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REACHABILITY OF PERTURBED SYSTEMS AND MIN SUP PROBLEMS*

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1. Introduction. The problem of reachability for control processes, that is, the problem of finding an admissible control which steers the system into some target set is a preliminary one in the study of optimal control problems. For linear control processes in both finite and infinite dimensions, reachability has been discussed by several authors [1], [2], [3], [4], [5], [6], [7], [8]. The reachability problem for nonlinear differential control processes has also recently been investigated [9].

Control problems in the presence of disturbances have usually been treated as stochastic control problems. However, in many control problems statistics of the disturbance are not available. For such problems a natural way to model the disturbance is to assume that it belongs to some fixed bounded set in the space of disturbances.

The objective of this paper is to study such control processes in the presence of additive disturbances. We introduce the concept of strong reachability. A control process is said to be strongly reachable if there exists an admissible control which steers the system to the given target set in the presence of the worst disturbance. We show that the problem of finding the best open loop control in the presence of the worst disturbance is related to the concept of strong reachability. For the problem studied in this paper, the operations of finding infimum and supremum are not interchangeable and hence game-theoretical techniques used for example in [10], [11], [12], [13] are not applicable. We present a geometrical necessary and sufficient condition for strong reachability. For linear control processes with closed, bounded, convex control constraint and disturbance sets and a closed, convex target set, the geometrical condition can be translated into analytical form by using separation and embedding theorems for convex sets. By specializing to the case where the control constraint set, the disturbance set and the target set are balls, we can obtain an analytical necessary and sufficient condition in explicit form and also obtain expressions for the minimum norm control, maximum norm disturbance and the minimum target set radius in terms of the control process data. The final section considers applications to control processes described by differential equations. For some work related to this paper see [14].

Notation. For a map $f:X \to Y$, if $A \subset X$, $f(A) = \{f(x) | x \in A\}$. $\{x\}$ is the set consisting of the single element x. For two sets A, B which are subsets of a Banach Space X, $A + B = \{a + b | a \in A, b \in B\}$.

Let X be a Banach space and let X^* be its topological dual space. We define the symbol $\langle x, x^* \rangle$ by $\langle x, x^* \rangle = x^*(x)$, where the right-hand side is the value of the linear form x^* at the point x. The map $(x, x^*) \mapsto \langle x, x^* \rangle$ is a bilinear form on $X \times X^*$. $\mathscr{L}(X, Y)$ denotes the space of continuous linear maps from X into Y. For $S \in \mathscr{L}(X, Y)$, S^* is the adjoint linear map and $S^* \in \mathscr{L}(Y^*, X^*)$.

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2. Basic definitions.

2.1. Mathematical definition of the system. Let X_1, X_2, be reflexive Banach spaces and let X_3 be a Banach space. X_1 is to be thought of as the *control space*, X_2 the *disturbance space* and X_3 the *state space* of a control system. Let U, W and B be subsets of X_1, X_2 and X_3 respectively. R denotes the real line.

Let $S:X_1 \to X_3$ (or $u \mapsto Su$), and $T:X_2 \to X_3$ (or $w \mapsto Tw$) be continuous (not necessarily linear) maps and let s be a given element in X_3. We shall be concerned with the *abstract control process*, (C), defined by the equation,

$$x = s + Su + Tw. \tag{C}$$

A control $u \in U$ will be called an *admissible control* and a disturbance $w \in W$ will be called an *allowable disturbance*. The set B will be referred to as the *target set*.

2.2. Definition of strong reachability and reachability. For the system (C),

(i) the target set B is said to be *strongly reachable* from s if there exists a $\bar{u} \in U$ such that $s + S\bar{u} + Tw \in B$ for all w in W;

(ii) the target set B is said to be *reachable* from s if for any w in W, there exists a $\bar{u} \in U$ such that $s + S\bar{u} + Tw \in B$.

We shall refer to the system (C) as being strongly reachable (reachable) when we mean B is strongly reachable (reachable) from s.

2.3. Definition of the min sup problem. For the function $f:X_3 \to R$ let $q:X_1 \times X_2 \to R$ be the mapping defined by $(u, w) \mapsto f(s + Su + Tw)$.

The min sup problem can now be formulated as follows: Given the function f, does there exist a $\hat{u} \in U$ such that

$$\sup\{q(\hat{u}, w):w \in W\} = \inf\{\sup[q(u, w):w \in W]:u \in U\}.$$

3. Geometrical necessary and sufficient condition for strong reachability and reachability. To obtain a necessary and sufficient condition for strong reachability we introduce two sets.

DEFINITION 3.1. For the control system (C), the *unperturbed attainable set* is defined as the set

$$A = \{s\} + S(U) \tag{3.1}$$

and the *modified target set* is defined as the set

$$M = \{x \in X_3 | \{x\} + T(W) \subset B\}. \tag{3.2}$$

Our first theorem identifies in geometrical form the necessary and sufficient conditions for strong reachability.

THEOREM 3.2. *The system* (C) *is strongly reachable if and only if* $A \cap M$ *is not empty.*

Proof. If (C) is strongly reachable, there exists an admissible control $\bar{u}$ such that

$$\{s\} + \{S\bar{u}\} + T(W) \subset B$$

and hence $s + S\bar{u} \in M$. However, $\bar{u}$ being admissible implies $s + S\bar{u} \in A$ and hence $A \cap M \neq \varnothing$. Conversely $A \cap M \neq \varnothing$ implies that there exists an $x \in X_3$ such that $x \in A$ and $x \in M$. Since $x \in A$, there exists a $\bar{u}$ admissible such that $x = s + S\bar{u}$. Likewise $x \in M$ implies $\{x\} + T(W) \subset B$ and hence $s + S\bar{u} + Tw \in B$, for every $w \in W$.

COROLLARY 3.3. *The system* (C) *is reachable if and only if for every* $w \in W$ $A \cap (B - \{Tw\}) \neq \emptyset$.

Remark. Theorem 3.2 does not make use of any topological properties and hence is true for a control process defined on linear spaces.

4. The min sup problem. This section contains two theorems. The first theorem exhibits the relationship between the problem of existence of a solution to the min sup problem with that of strong reachability with respect to an appropriately constructed target set. The second theorem shows that under certain assumptions on the function f and the unperturbed attainable set A, the min sup problem has a solution.

THEOREM 4.1. *Let,*

$$\varepsilon^* = \inf\{\sup[f(s + Su + Tw) : w \in W] : u \in U\} \tag{4.1}$$

and for any $\varepsilon \in R$, *define* $B(\varepsilon) = f^{-1}((-\infty, \varepsilon])$. *Then*

(i) *for every* $\varepsilon \in R$, *the control process* (C) *is strongly reachable with respect to the target* $B(\varepsilon)$ *if and only if* $\varepsilon^* = -\infty$.

(ii) *There exists no* $\varepsilon \in R$ *such that the control process* (C) *is strongly reachable with respect to the target* $B(\varepsilon)$ *if and only if* $\varepsilon^* = +\infty$.

(iii) *If there exists an* $\bar{\varepsilon} \in R$, *such that for any* $\varepsilon \in R$, (C) *is strongly reachable for the target* $B(\varepsilon)$ *if and only if* $\varepsilon \geqq \bar{\varepsilon}$, *then there exists a* $\hat{u} \in U$ *such that*

$$\sup[f(s + S\hat{u} + Tw) : w \in W] = \varepsilon^* = \bar{\varepsilon}. \tag{4.2}$$

(iv) *Conversely, if there exists a* $\hat{u} \in U$ *such that*

$$\sup[f(s + S\hat{u} + Tw) : w \in W] = \varepsilon^*$$

and $\varepsilon^* \in R$, *then for any* $\varepsilon \in R$, (C) *is strongly reachable for the target* $B(\varepsilon)$ *if and only if* $\varepsilon \geqq \varepsilon^*$.

Proof. (i) If $\varepsilon^* = -\infty$, then for every $\varepsilon \in R$, there exists a $u \in U$ such that $\sup\{f(s + Su + Tw) : w \in W\} \leqq \varepsilon$; that is, for every $\varepsilon \in R$, the control process (C) is strongly reachable for the target $B(\varepsilon)$. Conversely, for an arbitrary $\varepsilon \in R$, $A \cap M(\varepsilon) \neq \emptyset$ implies the existence of a $u \in U$ such that $\{s\} + \{Su\} + T(W) \subset B(\varepsilon)$; that is $\sup\{f(s + Su + Tw) : w \in W\} \leqq \varepsilon$. Since ε is arbitrary, take $\varepsilon = -n$, where n is a positive integer. For each n, there exists a $u_n \in U$ such that $\sup\{f(s + Su_n + Tw) : w \in W\} \leqq -n$ and hence

$$\varepsilon^* \leqq \lim_{n \to \infty} \{\sup[f(s + Su_n + Tw) : w \in W]\} \leqq \lim_{n \to \infty} (-n) = -\infty$$

which implies that $\varepsilon^* = -\infty$.

(ii) The following chain of statements are equivalent to $\varepsilon^* = +\infty$:
for every $u \in U$, $\sup\{f(s + Su + Tw) : w \in W\} = +\infty$,
for every $u \in U$, $\varepsilon \in R$, $\{s\} + \{Su\} + T(W)$ is not a subset of $B(\varepsilon)$,
for every $\varepsilon \in R$, there exists no $u \in U$ such that $\{s\} + \{Su\} + T(W) \subset B(\varepsilon)$,
for every $\varepsilon \in R$, $A \cap M(\varepsilon) = \phi$; and hence the desired conclusion.

(iii) For $\varepsilon < \bar{\varepsilon}$, there exists no $u \in U$ such that $M(\varepsilon) \cap A \neq \emptyset$; that is for every $u \in U$, $\sup \{f(s + Su + Tw) : w \in W\} \geqq \bar{\varepsilon}$, and then $\varepsilon^* \geqq \bar{\varepsilon}$. However, $M(\bar{\varepsilon}) \cap A \neq \emptyset$ guarantees the existence of a $u^* \in U$ such that $\sup \{f(s + Su^* + Tw) : w \in W\} \leqq \bar{\varepsilon}$. Hence for this particular $u^* \in U$, $\sup \{f(s + Su^* + Tw) : w \in W\} = \bar{\varepsilon} = \varepsilon^*$.

(iv) Since $A \cap M(\varepsilon^*) \neq \emptyset$, for every $\varepsilon \geqq \varepsilon^*$, $B(\varepsilon) \supset B(\varepsilon^*)$ and $M(\varepsilon) \supset M(\varepsilon^*)$ and finally $M(\varepsilon) \cap A \supset M(\varepsilon^*) \cap A \neq \emptyset$. So (C) is strongly reachable for $\varepsilon \geqq \varepsilon^*$. Now if $\varepsilon < \varepsilon^*$, there exists no $u \in U$ such that $\sup \{f(s + Su + Tw) : w \in W\} \leqq \varepsilon$; that is $\{s\} + \{Su\} + T(W)$ is not a subset of $B(\varepsilon)$ and $M(\varepsilon) \cap A = \emptyset$ for $\varepsilon < \varepsilon^*$.

THEOREM 4.2. *Let $f: X_3 \to R$ be lower semicontinuous and let the unperturbed attainable set A be compact* (*in an appropriate topology of X_3*). *Then there exists a $\hat{u} \in U$ such that*

$$\sup [f(s + S\hat{u} + Tw) : w \in W] = \inf [\sup \{f(s + Su + Tw) : w \in W\} : u \in U]. \tag{4.3}$$

Proof.

$$\inf \{\sup [f(s + Su + Tw) : w \in W] : u \in U\} = \inf \{\sup [f(x + y) : y \in T(W)] : x \in A\}.$$

Let $q(x, y) = f(x + y)$. Since f is lower semicontinuous, for fixed x the function $q_x(y) = q(x, y)$ is lower semicontinuous. Since the upper envelope of a family of lower semicontinuous functions is lower semicontinuous [15, p. 362, Theorem 4] and A is compact, there exists an $\hat{x} \in A$ such that

$$\sup [q(\hat{x}, y) : y \in T(W)] = \inf \{\sup [q(x, y) : y \in T(W)] : x \in A\}.$$

Hence from the definition of A there exists a $\hat{u} \in U$ such that (4.3) is true.

COROLLARY 4.3. *Let $f: X_3 \to R$ be continuous and let the sets A and $T(W)$ be compact* (*in an appropriate topology of X_3*). *Then there exists a $\hat{u} \in U$ and a $\hat{w} \in W$ such that*

$$f(s + S\hat{u} + T\hat{w}) = \inf \{\sup [f(x + Su + Tw) : w \in W] : u \in U\}.$$

Remark. Theorems 4.1 and 4.2 and Corollary 4.3 are true, for example in linear topological spaces which are Hausdorff.

COROLLARY 4.4. *Let $f: X_3 \to R$ be convex and strongly lower semicontinuous and A be weakly compact. Then Theorem* 4.2 *holds.*

Proof. The proof follows from the fact that a lower semicontinuous convex function defined on a Banach space is weakly lower semicontinuous.

5. Strong reachability of linear control processes. In this section we investigate strong reachability for linear control processes in a Banach space setting.

Let the assumptions of § 2.1 on the control system (C) hold. Further, let U be a closed, bounded, convex subset of X_1, W a closed, bounded, convex subset of X_2 and B a closed, convex subset of X_3. The maps S and T defined in § 2.1 are now assumed to belong to the spaces $\mathscr{L}(X_1, X_3)$ and $\mathscr{L}(X_2, X_3)$ respectively. The linear control system so obtained will now be referred to as (L).

We shall use the separation theorem and embedding theorem for convex sets to translate Theorem 3.2 into analytical form. The following *separation theorem* is an immediate consequence of the strong separation theorem for convex sets in locally convex topological vector spaces [16, p. 119, Corollary 14.4 and p. 23, Theorem 3.9 and p. 14].

THEOREM 5.1. *Let X be a Banach space, let A be a weakly compact, convex subset of X and let B be a closed, convex subset of X. Then*

$$A \cap B \neq \varnothing$$

if and only if

$$\sup \{\inf [\langle x, x^* \rangle : x \in B] - \sup [\langle x, x^* \rangle : x \in A] : \|x^*\|_{X^*} = 1\} \leqq 0.$$

We also quote the following embedding theorem due to Hörmander [17].

DEFINITION 5.2. Let X be a locally convex topological vector space and let K be a nonempty, closed, convex subset of X. The *support functional* $H(x^*)$ of K, $x^* \in X^*$, is defined by

$$H(x^*) = \sup [\langle x, x^* \rangle : x \in K].$$

THEOREM 5.3. *Let K_1 and K_2 be two closed convex sets, and let $H_1(x^*)$ and $H_2(x^*)$ be their corresponding support functionals. Then*

(i) $K_1 \subseteq K_2$ *if and only if* $H_1(x^*) \leqq H_2(x^*)$, *for every* $x^* \in X^*$

(ii) $K_1 = K_2$ *if and only if* $H_1(x^*) = H_2(x^*)$, *for every* $x^* \in X^*$.

5.1. Analytical necessary and sufficient conditions. In order to invoke the above theorems, it is necessary to establish some topological properties for the sets A and M.

PROPOSITION 5.4. *For the control system* (L), *the unperturbed attainable set A and the set $\{x\} + T(W)$ are convex and weakly compact.*

Proof. The proof of this proposition is an immediate consequence of the linearity and weak continuity of the maps S and T[18, p. 422, Theorem 15] and the weak-compactness of the sets U and W[18, p. 425, Corollary 8].

PROPOSITION 5.5. *The target set B and the modified target set M are weakly closed and convex.*

Proof. Since B is convex and closed, it is weakly closed. The convexity of M is obvious. We shall show M is a strongly closed subset of X_3. Consider a strong Cauchy sequence $\{x_n\}$ in M. Since $M \subset X$, $x_n \to x$, where $x \in X$. For any $w \in W$, the translated sequence $\{x_n + Tw\}$ is Cauchy and $x_n + Tw \to x + Tw$. However, as points of M the x_n's are such that $x_n + Tw \in B$, for every $w \in W$. But since B is strongly closed, $x + Tw \in B$, for every $w \in W$, and $x + T(W) \subset B$ which implies $x \in M$. Hence M is strongly closed and being convex is thus weakly closed.

PROPOSITION 5.6. *Given the system* (L), *the set M is given by*

$$M = \{x \in X_3 : h(x) \leqq 0\},$$

where

$$h(x) = \sup [\langle x, x^* \rangle + \sup (\langle w, T^* x^* \rangle : w \in W) - \sup (\langle y, x^* \rangle : y \in B) : \|x^*\|_{X_3^*} = 1]. \tag{5.1}$$

Further, M is nonempty if and only if

$$\inf [h(x) : x \in X_3] \leqq 0. \tag{5.2}$$

Proof. From Theorem 5.3, $\{x\} + T(W) \subset B$ if and only if $H_{\{x\}+T(W)}(x^*) \leqq H_B(x^*)$, for every $x^* \in X_3^*$. That is,

$$\sup [\langle x, x^* \rangle + H_{T(W)}(x^*) - H_B(x^*) : \|x^*\|_{X_3^*} = 1] \leqq 0, \tag{5.3}$$

and hence from the definition of $H_{T(W)}(x^*)$ and $H_B(x^*)$ we obtain (5.1) and the definition of the elements of M.

If M is nonempty, (5.2) clearly holds. To prove the proposition in the other direction we use the fact that $h(x)$ satisfies $|h(x_2) - h(x_1)| \leqq \|x_2 - x_1\|_{X_3}$, for every $x_1, x_2 \in X_3$ and hence $h(x)$ is continuous.

There are two cases to consider. If $\inf[h(x): x \in X_3] < 0$, then clearly there exists $x \in X_3$ such that $x + T(W) \subset B$, and M is not empty. If $\inf[h(x): x \in X_3] = 0$, then there exists a sequence $\{x_n\}$ such that $h(x_n) \to 0$; since h is continuous on X_3, there exists an $x \in X_3$ such that $x_n \to x$ and $h(x) = 0$. So again M is nonempty.

THEOREM 5.7. *The system* (L) *is strongly reachable if and only if*

(5.4)
$$\inf\{\sup[\langle x, x^*\rangle + \sup(\langle w, T^*x^*\rangle : w \in W) - \sup(\langle y, x^*\rangle : y \in B) : \|x^*\|_{X_3^*} = 1] : x \in X_3\} \leqq 0$$

and

(5.5)
$$\sup(\langle s, x^*\rangle + \inf(\langle u, S^*x^*\rangle : u \in U) - \sup(\langle y, x^*\rangle : y \in M) : \|x^*\|_{X_3^*} = 1) \leqq 0,$$

where

(5.6)
$$M = \{x \in X_3 : \sup[\langle x, x^*\rangle + \sup(\langle w, T^*x^*\rangle : w \in W) - \sup(\langle y, x^*\rangle : y \in B) : \|x^*\|_{X_3^*} = 1] \leqq 0\}.$$

Proof. From Theorem 3.2, (L) is strongly reachable if and only if $A \cap M \neq \emptyset$. Using Propositions 5.4 and 5.5 the proof now follows directly from Theorem 5.1 and Proposition 5.6.

COROLLARY 5.8. *The system* (L) *is reachable if and only if*

(5.7)
$$\sup\{\langle s, x^*\rangle + \inf(\langle u, S^*x^*\rangle : u \in U) + \sup(\langle w, T^*x^*\rangle : w \in W) - \sup(\langle y, x^*\rangle : y \in B) : \|x^*\|_{X_3^*} = 1\} \leqq 0.$$

6. Specialization of the results of § 5. The results of the previous section will now be specialized to the case where

(6.1)
$$U = \{u \in X_1 : \|u\|_{X_1} \leqq \rho\}, \quad 0 \leqq \rho < \infty,$$
$$W = \{w \in X_2 : \|w\|_{X_2} \leqq \beta\}, \quad 0 \leqq \beta < \infty,$$
$$B = \{x \in X_3 : \|x - x_d\|_{X_3} \leqq \varepsilon, x_d \in X_3 \text{ given}\}, \quad 0 \leqq \varepsilon < \infty.$$

6.1. Strong reachability and reachability.

THEOREM 6.1. *The system* (L) *with U, W and B as defined in* (6.1) *is strongly reachable if and only if*

(6.2)
$$\beta\|T\| \leqq \varepsilon,$$

(6.3)
$$\sup\{\langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in M_{x_d}] : \|x^*\|_{X_3^*} = 1\} \leqq 0,$$

where $M_{x_d} = M - \{x_d\}$ and

(6.4)
$$M_{x_d} = \{x \in X_3 : \sup[\langle x, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1] \leqq \varepsilon\}.$$

Proof. The proof follows from Theorem 5.7 by performing the necessary computations. In particular (5.4) becomes (6.2) as shown below. Equation (5.4) reduces to

$$\inf\{\sup[\langle x, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} - \varepsilon : \|x^*\|_{X_3^*} = 1] : x \in X_3\} \leqq 0.$$

But $k(x) = \sup[\langle x, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1]$ is an even function of x. Thus

$$k(x) \geqq \sup[\beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1] = \beta\|T^*\| = \beta\|T\|.$$

Hence $\beta\|T\| \leqq \varepsilon$.

Theorem 6.1 can be sharpened somewhat in the sense that in calculating $\sup[\langle x, x^*\rangle : x \in M_{x_d}]$ we may restrict ourselves to x's which belong to the boundary of M_{x_d}. Moreover we can find an analytical expression for the boundary of M_{x_d}. This is done in the following two propositions.

PROPOSITION 6.2. *If $M \neq \emptyset$, its boundary ∂M (in the norm topology) is defined by*

$$\partial M = \{x \in X_3 : \sup[\langle x - x_d, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1] = \varepsilon\}. \tag{6.5}$$

Proof. Consider the function

$$f : X_3 \to R : x \mapsto \sup\{\langle x - x_d, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\}.$$

From Theorem 5.3,

$$M = \{x \in X_3 : f(x) \leqq \varepsilon\}. \tag{6.6}$$

(a) Let $x_0 \in M$ such that $f(x_0) = \varepsilon$ and assume x_0 is an interior point of M. Then there exists an open ball $B(x_0; \delta)$ with center at x_0 and radius δ such $B(x_0; \delta) \subset M$. However, $\sup\{f(x) : x \in B(x_0; \delta)\} \geqq \varepsilon + \delta$ which shows that for some $x \in B(x_0; \delta)$, $f(x) > \varepsilon$ which contradicts (6.6).

(b) Now let $x_0 \in \partial M$ and assume $\alpha = f(x_0) < \varepsilon$. Let $\delta = (\varepsilon - \alpha)/2$ and consider the open ball $B(x_0; \delta)$. For every $y \in B(x_0; \delta)$,

$$f(y) \leqq \alpha + \|y - x_0\|_{X_3} \leqq \alpha + \frac{\varepsilon - \alpha}{2} < \varepsilon,$$

which contradicts that $x_0 \in \partial M$.

PROPOSITION 6.3. *Let $M_{x_d} \neq \emptyset$. Then*

$$\sup\{\langle x, x^*\rangle : x \in M_{x_d}\} = \sup\{\langle x, x^*\rangle : x \in \partial M_{x_d}\}.$$

Proof. By Theorem 5.6, for any $x \in M_{x_d}$

$$\alpha = \sup\{\langle x, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\} \leqq \varepsilon.$$

Let $x \in M_{x_d}$ and assume $x \neq 0$ (otherwise $M_{x_d} = \partial M_{x_d} = \{0\}$). Consider the real valued function f on $[0, \infty)$, defined by

$$f(c) = \sup\{c\langle x, x^*\rangle + \beta\|T^*x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\};$$

it is monotone increasing, convex and continuous on $[0, \infty)$. Moreover $f(1) \leqq \varepsilon$ and for $c_0 = (\varepsilon + \beta\|T\| + 1)/\|x\|_{X_3}$, $f(c_0) > \varepsilon$. Hence there exists a unique $\bar{c}$ in

$[1, \infty)$ such that $f(\bar{c}) = \varepsilon$. By Proposition 6.2, $\bar{c}x \in \partial M_{xd}$. We then have for any x in M_{xd}

$$|\langle x, x^*\rangle| \leqq |\langle \bar{c}x, x^*\rangle|,$$

$$\sup\{|\langle x, x^*\rangle| : x \in M_{xd}\} \leqq \sup\{|\langle x, x^*\rangle| : x \in \partial M_{xd}\}.$$

The theorem now follows from the linearity of the functional $\langle x, x^*\rangle$ and from the fact that M_{xd} is symmetrical about 0 in X_3.

COROLLARY 6.4. *If X_3 is a reflexive Banach space, Theorem* 6.1 *holds with M_{xd} in* (6.3) *replaced by the set of extreme points of M_{xd}.*

Proof. The proof follows from the Krein–Millman theorem [16, p. 131, Theorem 15.1] and from a proposition in Bourbaki [19, p. 106, Proposition 1].

Remark. It might be useful to find a representation for the extreme points of M_{xd}.

6.2. The characterization problem. If the system (L) is strongly reachable, then it is useful to characterize the minimum values of ρ and ε and the maximum value of β for which the system remains strongly reachable. This is done in the following theorems.

THEOREM 6.5 (minimum norm control). *For given β and ε assume that* (L) *is strongly reachable for some $\tilde{\rho}$, $0 \leqq \tilde{\rho} < \infty$. Then there exists a minimum bound ρ^* for which* (L) *is strongly reachable. Moreover ρ^* is given by*

(6.7) (i) $\rho^* = 0$, *if* $g(0) \leqq 0$,

(ii) *ρ^* is the unique solution in $[0, \tilde{\rho}]$ of the equation $g(\rho) = 0$ if $g(0) > 0$, where*

$$g(\rho) = \sup\{\langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in M_{xd}] : \|x^*\|_{X_3^*} = 1\}.$$

Proof. Consider the function

$$f : (R^+ \cup \{0\}) \times X_3^* \to R : (\rho, x^*) \mapsto \langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in M_{xd}]$$

and the function

$$g : R^+ \cup \{0\} \to R : \rho \mapsto \sup\{f(\rho, x^*) : \|x^*\|_{X_3^*} = 1\}.$$

We show that g is a monotonically decreasing, continuous convex function of ρ. For $\rho_2 \geqq \rho_1 \geqq 0$, $f(\rho_2, x^*) \leqq f(\rho_1, x^*)$, for every $x^* \in X_3^*$ and hence $g(\rho_2) \leqq g(\rho_1)$ showing that g is monotonically decreasing. For $\rho_2 \neq \rho_1$ and $\lambda \in [0, 1]$,

$$f(\lambda\rho_1 + (1 - \lambda)\rho_2, x^*) = \lambda f(\rho_1, x^*) + (1 - \lambda)f(\rho_2, x^*) \qquad \text{for every } x^* \in X_3^*,$$

which implies that g is convex.

Finally for $\rho_2 \geqq \rho_1 \geqq 0$,

$$|g(\rho_2) - g(\rho_1)| = g(\rho_1) - g(\rho_2) \leqq \sup[(\rho_2 - \rho_1)\|S^*x^*\|_{X_1^*} : \|x^*\|_{X_3^*} = 1]$$
$$\leqq |\rho_2 - \rho_1| \cdot \|S\|.$$

Since S is linear and continuous $\|S\| < \infty$. This shows that g is a continuous function of ρ.

Since (L) is strongly reachable for $\tilde{\rho} \geqq 0$, we have $g(\tilde{\rho}) \leqq 0$. There are two cases to consider:

(a) $g(0) \leqq 0$. Then the minimum bound $\rho^* = 0$.

(b) $g(0) > 0$. Then by virtue of the properties of the function $g(\rho)$, ρ^* is given by the unique solution in $[0, \tilde{\rho}]$ of $g(\rho) = 0$.

THEOREM 6.6 (maximum norm disturbance). *Given the bounds ε and ρ, assume that* (L) ($T \not\equiv 0$) *is strongly reachable for $\beta = 0$. Then there exists a maximum bound β^* such that* (L) *is strongly reachable if and only if $\beta \leqq \beta^*$. Moreover defining*

$$\bar{\beta} = \frac{\varepsilon}{\|T\|},$$

(i) $\beta^* = \bar{\beta}$, *if* $f(\bar{\beta}) \leqq 0$

(ii) $\beta^* = \hat{\beta}$, *if* $f(\bar{\beta}) > 0$, *where $\hat{\beta}$ is the unique solution of $f(\beta) = 0$ in $[0, \bar{\beta})$. $f(\beta)$ is defined by*

$$f(\beta) = \sup\{\langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in M_{x_d}(\beta)] : \|x^*\|_{X_3^*} = 1\}.$$

Proof. A necessary condition for strong reachability of (L) is $M \neq \emptyset$ which implies $\beta \leqq \bar{\beta}$. We shall show that f is monotonically increasing, convex and continuous on $[0, \bar{\beta})$. Hence there are two cases:

(i) $f(\bar{\beta}) \leqq 0$. In that case (L) is strongly reachable and $\beta^* = \bar{\beta}$.

(ii) $f(\bar{\beta}) > 0$. Since $f(0) \leqq 0$ by hypothesis and $f(\bar{\beta}) > 0$, if we prove the asserted properties of the function $f(\beta)$, $f(\beta)$ has a unique solution $\hat{\beta}$ on $[0, \bar{\beta})$ and $\beta^* = \hat{\beta}$.

Now if $0 \leqq \beta_1 \leqq \beta_2 \leqq \bar{\beta}$, $\beta_1\|T\| \leqq \beta_2\|T\|$, which implies $M(\beta_1) \supset M(\beta_2)$ and using Theorem 5.3, $f(\beta_1) \leqq f(\beta_2)$ which shows that f is monotonically increasing.

Since $\lambda M(\beta_1) + (1 - \lambda)M(\beta_2) \subset M(\lambda\beta_1 + (1 - \lambda)\beta_2)$, for every $\lambda \in [0, 1]$, it follows from Theorem 5.3 that

$$f(\lambda\beta_1 + (1 - \lambda)\beta_2) \leqq \lambda f(\beta_1) + (1 - \lambda)f(\beta_2) \qquad \text{for every } \lambda \in [0, 1],$$

which shows that f is convex on $[0, \bar{\beta}]$.

The convexity of f implies continuity on $(0, \bar{\beta})$. Continuity at 0^+ can be demonstrated in a manner analogous to Theorem 6.5.

THEOREM 6.7 (minimum miss distance). *Given the bounds ρ and β there exists a minimum bound ε^* such that* (L) *is strongly reachable if and only if $\varepsilon \geqq \varepsilon^*$. Moreover defining $\bar{\varepsilon} = \beta\|T\|$,*

(i) $\varepsilon^* = \bar{\varepsilon}$ *if* $f(\bar{\varepsilon}) \leqq 0$,

(ii) $\varepsilon^* = \hat{\varepsilon}$, *if* $f(\bar{\varepsilon}) > 0$, *where $\hat{\varepsilon}$ is the unique solution of $f(\varepsilon) = 0$ in $(\bar{\varepsilon}, \infty)$. $f(\varepsilon)$ is defined by*

$$f(\varepsilon) = \sup\{\langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in M_{x_d}(\varepsilon)] : \|x^*\|_{X_3^*} = 1\}.$$

Proof. We first show that for some $\varepsilon_\alpha \geqq 0$, the system (L) is strongly reachable. Let α and ε_α be defined as

$$\alpha = \sup\{\langle s - x_d, x^*\rangle - \rho\|S^*x^*\|_{X_1^*} : \|x^*\|_{X_3^*} = 1\},$$

$$\varepsilon_\alpha = |\alpha| + \bar{\varepsilon}.$$

Clearly $B(|\alpha|) = \{x \in X_3 : \|x\|_{X_3} \leqq |\alpha|\} \subset M_{x_d}(\varepsilon_\alpha)$ and for any $x^* \in X_3^*$, such that $\|x^*\| = 1$,

$$|\alpha| = \sup\{\langle x, x^*\rangle : x \in B(|\alpha|)\} \leqq \sup\{\langle x, x^*\rangle : x \in M_{x_d}(\varepsilon_\alpha)\}$$

which implies that (L) is strongly reachable for $\varepsilon = \varepsilon_\alpha$ since $f(\varepsilon_\alpha) \leqq 0$. In a manner analogous to the previous two theorems, it may be shown that f is monotonically decreasing, convex on $(\bar{\varepsilon}, \infty)$ and continuous on $(\bar{\varepsilon}, \varepsilon_\alpha]$. A necessary condition for strong reachability of (L) is $M \neq \emptyset$ which implies $\varepsilon \geqq \bar{\varepsilon}$. There are two cases to consider:

(i) $f(\bar{\varepsilon}) \leqq 0$. In that case (L) is strongly reachable and hence $\varepsilon^* = \bar{\varepsilon}$.

(ii) $f(\bar{\varepsilon}) > 0$. Since $f(\varepsilon_\alpha) \leqq 0$ and $f(\bar{\varepsilon}) > 0$, in view of the properties of f, $f(\varepsilon) = 0$ has a unique solution $\hat{\varepsilon}$ on $(\bar{\varepsilon}, \varepsilon_\alpha]$ and $\varepsilon^* = \hat{\varepsilon}$.

Remark. Theorems 6.5, 6.6 and 6.7 have obvious corollaries when strong reachability is replaced by reachability.

7. Applications to control processes described by differential equations. We shall illustrate the theory presented in the previous sections by considering its application to control processes described by differential equations.

7.1. Existence theorem for min sup problem. We consider an existence theorem for a min sup problem analogous to the existence theorem for optimal control problems.

Consider the perturbed control process in R^n

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, u(t)) + g(t, w(t)), \qquad t \in [0, t_1], \tag{7.1}$$

where $A(t)$ is a $n \times n$ measurable and bounded matrix on $[0, t_1]$, f is in C^1 in R^{1+m} and g is in C^1 in R^{1+p}, (n, m and p are integers $\geqq 1$). Furthermore,

(i) The initial state x_0 at time 0 is given.

(ii) The admissible controllers $\mathscr{F}$ consist of all Lebesgue measurable functions $t \mapsto u(t)$ on the compact interval $[0, t_1]$ such that $u(t) \in U$, (almost everywhere on $[0, t_1]$), where U is a compact set in R^m.

(iii) The admissible disturbances $\mathscr{G}$ consist of all Lebesgue measurable functions $t \mapsto w(t)$ on the compact interval $[0, t_1]$ such that $w(t) \in W$, almost everywhere on $[0, t_1]$, where W is a compact set in R^p.

(iv) The cost function for each admissible u and w is given by $C(u, w) = g(x(t_1))$, where g is a continuous function in R^n.

THEOREM 7.1. *For the above system, there exists a $\hat{u} \in \mathscr{F}$ and a $\hat{w} \in \mathscr{G}$ such that*

$$C(\hat{u}, \hat{w}) = \inf[\sup\{C(u, w) : w \in \mathscr{G}\} : u \in \mathscr{F}].$$

Proof. Since the differential equation (7.1) is linear in x, there exists an absolutely continuous function $t \mapsto x(t)$ defined on $[0, t_1]$ which satisfies (7.1) almost everywhere. Moreover by using the variation of parameters formula, the solution x of (7.1) at time t_1 may be written as,

$$\begin{aligned} x(t_1) = \phi(t_1)x_0 + \phi(t_1)\int_0^{t_1} \phi^{-1}(s) f(s, u(s))\, ds \\ + \phi(t_1)\int_0^{t_1} \phi^{-1}(s) g(s, w(s))\, ds, \end{aligned} \tag{7.2}$$

where ϕ is the usual transition matrix associated with (7.1).

Let $s \in R^n$ be defined by, $s = \phi(t_1)x_0$, and define the continuous nonlinear operators

$$S: L_1(R^m; 0, t_1) \to R^n : u \mapsto \phi(t_1) \int_0^{t_1} \phi^{-1}(s) f(s, u(s))\, ds,$$

$$T: L_1(R^p; 0, t_1) \to R^n : w \mapsto \phi(t_1) \int_0^{t_1} \phi^{-1}(s) g(s, w(s))\, ds,$$

where $L_1(X; 0, t_1)$ is the space of all integrable functions $t \mapsto x(t)$ with values in X. Equation (7.2) may then be written as:

$$x(t_1) = s + Su + Tw. \tag{7.3}$$

It follows from a result of Neustadt [20] that for the system (7.3) the unperturbed attainable set $\{s\} + S(U)$ and the set $T(W)$ are compact and hence the theorem follows from Corollary 4.3.

7.2. Strong functional reachability. Consider the linear differential control process

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + C(t)w(t), \tag{7.4}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $w(t) \in R^k$ (n, m, k are integers $\geqq 1$) and $A(t)$, $B(t)$, $C(t)$ are matrices of appropriate order which are measurable and bounded on the given compact interval $[0, t_1]$.

Let $1 < p < \infty$ and let $L^p(R^m; 0, t_1)$ be the reflexive Banach space of R^m-valued measurable functions such that

$$\int_0^{t_1} \|u(t)\|_{R^m}^p\, dt < \infty.$$

The Banach space $L^p(R^m; 0, t_1)$ is normed by

$$\|u\|_p = \left(\int_0^{t_1} \|u(t)\|_{R^m}^p\, dt\right)^{1/p}.$$

In a similar manner define $L^q(R^k; 0, t_1)$, $1 < q < \infty$, as the reflexive Banach space of all R^k-valued measurable functions with norm

$$\|w\|_q = \left(\int_0^{t_1} \|w(t)\|_{R^k}^q\, dt\right)^{1/q}$$

and $L^r(R^n; 0, t_1)$, $1 \leqq r \leqq \infty$, as the Banach space of all R^n-valued measurable functions with norm

$$\|x\|_r = \left(\int_0^{t_1} \|x(t)\|_{R^n}^r\, dt\right)^{1/r}.$$

Let the control restraint set Ω_u, the disturbance set Ω_w and the target set B be defined by

$$\begin{aligned} \Omega_u &= \{u : \|u\|_p \leqq \rho\}, \\ \Omega_w &= \{w : \|w\|_q \leqq \beta\}, \\ B &= \{x : \|x - x_d\|_r \leqq \varepsilon,\ x_d \text{ given element in } L^r(R^n; 0, t_1)\}. \end{aligned} \tag{7.5}$$

Since the differential equation is linear for given $x(0) \in R^n$, $u \in L^p(R^m; 0, t_1)$ and $w \in L^q(R^k; 0, t_1)$, there exists an absolutely continuous function $t \mapsto x(t)$ defined on the compact interval $[0, t_1]$ which satisfies (7.4) almost everywhere. Since $t \mapsto x(t)$ is absolutely continuous, $x \in L^r(R^n; 0, t_1)$. The solution of (7.4) is given by

$$x(t) = \phi(t)x(0) + \phi(t)\int_0^t \phi^{-1}(s)B(s)u(s)\,ds + \phi(t)\int_0^t \phi^{-1}(s)C(s)w(s)\,ds, \qquad t \in [0, t_1]. \tag{7.6}$$

Let $S: L^p(R^m; 0, t_1) \to L^r(R^n; 0, t_1)$ be the linear bounded transformation defined by

$$(Su)(t) = \phi(t)\int_0^t \phi^{-1}(s)B(s)u(s)\,ds, \qquad 0 \leqq t \leqq t_1,$$

and let $T: L^q(R^k; 0, t_1) \to L^r(R^n; 0, t_1)$ be the linear bounded transformation defined by

$$(Tw)(t) = \phi(t)\int_0^t \phi^{-1}(s)C(s)w(s)\,ds, \qquad 0 \leqq t \leqq t_1,$$

and let $s \in L^r(R^n; 0, t_1)$ be defined by $s(t) = \phi(t)x(0), 0 \leqq t \leqq t_1$.

Then (7.6) may be written as the operator equation

$$x = s + Su + Tw. \tag{7.7}$$

DEFINITION 7.2. The control process (7.7) is *strongly functionally reachable* with respect to $(s, \Omega_u, \Omega_w, B, t_1)$ if there exists a $\bar{u} \in \Omega_u$ such that $x(\cdot\,; \bar{u}, w) \in B$, for every $w \in \Omega_w$.

Necessary and sufficient conditions for strong functional reachability can now be obtained using the theory developed in previous sections.

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