Large deviations for non-zero initial conditions in linear systems

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A B S T R A C T

Transient response of linear systems with non-zero initial conditions was at the center of attention for engineers and researchers at early stages of classical control theory. However this field was not intensively investigated later. For instance, the breakthrough result on unavoidable peaking effect for systems with strong damping factor was obtained by Izmailov in 1987, but it did not attract much attention. We try to continue this line of research and provide explicit worst-case lower bound. Then we exhibit large deviation effects for other pole locations and estimate lower bounds for them. The upper bounds for deviations of trajectories are much better studied; to obtain the smallest deviations by static linear feedback the techniques of linear matrix inequalities can be exploited. We demonstrate that for such closed-loop systems the upper and lower bounds have the same asymptotic behavior.

1. Introduction

Transient response of linear systems with non-zero initial conditions was at the center of attention for engineers and researchers at early stages of classical control theory. However this field was not intensively investigated later. We try to continue this line of research.

Consider a single-input continuous time linear control system

\[ \dot{x} = Ax + bu, \quad u \in \mathbb{R}, \quad b \in \mathbb{R}^n. \]  \hspace{1cm} (1)

If the vectors \( b, Ab, \ldots, A^{n-1}b \) are linearly independent, then the system is controllable and, by the pole assignment theorem, there exists a linear feedback defined by a vector \( K \) such that the equilibrium position \( x = 0 \) of the closed-loop system

\[ \dot{x} = Ax + bk^Tx = Fx \]  \hspace{1cm} (2)

is asymptotically stable. Moreover, one can generate a linear system with any given set \( \Lambda \in \mathbb{C} \) of eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \). (We assume that if \( \lambda \in \Lambda \) and \( \text{Im} \lambda \neq 0 \), then \( \text{Re} \lambda - i \text{Im} \lambda \in \Lambda \).) Therefore, by choosing an appropriate linear feedback it is possible to obtain a closed-loop system with an arbitrary given damping speed. However, the trajectories of the closed-loop system with fast damping significantly deviate from the equilibrium position during the initial phase of the stabilization for some non-zero initial conditions. This phenomenon is called the peak effect and the large deviation is referred to as an overshoot. Although this phenomenon always attracted attention of scientific community (see, e.g., Feldbaum, 1948), the estimate for the peak effect in general situation was proved in Izmailov (1987). More precisely, Izmailov showed that there exists a constant \( \gamma = \gamma(A, b) > 0 \), such that, if \( \{\lambda_1, \ldots, \lambda_n\} \), are the eigenvalues of \( F = A + bk^T \), then the condition \( \text{Re} \lambda_j \leq -\sigma < 0, j = 1, n \), implies

\[ \sup_{0 \leq t \leq \frac{n}{\sigma}} \sup_{|x(0)| = 1} |x(t)| \geq \gamma \sigma^{n-1} \] \hspace{1cm} (3)

for solutions of the closed-loop system (2). Here \( |x| \) is an arbitrary norm of vector \( x \) (below we shall specify it). The proof given by Izmailov was significantly simplified in Bushenkov and Smirnov (1997). In Sussman and Kokotovic (1991) Izmailov’s result was generalized to obtain estimates for peaking effect for outputs. From (3) we also conclude that even for moderate decay rate (\( \sigma = 2 \sim 5 \)) the peak effect grows exponentially as a function of the system dimension \( n \).

In what follows we always treat lower bounds in the sense of (3), i.e. as worst-case transient response with respect to initial...
conditions. Of course it does not imply large deviations for all initial conditions; for any stable system $\dot{x} = Fx$ there exist initial conditions such that $|x(t)| \leq |x(0)|$ for all $t > 0$.

The effect of possible large deviations of trajectories is generally recognized beyond control community, for instance, in numerical analysis. The solution of the closed-loop system (2) is $x(t) = \exp(Ft)x(0)$, $F = A + bK^T$ thus $\max_{|x(0)|=1}|x(t)| = \| \exp(Ft) \|$, where $\| \cdot \|$ is matrix norm associated with vector norm $| \cdot |$. Hence deviations of trajectory are closely related to behavior of matrix exponent. In the famous paper Moler and Van Loan (1978) and its continuation Moler & Van Loan, 2003 there are numerous examples of matrix exponents, having big humps. Fig. 1, borrowed from Moler and Van Loan (2003), exhibits transition process related to stabilization of Boeing 767 example ($\|e^{Ft}\|$ as function of $t$). Notice that the values of the humps are of order $10^5$.

The problem of large deviations is highly significant, because non-zero initial conditions naturally arise in many applications. Typical example—observers, where true coordinates of the observed trajectory are never known. It is interesting to mention that Izmaiov’s result (as well as previous publications on peak effect Polotskij, 1981) are formulated for observers. Similar situation is met in switching systems (Liberon, 2003), where the trajectory is always in non-zero position after switching. The large deviations of control systems trajectories from the equilibrium position during the transition process represent a serious obstacle to the design of cascade control systems (Sussman & Kokotovic, 1991) and to guidance stabilization Bushenkov and Smirnov (1997).

Therefore, it is an important problem to estimate possible characteristics of the transition process and to describe the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ causing large deviations. Surprisingly this effect is not specific for large eigenvalues only. In Smirnov, Bushenkov, and Miranda (2009) it was shown that the large deviation of the solutions from the equilibrium position at the beginning of stabilization occurs for the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ with $|\lambda_k| \ll 1$ and $|\lambda_k| \gg 1$, $k = 1, \ldots, n$.

It is necessary to note that the effect of significant growth of solution at transient times, similar to the peak effect, is also observed for two- and three-dimensional Poiseuille and Couette flows, even when all the eigenmodes decay exponentially (Reddy & Henning, 1993). This phenomenon caused by the near-linear dependence of the eigenfunctions is of importance in the study of hydrodynamic stability and transition to turbulence and is discussed in the control theory framework in Bewley and Liu (1998). Transient effects play significant role in distributed control, where system dimension is large, see e.g. recent research on vehicular platoons (Martinec, Z, & Sebek, 2015).

In this work we analyze the situation with transient response in linear systems with non-zero initial conditions more deeply. First we treat the systems in companion form and provide examples with all eigenvalues equal real $-\sigma < 0$ where deviations of the trajectory for specific initial conditions can be estimated explicitly (Section 2). We show that the large deviation effect is present both for $\sigma$ large or small. In Section 3 we focus on main results—lower bounds for worst deviations for systems in canonical form. First we calculate constant $\gamma$ in (3). Next, the cases of other

![Boeing 767 stabilization, n = 55.](image)

**Fig. 1.** Boeing 767 stabilization, $n = 55$. 
locations of eigenvalues are examined, for instance if some of the eigenvalues have very big or very small moduli. Extension of the results for systems in general (not canonical) form is given in Section 4. In Section 5 we address upper bounds for the peak effect. These problems are much better investigated, see Balandin and Kogan (2009), Bulgakov (1980), Hinrichsen, Plischke, and Wurth (2002), Whidborne and Amar (2011) and Whidborne and McKernan (2007). We provide a version of upper bounds, based on construction of invariant ellipsoids for the closed-loop system using semidefinite programming (SDP) approach (Boyd, El Ghaoui, Ferron, & Balakrishnan, 1994), and compare numerically various bounds for systems in companion form with desired damping. Theoretical comparison of the bounds for arbitrary linear systems is also discussed. Finally, we consider some open problems and directions for future research.

The conference version of the paper has been submitted at 19th IFAC World Congress (Polyak & Smirnov, 2014). Recently the authors with several additional coauthors have published the paper Polyak, Tremba, Khlebnikov, Scherbakov, and Smirnov (2015) also related to large deviations problem. However there are serious differences if compared with the present work. (a) The detailed proofs of main results (Theorems 1 and 2) are obtained here for the first time, while in Polyak et al. (2015) related propositions are given with no proofs. (b) The same is true for constants in lower bounds, which are estimated below explicitly. (c) Some important results on mixed eigenvalue locations (Theorems 3 and 4) have not been addressed in Polyak et al. (2015). (d) Same is true for basic statement on asymptotic comparison of the bounds (Theorem 7). On the other hand, the paper Polyak et al. (2015) is focused on connections of the problem with Feldbaum’s result (Feldbaum, 1948) and on large deviations in the case of zero initial conditions and non-zero inputs.

**Notations.** By default $|x|$ for vector $x \in \mathbb{R}^n$ with components $x_k, k = 1, \ldots, n$ stands for $|x|_\infty = \max_k |x_k|$ norm. When needed, the Euclidean norm is denoted by $|x|_2$. The same notation $|x|$ for a complex number $\lambda$ is used for its modulus. The entries of a matrix $M$ are denoted $m_{kj}$ and its transpose is $M^T$. The $k$th column of the identity matrix $I$ is $e_k$. The spectrum of a closed-loop matrix $F$ is the set $\{\lambda \in \mathbb{C} | \lambda\}$, and we use the following characteristics of eigenvalues:

$$\sigma = - \max_j \Re \lambda_j; \quad \omega = \max_j |\lambda_j|; \quad \rho = \min_j |\lambda_j|.$$ 

That is $\sigma$ is stability degree of $F$, $\omega$ is its spectral radius and $\rho$ is inverse of spectral radius of $F^{-1}$.

**2. Motivating examples**

First we provide some examples on the lower bounds for deviations for linear systems in input–output form. The characteristic polynomial has all roots equal $-\sigma < 0$. We observe that for some specific initial conditions large deviations arise both for $\sigma$ small and $\sigma$ large.

**Example 1.** Linear differential equation

$$\begin{align*}
(s + \sigma)^n y(t) &= 0, \\
\quad s &= \frac{d}{dt}
\end{align*}$$

where $\sigma > 0$ and $n \geq 2$, with the initial conditions $y(0) = 1$, $y^{(k)}(0) = 0, k = 1, \ldots, n - 1$.

The respective solution is given by the formula

$$y(t) = \left( \sum_{j=0}^{n-1} \frac{(\sigma t)^j}{j!} \right) e^{-\sigma t}. \quad (5)$$
Since \( y'(t) = -\frac{a_n t^{-1}}{n-1} e^{-\sigma t} \), we get \( y^{(n-1)}(t) = -\frac{n!}{n-1} \sigma^{n-1} e^{-\sigma t} \) and \( \lambda_j = \frac{\sigma}{h_j} \) for \( j = 1, \ldots, n \) is the characteristic polynomial of \( A \). The solution for \( \sigma > 0 \) achieves its maximum at a point \( \theta_\sigma \). Thus for \( \gamma_n = \frac{n!}{n-1} \sigma^{n-1} \) we have
\[
\left| y^{(n-1)}(\theta_\sigma) \right| = \gamma_n \sigma^{n-1}.
\]

3. Lower bounds for systems in companion form

In this section we consider system (1) in companion form, i.e. with
\[
A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad b = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
The closed-loop system with the feedback \( u(x) = (K, x) \) has the system matrix \( F = A + bK^T \)
\[
= \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}, \quad a_{n-i+1} = -K_i
\]
and its characteristic polynomial is
\[
\Delta(s) = s^n + a_1 s^{n-1} + \cdots + a_n.
\]

3.1. Large eigenvalues

One of the first papers on dependence of transient response on spectrum location is due to Feldbaum (1948). He investigated the behavior of the first component of the solution \( \chi_i(t) \); it decreases monotonically and exhibits no peaking. However it is not hard to see that the last component \( \chi_n(t) \) behaves differently and peaking effect is strong for eigenvalues with large negative real parts (Polyak et al., 2015). The results in Feldbaum (1948) are true for all real eigenvalues (with one possible complex pair); below we get rid of this assumption. The proof of the theorem (as well as of all other results) can be found in the Appendix; its statement without the proof has been published in Polyak et al. (2015).

**Theorem 1.** If \( \lambda_1, \ldots, \lambda_n \), with \( \Re \lambda_j < 0 \), \( j = 1, n \), are the eigenvalues of \( F \), \( \rho = \min(\{\lambda_j : j = 1, n\} \), \( \omega = \max(\{\lambda_j : j = 1, n\} \), and \( x(0) = e_n \), then
\[
| x \left( \log 2 / n \rho \right) | \geq 2 \log 2 - 1 - \rho^{n-1} > \frac{0.3862}{n} \rho^{n-1}.
\]

We conclude that for large eigenvalues and large \( n \) unavoidable deviations can exceed any practically acceptable values. For instance, \( \sigma = 4, n = 10 \) imply deviations of order \( 10^5 \).

3.2. Small eigenvalues

Now we address the case with \( \omega = \max(\{\lambda_j : j = 1, n\} \), small, where \( \{\lambda_1, \ldots, \lambda_n\} \), are the eigenvalues of \( F \).

**Theorem 2.** If \( \Re \lambda_j < 0 \), \( j = 1, n \) and \( x(0) = e_n \) then
\[
| x \left( \frac{\theta_\rho}{\omega} \right) | \geq \gamma_n \frac{1}{\omega^{n-1}},
\]
where \( \theta_\rho = \frac{n-1}{n} \log 2 \) and \( \gamma_n = \frac{(\log 2)^{n-1}}{\rho^{n-1}} \).

We conclude that small eigenvalues (with \( \omega < 0.2/n \), for example) cause large deviations for \( t \) large.

3.3. Mixed eigenvalues

We consider the case when both small and large eigenvalues are present. Theorem 3 exhibits that dominating (see (15)) large eigenvalues cause peak effect in spite of existence of small eigenvalues, while Theorem 4 demonstrates large deviations when small eigenvalues dominate (18). Notice that the corresponding initial conditions are different.

Assume that the eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfy the inequalities
\[
\omega > |\lambda_1| \geq \cdots \geq |\lambda_\rho| \geq \eta > \xi > |\lambda_{\rho+1}| \geq \cdots \geq |\lambda_n|
\]
and \( \Re \lambda_j < 0, j = 1, \ldots, n \).
Theorem 3. Let \( \nu \geq 2 \) and
\[
\eta > \left( \frac{10(1 + \xi)}{\omega n} \right)^{\frac{1}{\nu}}.
\] (15)
Then for \( x(0) = e_{n-v+1} \) the following inequality holds:
\[
\left| x \left( \log \frac{2}{\omega n} \right) \right| \geq \gamma \eta^{n-1}, \quad \gamma_n = \frac{1 - 1.1 \log 2}{n} \log 2.
\] (16)

Assume now that the eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfy more restrictive inequalities
\[
|\text{Re} \lambda_1| \geq \cdots \geq |\text{Re} \lambda_\nu| \geq \eta > \xi > |\lambda_{\nu+1}| \geq \cdots \geq |\lambda_n|,
\] (17)
where \( \omega > |\lambda_\nu| \) and \( \text{Re} \lambda_j < 0 \), \( j = 1, \ldots, n \).

Theorem 4. Suppose that
\[
\nu \leq \min(n-3, (n-2)/2), \quad \eta \geq 3,
\] (18)
\[
\xi \leq \min(c/\omega, 1),
\]
where \( c = \left( \frac{2}{\log 2} \right)^{1/(\nu-1)} \), and \( x(0) = e_n \). Then the following inequality is satisfied
\[
\left| x \left( \frac{1}{\xi} \right) \right| \geq \frac{\gamma_n}{\xi^{n-2}}.
\] (19)
Here \( \gamma_n \) can be estimated explicitly (see the proof). From this theorem we see that if small eigenvalues dominate (that is their number is large enough and their values are small \( (18) \)), then condition \( (17) \) implies the effect of large deviations for large \( t \).

4. Lower bounds for systems in general form

Let \( l \in \mathbb{R}^n \) satisfy \( (l, A^{n-1}b) = 1 \) and \( (l, A^{n-1}b) = 0 \), \( j = 1, n-1 \) (such \( l \) exists due to the controllability assumption). In the coordinates \( z_i = (l, A^{n-1}x_j), j = 1, n \) system (1) is converted into companion form and for closed-loop system \( \dot{z} = Gz \), the eigenvalues of matrices \( G \) and \( F \) coincide. We have \( z = Mx \), where matrix \( M \) has the rows \( l^TA^{n-1}, j = 1, n \). Since \( |x|_2 \geq |x|_2/\|M\| \) \( \geq |z|_2/\kappa \), where \( \kappa \) is spectral norm of \( M \), that is the maximal singular value of the matrix \( M \), all the estimates, previously obtained for systems in canonical form, can be reformulated in terms of Euclidean norm in the original space after division by \( \kappa \). Notice that matrix \( M \) depends only on the pair \( (A, b) \) and does not depend on the choice of the eigenvalues (i.e. on the feedback \( K \)), see Wonham (1979). For example, combining Theorem 1 with this result, we get

Theorem 5. For any solution of (2) the constraint \( \text{Re} \lambda_j \leq -\sigma < 0 \), \( j = 1, n \) for eigenvalues of the closed-loop system implies the estimate
\[
\sup_{0 \leq t \leq \frac{1}{\xi}} \sup_{|x(0)| = 1} |x(t)|_2 \geq \gamma \sigma^{n-1} \] (20)
with \( \gamma = (2 \log 2 - 1)/(n \kappa) \).

This is Izmailov’s theorem with specified constant in lower bound. Of course this estimate can be more conservative than ones given by Theorem 1 for systems in companion form. It is of interest to obtain less conservative lower bounds for some particular classes of matrices, for instance, for matrices in Jordan form or for tridiagonal matrices.

Illustrative examples

To illustrate the previous theorems we address a series of examples of control systems in companion form. We choose initial conditions provided by the theorems, they are not necessarily “the worst-case” conditions. The results are presented as infinity-norm of \( x(t) \), thus non-smooth plots are due to relative domination of different states at different time periods.

First consider a two dimensional system. The eigenvalues of the closed loop system matrix are \( \lambda_1 = \lambda_2 = -10 \). The norm \( |x(t)| \) of the solution starting at \( (0, 0) \) is shown in Fig. 2. The true deviation is compared with lower bounds provided by (6) and (12). We see that lower bound (6), shown by horizontal line, is tight (it is specially oriented for equal eigenvalues, as in this example), while estimate provided by Theorem 1 (shown here and in the other examples with an asterisk in a circle) is conservative. However, it approximately predicts the time instant with peaking effect and its value.

When \( n \) increases, the peak effect is strongly exhibited for moderate stability degree. In Fig. 3, one can see that for \( n = 6 \) large deviations are met for \( \sigma = 2 \).

To illustrate Theorem 2 consider the closed loop system with \( \Lambda = \{-0.1, -0.1\} \); norm of the solution starting at \( (0, 0) \) is shown in Fig. 4.

The case of mixed eigenvalues is presented in Fig. 5. Here \( n = 5 \), \( \Lambda = \{-1, -1000, -1000, -1000, -1000\} \) and Theorem 3 can be applied with \( v = 4, \eta = 1000, \xi = 1, n - v + 1 = 2 \). The norm of trajectory with \( x(0) = e_2 \) demonstrates peak effect for \( t \) extremely small.

5. Upper bounds

In contrast with lower bounds, upper bounds for transient process in linear systems with non-zero initial conditions are much better studied, see Balandin and Kogan (2009), Bulgakov (1980), Hinrichsen et al. (2002), Nechepurenko (2002), Van Dorsselaer,
Here, the invariant ellipsoid $V$ is the set of points $x$ such that $x = P^{-1} y$ for some $y$ in the range of $P$. Our main goal is to compare lower and upper bounds (Theorem 7).

If we have a stable closed-loop linear system $\dot{x} = Fx$, then we can try to find the quadratic Lyapunov function $V(x) = \langle P^{-1} x, x \rangle$ such that the invariant ellipsoid $E = \{x : V(x) \leq 1\}$ contains the unit ball $\{x : |x|_2 \leq 1\}$ and has minimal ratio of the semi-major and the semi-minor axes (that is condition number of $P$ is minimal). This is equivalent to solving Semi-Definite Programming (SDP Boyd et al., 1994) problem

$$\min \|P\| \quad \text{s.t.} \quad PF^T + FP \preceq 0, \quad I \preceq P. \tag{21}$$

Here $P = P^T \in \mathbb{R}^{n \times n}$ is matrix variable, $M \preceq 0$ denotes semidefinite matrix $M$, and $\|P\|$ stands for spectral norm of $P$. Then we can guarantee the estimate

$$\max_{t \geq 0} \max_{|x(0)| \leq 1} |x(t)|_2 \leq \|\hat{P}\|. \tag{22}$$

where $\hat{P}$ is the solution of the above SDP. This is a well known approach to get upper bounds for deviations in closed-loop systems.

For open-loop systems we can similarly design a linear feedback in order to guarantee the minimal possible deviations with desired damping. Consider SDP

$$\min \quad PF^T + FP \preceq 0, \quad I \preceq P. \tag{23}$$

$$P(A + bK^T)^T + (A + bK^T)P \preceq -2\sigma P, \tag{24}$$

$$I \preceq P. \tag{25}$$

Lyapunov inequality (24) guarantees that the decay rate of the closed-loop system exceeds $\sigma$ and that the ellipsoid $E = \{x \in \mathbb{R}^n : (x, P^{-1} x) \leq 1\}$ is its invariant set, while conditions (23) and (25) imply that $E$ contains the unit ball $\{x : |x| \leq 1\}$ and has minimal ratio of the semi-major and the semi-minor axes. Inequality (24) is nonlinear with respect to the unknown variables $P$ and $K$. This difficulty can be easily overcome by introducing new variable $Y = PK$.

In terms of variables $P$ and $Y$ (24) reads

$$AP + PA^T + bY^T + Yb^T \preceq -2\sigma P. \tag{26}$$

This is a typical SDP problem and it can be solved numerically (Grant & Boyd, n.d.). Its solution $K = \hat{P}^{-1} \hat{Y}$ can be a good candidate for a feedback with desired stability degree and small deviation for all non-zero initial conditions. Similar approaches can be found in the references mentioned above.

It is of interest to compare lower and upper bounds for systems in companion form with desired decay rate $\sigma$. The lower bounds are given by Theorem 1; having in mind that $\rho \geq \sigma$, estimate (12) implies

$$\max_{\sigma \geq 0} \max_{|x(0)| \leq 1} |x(t)|_2 \geq \gamma_n\sigma^{n-1} = \kappa_n, \quad \gamma_n = \frac{0.3863}{n}. \tag{27}$$

To obtain upper bounds we fix the desired $\sigma$ and solve SDP (23), (25), (26) with its solution $\hat{P}$. Then we guarantee the estimate

$$\max_{\sigma \geq 0} \max_{|x(0)| \leq 1} |x(t)|_2 \leq \|\hat{P}\| = \kappa_n. \tag{28}$$

Minor incompatibility of the bounds is that we used $\infty$-norm for lower bounds and 2-norm for upper bounds; however $\|x\|_2 \leq \|x\|_\infty$ and this leads to more conservative results.

Finally, we calculate the true maximal deviation in 2-norm as

$$\kappa_n = \max_{\sigma \geq 0} \max_{|x(0)| \leq 1} |x(t)|_2 = \max_{\sigma \geq 0} \|e^{F_t}\|_2 \tag{29}$$

where $F$ is closed-loop system matrix with the controller $K = \hat{P}^{-1} \hat{Y}$.

In numerical calculations we fixed $\sigma = 2$, the results are collected in Table 1:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_n$</td>
<td>0.3863</td>
<td>0.5151</td>
<td>0.7726</td>
<td>1.2362</td>
</tr>
<tr>
<td>$\kappa_n$</td>
<td>1.60</td>
<td>5.75</td>
<td>17.18</td>
<td>66.46</td>
</tr>
<tr>
<td>$\kappa_n$</td>
<td>17.94</td>
<td>337.00</td>
<td>6.8 \cdot 10^3</td>
<td>1.47 \cdot 10^4</td>
</tr>
</tbody>
</table>

We see that both lower and upper bounds are highly conservative even for small dimensions $n$.

Another problem of interest is to investigate the closed-loop system obtained by solving upper bound (23), (25), (26) with fixed $\sigma = 2$. For system in canonical form we calculate the matrix $F = A + bK^T$, $K = \hat{P}^{-1} \hat{Y}$, its eigenvalues $\lambda(F)$ and stability degree $\sigma(F) = -\max_i \Re \lambda_i$; they are shown in Table 2 for several $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(F)$</td>
<td>-2.31 ± 2.21i</td>
<td>-18.27</td>
<td>-2.40 ± 1.26i</td>
<td>-2.17 ± 0.91i</td>
</tr>
<tr>
<td>$\sigma(F)$</td>
<td>-2.3 ± 4.3i</td>
<td>-2.4 ± 1.3i</td>
<td>-2.5 ± 1.3i</td>
<td>-2.4 ± 1.3i</td>
</tr>
</tbody>
</table>

We conclude that true stability degree $\sigma(F)$ is close to desired decay rate $\sigma = 2$. Surprisingly the spectrum $\lambda(F)$ exhibits no regularity; eigenvalues are both real and complex and some of them (e.g. for $n = 3$) have large modulus.

(relating to systems in general form and large \(\sigma\)) in terms of quadratic Lyapunov functions \(V(x) = (P^{-1}x, x)\). We denote \(\chi(P)\) the condition number of matrix \(P\), that is \(\chi(P) = \|P\||P^{-1}\| = M/m\), \(M, m\) being the largest and the smallest eigenvalues of \(P\). Obviously large \(\chi(P)\) correspond to ellipsoids \(E = \{x \in \mathbb{R}^n : (x, P^{-1}x) \leq 1\}\) with large ratio of its biggest and smallest half-axis.

**Theorem 6.** For any feedback \(u = K^Tx\) such that the closed-loop matrix \(F = A + bK^T\) has eigenvalues \(\lambda_i\) with \(\Re\lambda_i \leq -\sigma < 0\) and for any \(V(x) = (P^{-1}x, x)\) which is Lyapunov function for closed-loop system \(x = Fx\) the following estimate holds:

\[
\chi(P) \geq \gamma \sigma^{2(n-1)}.
\]  

(30)

Here \(\gamma > 0\) depends on \(A, b\) but not on \(K\).

Indeed for \(|x(0)|^2 = 1\) due to Theorem 6 and the Lyapunov function properties, we have

\[
1/m = (1/m)|x(0)|^2 \geq V(x(0)) \geq \max_{t\leq 0} V(x(t)) \geq (1/M) \max_{t\geq 0} |x(t)|^2 \geq (1/M) \max_{t\geq 0} |x(t)|^2 \geq (1/M) \gamma \sigma^{2(n-1)}.
\]

(31)

Thus condition number of \(P\) can be used for comparison of upper and lower bounds for deviations of the trajectories in 2-norm. Now we are in a position to present the main result of this section.

**Theorem 7.** For controllable system (1) and any \(\sigma > 0\) there exist a feedback \(u = K^Tx\) and a quadratic form \(V(x) = (P^{-1}x, x)\) such that the closed-loop matrix \(F = A + bK^T\) has eigenvalues \(\lambda_i\) with \(\Re\lambda_i \leq -\sigma\) and \(V(x)\) is its Lyapunov function while \(\chi(P) \leq \chi(\sigma + 3/2)^{2(n-1)}\).

(32)

where \(\chi\) depends on \(A, b\) only.

Of course dependence of \(\gamma\) and \(\chi\) on \(n\) for instance for systems in companion form can be different; moreover terms \(\sigma^{2(n-1)}\) and \((\sigma + 3/2)^{2(n-1)}\) in lower and upper bounds (30), (32) are different. However both bounds are of order \(O(\sigma^{2(n-1)})\) for \(\sigma\) large.

6. Conclusions and future research

We provided some results on transient response in linear systems with non-zero initial conditions. First we investigated systems in companion form and obtained lower bounds for maximal deviations in such systems. The bounds depend on pole location of closed-loop systems; large deviations are unavoidable for various locations. The results are extended to systems in general form. Comparison confirms asymptotic equivalence of lower and upper bounds.

However these are just the first steps in the challenging area. We can mention few open problems relating to transient response research:

(1) The results on lower bounds for systems in companion form (Section 3) are formulated for specific eigenvalue locations. But what can be said for arbitrary eigenvalues? How to estimate \(\alpha_L = \inf_{x_0} \sup_{t \geq 0} |x(t)|?\) Here \(x(t)\) is the solution of equation with matrix (10) having spectrum \(\Lambda\. Our initial conjecture was that \(\Lambda = [-\sigma, \ldots, -\sigma]\) with real \(\sigma > 0\) depending on \(n\) is the answer. However this conjecture is false; counterexamples can be constructed for small \(n\). Probably the question has sense only if some restrictions on \(\Lambda\) are added, for example \(|\lambda_i| \leq \omega, i = 1, \ldots, n\). To the best of our knowledge, the problem remains open.

(2) Similar question: what are the best constants in all theorems of Section 3?

(3) Lower bounds for systems in general form (Section 4) can be rather conservative. Are there some classes of systems (beyond canonical) which allow more accurate estimates?

(4) The case of discrete-time systems is challenging. It is well known that for Shur stable matrix \(A\) its powers \(A^k\) can increase at initial iterations. But what is an analog for lower bound results of Section 3?

(5) Lower bounds for deviations of transient response from steady-state outputs for systems with zero initial conditions and harmonic inputs deserve similar analysis. Some results in this direction are presented in Polyak et al. (2015).

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Appendix. Proofs

A.1. Auxiliary results

Assume that all the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of system (2) are different. Consider the Cauchy problem

\[
\dot{\Phi}(t) = F\Phi(t), \quad \Phi(0) = I,
\]

(33)

where \(I\) is the identity matrix. The components of \(\Phi(t)\) are denoted by \(\phi_{ij}(t)\). Recall some results from Smirnov et al. (2009) used below. Denote by \(\mathcal{L}(\Phi)(s)\) the Laplace transform of the matrix \(\Phi(t)\), the solution of problem (33). Applying the Laplace transform to differential equation (33), we obtain \(\mathcal{L}(\Phi)(s) = F\mathcal{L}(\Phi)(s)\). From this we get

\[
\mathcal{L}(\Phi)(s) = (sl - F)^{-1} = \Delta(s)^{-1}M(s),
\]

(34)

where \(M(s)\) is a matrix with polynomial elements and \(\Delta(s) = \det(sl - F) = \prod_{i=1}^n (s - \lambda_i)\). Using the inversion theorem, we have

\[
\Phi(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(st) \frac{M(s)}{\Delta(s)} ds,
\]

(35)

where \(\Gamma\) is a closed curve containing inside all points \(\lambda_k, k = 1, n\). From this we have

\[
\phi_{n,0}(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(st) \sum_{\beta=0}^{k-1} a_{\beta} s^{\beta+\beta-1} ds
\]

(36)

and

\[
\phi_{1,n}(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(st) \frac{ds}{(s - \lambda_1) \ldots (s - \lambda_n)}.
\]

(37)
Expanding the exponential function in series, using the equality
\[(1 - h_1)^{-1} \ldots (1 - h_n)^{-1} = \sum_{q=0}^{\infty} \sum_{q_1 + \ldots + q_n = q} h_1^{q_1} \ldots h_n^{q_n}, \quad |h_j| < 1, j = 1, n,\] (38)
with \(h_j = \lambda_j / \lambda_n\), and calculating the residue at infinity, from (36) and (37) we obtain
\[\phi_{n,l}(t) = \frac{- \sum_{\beta = 1}^{l - 1} \lambda_{1l} \ldots \lambda_{n-l}}{\sum_{\alpha = l - \beta}^{\infty} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha + \beta - 1}} \] (39)
and
\[\phi_{n,1}(t) = \sum_{\alpha = n-1}^{\infty} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha - 1} \lambda_1^{q_1} \ldots \lambda_n^{q_n} \] (40)
respectively. The details can be found in Smirnov et al. (2009).

A.2. Proof of Theorem 1

Assume that all the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of system (2) are different. From (39) we have
\[\phi_{n,1}(t) = -\lambda_1 \ldots \lambda_n \sum_{\alpha = 2}^{\infty} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha - 1} \lambda_1^{q_1} \ldots \lambda_n^{q_n}. \] (41)
Let \(\omega = \max(|\lambda_j| : j = 1, n)\) and \(t_* = \theta / (\omega n)\), \(\theta\) being a parameter. Then we have
\[|\phi_{n,1}(t_*)| \leq \frac{\lambda_1 \ldots \lambda_n}{\omega n} \delta(\theta), \] (42)
where
\[\delta(\theta) = \theta \left[ 1 + \sum_{\alpha = 2}^{\infty} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha - 1} \lambda_1^{q_1} \ldots \lambda_n^{q_n} \right]. \] (43)
The number of terms in the last sum equals
\[\binom{\alpha + n - 2}{n - 1} = \left(1 + \frac{n - 1}{\alpha - 1}\right) \left(1 + \frac{n - 1}{\alpha - 2}\right) \ldots \left(1 + \frac{n - 1}{1}\right) \leq n^{\alpha - 1}. \] (44)
Since \(|\lambda_j| \leq \omega\), we get
\[\delta(\theta) \geq \theta \left[ 1 - \sum_{\alpha = 2}^{\infty} \frac{t^\alpha}{\alpha! (\omega n)^{\alpha - 1}} q_{1+\ldots+q_\alpha = \alpha - 1} \omega^{\alpha - 1} \right] \]
\[\geq \theta \left[ 1 - \sum_{\alpha = 2}^{\infty} \frac{t^\alpha}{\alpha! (\omega n)^{\alpha - 1}} \omega^{\alpha - 1} \right] = 2 \theta + 1 - e^\theta. \] (45)
A.3. Proof of Theorem 2

Assume that all the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of system (2) are different. From (40) we obtain
\[\phi_{1,n}(t) = \sum_{\alpha = 0}^{n-1} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha - n + 1} \lambda_1^{q_1} \ldots \lambda_n^{q_n} \]
\[= \frac{t^{n-1}}{(n-1)!} + \sum_{\alpha = 0}^{\infty} \frac{t^\alpha}{\alpha!} q_{1+\ldots+q_\alpha = \alpha - n + 1} \lambda_1^{q_1} \ldots \lambda_n^{q_n}. \] (46)
Let \(\omega = \max(|\lambda_j| : j = 1, n)\), \(t_* = \theta / (\omega n)\), where parameter \(\theta\) will be specified later. Then, as in the previous proof, we have
\[|\phi_{1,1}(t_*)| \geq \frac{\theta^{n-1}}{\omega^{n-1}(n-1)!} \sum_{\alpha = n}^{\infty} \frac{\theta^\alpha}{\alpha!} (n - 1) \]
\[= \frac{\theta^{n-1}}{\omega^{n-1}(n-1)!} \left[1 - \sum_{\alpha = 0}^{\infty} \frac{\theta^\alpha}{(\alpha - n + 1)!}\right]. \]
\[= \frac{\theta^{n-1}}{\omega^{n-1}(n-1)!} (2 - e^\theta). \] (47)
Since \(2 - e^\theta \geq 1 - \theta / \log 2\), whenever \(0 \leq \theta \leq \log 2\), maximizing the function \(\psi(\theta) = \theta^{n-1} (1 - \theta / \log 2)\) we obtain
\[\theta^* = \arg \max \psi(\theta) = \frac{n - 1}{n} \log 2, \quad \psi(\theta^*) \geq \frac{(\log 2)^{n-1}}{en}. \] (48)
This leads to (13). Thus, the theorem is proved for different \(\lambda_1, \ldots, \lambda_n\). We conclude as in the previous proof. \(\square\)

A.4. Proof of Theorem 3

We need the following auxiliary estimate.

Lemma 1. Assume that \(\Re\lambda_j < 0, j = 1, \ldots, n,\) and (14) is satisfied. Then the following inequality holds
\[\left| \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k} \right| \geq |\lambda_1 \ldots \lambda_n| \geq \eta^n. \] (49)

Proof. Since the coefficients of the characteristic polynomial \(\Delta(\lambda)\) are real, the roots have the following structure: \(\lambda_j \in R, j \in J = \{j_1, \ldots, j_k\},\) and \(\lambda_j = \alpha_j + \beta_j, \alpha_j, \beta_j \in R, j \in J.\) (The set \(J\) can be empty or can coincide with the set of all indices.) We obviously have
\[\Delta(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) = \lambda^n + \ldots + (\lambda - \alpha_j) \lambda^n - \ldots + (\lambda - \alpha_j) \lambda^n - \ldots + \left(\lambda - \alpha_j\right)^n = \prod_{r=1}^{\mu} (\lambda - \lambda_{j_r}) \prod_{r=1}^{\mu} (\lambda - \lambda_{j_r}) \left(\lambda - \alpha_j\right)^2 + \beta_j^2 \]
\[= \prod_{r=1}^{\mu} (\lambda - \lambda_{j_r}) \prod_{r=1}^{\mu} (\lambda - \lambda_{j_r}) \left(\lambda - \alpha_j\right)^2 + 2\lambda |\alpha_j|^2. \] (50)
From this we see that the coefficient of $\lambda^{n-v}$ of the polynomial $\Delta(\lambda)$ is a sum of positive terms and therefore has the form
\[(−1)^{v} \sum_{1 \leq i_1 < \ldots < i_v \leq n} \lambda_{i_1} \ldots \lambda_{i_v} = |\lambda_1 \ldots \lambda_v| + \delta, \quad (51)\]
where $\delta > 0$. \(\square\)

**Proof of Theorem 3.** Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are different. Set $l = n - v + 1$. Then from (39) we have
\[\phi_{n,n-v+1}(t) = \sum_{1 \leq i_1 < \ldots < i_v \leq n} \lambda_{i_1} \ldots \lambda_{i_v} t^{l - v} \sum_{\alpha \geq 2} \alpha! \sum_{q_1 + \ldots + q_n = 0} \lambda_q^1 \ldots \lambda_q^n. \quad (52)\]

Since $\exp(a) - (1 + a) \leq \exp(a^2)/2, a \geq 0$, we get
\[\sum_{\alpha \geq 2} \alpha! \sum_{q_1 + \ldots + q_n = 0} \lambda_q^1 \ldots \lambda_q^n \leq \exp(\alpha a - (1 + a)a^2/2), \quad (53)\]

(see the proof of Theorem 1). Similarly, we obtain
\[\sum_{\alpha \geq 2} \alpha! \sum_{q_1 + \ldots + q_n = 0} \lambda_q^1 \ldots \lambda_q^n \leq \alpha^{n-v-1} \exp(\alpha \theta - (1 + \alpha)\theta^2/2), \quad (54)\]

whenever $\beta \leq n - v - 1$ and $t < 1$. Using Lemma 1 and assumption (14) we get
\[\sum_{\beta = 0}^{n-v-1} \sum_{1 \leq i_1 < \ldots < i_v < \ldots < i_{n-\beta}} \lambda_{i_1} \ldots \lambda_{i_{n-\beta}} \leq \exp(\theta t^2/2), \quad (55)\]

Let $\zeta$ and $\bar{\zeta}$ be the points of intersection of the circumference $|s| = \omega + (\eta + \xi)/2$ with the line $\text{Res} = -(\eta + \xi)/2$. Consider the closed curves $\Gamma_1 = \{s \mid |s| = \omega + (\eta + \xi)/2, \text{Res} < -(\eta + \xi)/2 \cup [\zeta, \xi] \}$ and $\Gamma_2 = \{s \mid |s| = \xi, \text{Res} < 0 \cup [-\xi, \xi] \}$ (see Fig. 6).

From (40) we have $\phi_{1,n}(t) = I_1 + I_2$, where
\[I_k = \frac{1}{2\pi i} \int_{\Gamma_k} \exp(st) \frac{ds}{(s - \lambda_1) \ldots (s - \lambda_n)}, \quad k = 1, 2. \quad (57)\]

Obviously, we have
\[|I_1| \leq \frac{\pi (\omega + (\eta + \xi)/2)}{2\pi} e^{-((\eta + \xi)/2)^2}, \quad (58)\]

Taking $t = 1/\xi$ and having in mind that due to (18) $\eta - \xi \geq 2$ and $\xi < 1$, we obtain
\[|I_1| \leq c_1 \omega, \quad c_1 = 0.75e^{-2}. \quad (59)\]

Now let us estimate $I_2$. Expanding the exponential function and using (38), we get $I_2 = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\sum_{q_0, a_0, a_1, \ldots, a_{n-1}} \omega^q \prod_{j=1}^v (s - \lambda_j)^{a_j} ds}{\prod_{j=1}^v (s - \lambda_j)}$. \[\quad (60)\]
Calculating the residue at \( s = 0 \), we get
\[
I_2 = \frac{(-1)^v}{\prod \lambda_j} \sum_{a_j=0}^{v} \frac{r^a}{\alpha!} \lambda_{v+1}^{q_{v+1}} \ldots \lambda_n^{q_n}.
\]

(61)

Set \( t = 1/\xi \). Then we have
\[
I_2 = \frac{(-1)^v}{\prod \lambda_j} \frac{1}{(n-v-1)!} \sum_{a_j=0}^{n-v} \frac{(\frac{1}{\xi})^{n-v+1}}{\alpha!} \lambda_{v+1}^{q_{v+1}} \ldots \lambda_n^{q_n}. \]

(62)

Observe that
\[
\sum_{a_j=0}^{n-v+1} \frac{(\frac{1}{\xi})^{n-v+1}}{\alpha!} \lambda_{v+1}^{q_{v+1}} \ldots \lambda_n^{q_n} \leq \sum_{\alpha \geq n-v} \frac{(\frac{1}{\xi})^{n-v+1}}{\alpha!} \frac{\alpha^{n-v+1}}{\alpha} = \frac{e}{(n-v)!}.
\]

(63)

Hence we get
\[
|I_2| \geq \frac{c_2}{\xi^{n-v-1} \alpha! (n-v-1)}.
\]

(64)

where \( c_2 \geq 1 - e/(n-v) \geq 1 - e/3 \). Finally, we have
\[
|\phi_{s_1}(1/\xi)| \geq |I_2| - |I_1|.
\]

(65)

Combining the last inequality with (59) and (18) we obtain the result. \( \square \)

### A.6. Proof of Theorem 7

We will need a few auxiliary lemmas. Below \( s \in \mathbb{R} \) does not belong to spectrum of \( A \); it will be specified later (see (90)).

**Lemma 2.** The vectors \( b_k = (A - sl)^{-k}b, k = 0, n - 1 \), form a basis in \( K^s \).

**Proof.** Indeed, then the vectors \( b_k \) are well-defined. It suffices to show that they are linearly independent. Assume that \( 0 = \sum_{k=0}^{n-1} \beta_k b_k = \sum_{k=0}^{n-1} \beta_k (A - sl)^{-k}b \). Multiplying this equality by \( (A - sl)^{-1} \) we get
\[
0 = \sum_{k=0}^{n-1} \beta_k (A - sl)^{-k-1}b = \sum_{k=0}^{n-1} \beta_k (A - sl)^{-k-1}b.
\]

(66)

Since the vectors \( A^j b, j = 0, n - 1 \) are linearly independent, we have
\[
\sum_{k=0}^{n-1} \beta_k \left( \begin{array}{c} n - k - 1 \\ j \end{array} \right) (-s)^{n-k-1} A^j b = 0, j = 0, n - 1.
\]

\[
\sum_{k=0}^{n-1} \beta_k \left( \begin{array}{c} n - k - 1 \\ j \end{array} \right) (-s)^{n-k-1} A^j b = 0, j = 0, n - 1.
\]

Taking successively \( j = n - 1, n - 2, \ldots \), we obtain \( \beta_0 = 0, \beta_1 = 0, \) etc. \( \square \)

Any vector \( x \in \mathbb{R}^n \) can be represented as \( x = \sum_{m=0}^{n-1} \beta_m b_m \) and as
\[
x = \sum_{m=0}^{n-1} \gamma_m A^m b.
\]

(68)

The vectors \( \beta = (\beta_0, \ldots, \beta_{n-1}) \) and \( \gamma = (\gamma_0, \ldots, \gamma_{n-1}) \) satisfy the equality
\[
\beta(s, \gamma) = M(s)\gamma,
\]

(69)

where \( M(s) \) is an \((n \times n)\)-matrix.

**Lemma 3.** The following representation takes place:
\[
\beta_k(s, \gamma) = \sum_{r=0}^{k} \beta_r^k(-\gamma)^r, \quad k = 0, n - 1.
\]

(70)

**Proof.** Fix \( \gamma \neq 0 \) and set \( \beta_k(s) = \beta_k(s, \gamma), k = 0, n - 1 \). Multiplying the equality
\[
\sum_{k=0}^{n-1} \beta_k(A - sl)^{-k}b = \sum_{m=0}^{n-1} \gamma_m A^m b
\]

(71)

by \((A - sl)^{n-1}\) we get
\[
\sum_{k=0}^{n-1} \beta_k(A - sl)^{n-k-1}b = \sum_{m=0}^{n-1} \gamma_m (A - sl)^{n-1} A^m b
\]

(72)

or, equivalently,
\[
\sum_{k=0}^{n-1} \sum_{m=0}^{n-1} \beta_k(b)(\gamma)^{n-k-1} m \left( \begin{array}{c} n - k - 1 \\ m \end{array} \right) A^m b
\]

(73)

Switching the order of summation we have
\[
\sum_{m=0}^{n-1} \sum_{k=0}^{n-1} \beta_k \left( \begin{array}{c} n - k - 1 \\ m \end{array} \right) (-s)^{n-k-1} A^m b
\]

(74)

where the degree of the polynomials \( P_m(s) = \sum_{m=0}^{n-1} \gamma_m A^m b \) is less than or equal to \( n - 1 \). From this we successively determine \( \beta_0, \beta_1, \) etc. The functions \( \beta_k = \beta_k(s), k = 0, n - 1 \), have a form of polynomials
\[
\beta_k(s) = \sum_{r=0}^{n-1} \beta_r^k(-s)^r.
\]

(75)

From
\[
\sum_{m=0}^{n-1} \sum_{k=0}^{n-1} \beta_k \left( \begin{array}{c} n - k - 1 \\ m \end{array} \right) (-s)^{n-r-k-1} A^m b
\]

(76)

we see that the degree of polynomials
\[
\sum_{k=0}^{n-1} \sum_{r=0}^{n-1} \beta_k \left( \begin{array}{c} n - k - 1 \\ m \end{array} \right) (-s)^{n-r-k-1}
\]

(77)
is also less than or equal to \( n - 1 \). Set \( m = n - q, q = \frac{1}{n} \). We see that the degree of the polynomials
\[
\sum_{k=0}^{n-1} \sum_{r=0}^{k} \beta_k^r \left( \frac{n - k - 1}{n - q} \right) (-s)^{r+k-1}
\] (78)
does not exceed \( n - 1 \). Considering \( q = \frac{1}{n} \), we get \( \beta_0^r = 0 \) whenever \( r > n - q \). Then considering \( q = \frac{n}{n} \), we get \( \beta_1^r = 0 \) for \( r > n - q + 1 \), and so on. Finally, we obtain \( \beta_k^r = 0 \) whenever \( r > k, k = 0, n - 1 \). Thus we have (70). □

As a special case of this lemma we get

**Lemma 4.** The coordinates \( \alpha_k, k = 0, n - 1 \), of the vector \( Ab \) with respect to the basis \( b_k = (A - sI)^{-k}b \), \( k = 0, n - 1 \), are \( O(|s|^{n-1}) \) as \(|s| \to \infty\).

Recall that by matrix norm we intend the norm induced by the vector norm.

**Lemma 5.** The norm of the matrix \( H^{-1}(s) \) is \( O(1) \) as \(|s| \to \infty\).

**Proof.** Indeed, from Lemma 3 we see that \( \beta_k(s, y) s^{-k-m} = O(1), |s| \to \infty, m = \frac{1}{n}, n - 1 \). Dividing (73) by \((-s)^{n-1}\) we obtain
\[
\beta_0 b + c(s) = \sum_{m=0}^{n-1} \gamma_m A^m b + \sum_{m=0}^{n-1} \gamma_m d_m(s),
\] (79)
where \( |c(s)| = O(1) \) and \( |d_m(s)| = O(1), m = 0, n - 1 \), as \(|s| \to \infty\). From this we get
\[
\gamma_m = \left\{ \begin{array}{ll}
\beta_0 + o(1), & m = 0, \\
o(1), & m = 1, n - 1.
\end{array} \right.
\] (80)

**Proof of Theorem 7.** By Lemma 2 the vectors \( b_k = (A - sI)^{-k}b, k = 0, n - 1 \), form a basis in \( R^n \). Let \( x = \sum_{k=0}^{n-1} \beta_k b_k \) and \( Ab = \sum_{k=0}^{n-1} \alpha_k b_k \). Choose linear feedback \( u(x) = u(\beta) = s\beta_0 - (\beta_1 + \beta_0\alpha_0) \). Since \( x = \sum_{k=0}^{n-1} \beta_k b_k \) and
\[
A x + u(x)b = A \sum_{k=0}^{n-1} \beta_k b_k + (s\beta_0 - (\beta_1 + \beta_0\alpha_0))b
\] (81)
the system \( \dot{x} = Ax + u(x)b \) in the coordinates \( \beta \) takes the form
\[
\dot{\beta}_0 = s\beta_0,
\] (82)
\[
\dot{\beta}_1 = s\beta_1 + \beta_2 + \beta_0\alpha_1,
\] (83)
\[
\ldots
\] (84)
\[
\dot{\beta}_{n-2} = s\beta_{n-2} + \beta_{n-1} + \beta_0\alpha_{n-2},
\] (85)
\[
\dot{\beta}_{n-1} = s\beta_{n-1} + \beta_0\alpha_{n-1}.
\] (86)

Put
\[
\hat{\delta}^2 = (n - 1) \max_{k=1, n-1} \alpha_k^2
\] (87)
and
\[
W(\beta) = \sum_{k=1}^{n-1} \beta_k^2 + \delta^2 \beta_0^2.
\] (88)

Then we see that \( dW(\beta)/dt \) is equal to
\[
= 2sW(\beta) + 2 \sum_{k=1}^{n-2} (\beta_k \beta_{k+1} + \beta_k \beta_0 \alpha_k) + \beta_{n-1} \beta_0 \alpha_{n-1}
\leq 2sW(\beta) + \sum_{k=1}^{n-2} (\beta_k^2 + \beta_{k+1}^2) + \sum_{k=1}^{n-1} \left( \alpha_k^2 \beta_0^2 (n - 1) + \delta^2 \beta_0^2 \right)
\leq (2s + 3W(\beta)).
\] (89)

Now fix
\[
s = -\sigma - 3/2,
\] (90)
then the last inequality becomes
\[
\dot{W}(\beta) \leq -2\sigma W(\beta).
\] (91)

We introduced Lyapunov function \( W(\beta) \) in variables \( \beta \), now convert it into \( V(x) \). Recall that \( x = H^{-1}(s) y \) (see (68)), where \( H = [bAb \ldots A^{n-1}b] \) is the controllability matrix. Due to controllability assumption \( H^{-1} \) exists and both \( H, H^{-1} \) do not depend on \( \sigma \). On the other hand \( \beta = M(s)y \) (see (69)), and \( \|M(s)\| = O(|s|^{n-1}) \leq c_1(|\sigma + 3/2|^{n-1}) \) (see (70)), while \( \|M(s)\| = O(1) \) (Lemma 5). We have \( W(\beta) = (Q\beta, \beta) \), where \( Q = \text{diag}(\delta^2, 1, \ldots, 1) \) (88), with \( \delta^2 = O(\sigma^2) \) (see (87)). From Lemma 4 we conclude that the condition number satisfies the inequality \( \chi(Q) \leq c_2(\sigma + 3/2)^{2(n-1)} \). Collecting all these relations we obtain
\[
W(\beta) = W(MH^{-1}x) = V(x) = (P^{-1}x, x),
\] (92)
with
\[
P^{-1} = (H^{-1})^\dagger M^T Q M H^{-1},
\] (93)
and
\[
\chi(P) \leq c_1(\sigma + 3/2)^{2(n-1)}.
\] (94)

Final remark is that inequality
\[
\dot{V}(x) \leq -2\sigma V(x)
\] (95)
for quadratic Lyapunov function \( V(x) \) and linear system \( \dot{x} = Fx \) implies \( \text{Re} \lambda_i \leq -\sigma \) for all eigenvalues \( \lambda_i \) of the matrix \( F \).

Thus we have constructed linear feedback and quadratic Lyapunov function with the desired properties. □

**References**


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