+d!n!)}{2(d!n!)^2}$ fewer decision variables than the latter. Let us end with a concrete example.

Example IV.1: Consider the vector field

$$\dot{x}_1 = 0.36x_1 + 2x_2 - 0.32x_1^7 - 0.02x_1x_2^6 + 8x_2^7 + 3x_1^2x_2^5$$
$$\dot{x}_2 = -2x_1 - 0.44x_2 - 16x_1^7 - x_1x_2^6 - 0.16x_2^7 - 0.06x_1^2x_2^5.$$

We solve an SDP which searches for a polynomial Lyapunov function V of degree d that satisfies "$t.h.c.(V)$ sos and $-\dot{V}$ sos". This SDP is infeasible for $d = 2, 4, 6$. Because of Lemma IV.2, we know that the SDP with the more standard conditions "V sos and $-\dot{V}$ sos" would have been infeasible also. Moreover, since nonnegative bivariate forms are sos, we know that our constraint "$t.h.c.(V)$ sos" is lossless (while we wouldn't be able to make such a claim for the constraint "V sos").

For $d = 8$, our SDP returns the following solution:

$$\begin{aligned} lV(x) &= 2.195x_1^2 + 4.237x_1^8 + 2.183x_2^2 + 2.170x_2^8 \\ &\quad + 1.055x_1^2x_2^6 + 0.863x_1x_2 - 0.286x_1^7x_2 + 0.037x_1^6x_2^2 \\ &\quad + 0.042x_1^5x_2^3 - 0.011x_1^4x_2^4 - 0.021x_1^3x_2^5 + 0.039x_1x_2^7. \end{aligned}$$

By checking the eigenvalues of the Gram matrices in our SDP, we can see that the top homogeneous component of this polynomial, as well as $-\dot{V}$, are positive definite. Hence, by Lemma IV.2, we know that the system is GAS. For this example it happens that the returned solution V is sos although we asked for a weaker and cheaper condition. Indeed, the semidefinite constraint that we imposed to require $t.h.c.(V)$ to be sos has 105 fewer decision variables and 36 fewer equality constraints than the semidefinite constraint needed to impose V sos.

In general, by leaving out the constant term in the parametrization of the polynomial V, we can make sure that $V(0) = 0$ and hence (in view of the proof of Proposition IV.1) radial unboundedness would automatically imply positive definiteness of our Lyapunov function.

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