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Brief paper

Nussbaum functions in adaptive control with time-varying unknown control coefficients☆

Zhiyong Chen

School of Electrical Engineering and Computing, The University of Newcastle, Callaghan, NSW 2308, Australia

A R T I C L E I N F O

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A B S T R A C T

Nussbaum functions have been successfully used in adaptive controller design for dealing with unknown control direction since the original work in 1983. However, for time-varying control coefficients of an unknown sign (positive or negative), only a special Nussbaum function can be proved to be effective based on the explicit calculation on the particular function. It remains open whether a general Nussbaum function is sufficient in these scenarios and why. This paper gives a No answer with a counter example. Moreover, it introduces new types of Nussbaum functions and reveals their fundamental characteristics that are sufficient for dealing with time-varying unknown control coefficients in adaptive control. A multivariable version is also introduced.

1. Introduction

In a control system, when the sign of control coefficient is unknown, it is hard to design a controller because a control force of wrong direction may deteriorate the system away from the desired behavior. An interesting idea is to alternatively (periodically in most scenarios) change the sign of control force in an adaptive design. In the period with a wrong direction, the controller degrades the system but it would reward the system with quicker movement to the desired state with a higher gain when the sign is changed to a correct direction in the subsequent period. The success of the approach relies on an art of increasing the controller gain and the Nussbaum function invented in Nussbaum (1983) has offered such an art.

Over the past three decades, the Nussbaum gain technique has been extensively used for handling an unknown control direction in many papers. A fundamental tool is the technical lemma that guarantees the boundedness of a Lyapunov-like energy function when its derivative along a system is upper bounded by a Nussbaum function based manner. Most of these papers are mainly concerned about construction of such Lyapunov-like functions for various systems with appropriately designed controllers. Hence, applying the lemma concludes the desired behaviors.

The first complete formulation of such a lemma was given in Ye and Jiang (1998) for the case that the control coefficient is a *constant* whose sign or value is unknown; see Lemma 3.1. This lemma gives a clear picture how a Nussbaum function sufficiently works to guarantee boundedness of a Lyapunov-like function. In many other situations, researchers must deal with a *time-varying* control coefficient whose sign or value is unknown. A natural question has arisen whether the lemma still works to guarantee boundedness of a Lyapunov-like function. However, the question has been partially answered only for some particular Nussbaum functions. For instance, the result in Ye (1999) was achieved with the note that "Throughout this paper, we choose $v(\xi) = \cos(\pi\xi/2)\exp(\xi^2)$"; and that in Ge, Hong, and Lee (2004) with "$N(\zeta) = e^{\zeta^2}\cos((\pi/2)\zeta)$ is used throughout this paper". The proofs given in Ge et al. (2004) and Ye (1999) critically rely on the explicit calculation for the particularly chosen Nussbaum functions.

The aforementioned lemma in Ye and Jiang (1998) for a constant unknown control direction has been applied in many papers including Ge and Wang (2002), Jiang, Mareels, Hills, and Huang (2004), Liu and Tong (2017), Ramezani, Arefi, Zargarzadeh, and Jahed-Motlagh (2016), Xu and Huang (2010) and Pongvuthithum, Rattanamongkhonkun, and Lin (2018), while the techniques for time-varying control coefficients have let to many other applications in Bechlioulis and Rovithakis (2009), Liu and Huang (2006), Zhou, Wen, and Zhang (2005) and Xu, Qi, Jiang, and Yao (2017), etc., which were unavoidably on the same selection of particular Nussbaum functions. The applications include neuro-adaptive backstepping control, output regulation, fault tolerant control, etc. It remains open whether a general Nussbaum function is sufficient in time-varying scenarios and why. This paper gives a No answer with a counter example.

When multiple control inputs with unknown control coefficients are considered, for instance, in multiagent systems or interconnected large scale systems, the *multivariable* version of

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E-mail address: zhiyong.chen@newcastle.edu.au.

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the aforementioned lemma aims to guarantee boundedness of a Lyapunov-like function when its derivative is upper bounded by summation of multiple Nussbaum function based manners. Again, the lemma has been proved only for some particularly chosen Nussbaum functions. For instance, $N_0(k) = \cosh(\lambda k)\sin(k)$ with a lower bound condition on λ is used in Chen, Li, Ren, and Wen (2014), $N(k) = e^{k^2/2}(k^2+2)\sin(k)$ in Ding (2015), and $N(k) = e^{k^2}\cos(k)$ in Wang, Wen, and Lin (2017). The proofs again are critically based on the explicit calculation of the particularly chosen Nussbaum functions. The current situation is as pointed out by the authors of Chen et al. (2014) that "it is not clear how to use the existing (general) Nussbaum-type functions to prove (the lemma)".

The main contribution of this paper is not only to give a clear No answer to the long standing question. More importantly, it introduces an enhanced version of Nussbaum function and reveals its fundamental characteristics that are sufficient for dealing with time-varying and/or multivariable unknown control coefficients using clear-cut proofs that are not necessarily based on explicit calculation of a particular function. The necessity of using the enhanced version of Nussbaum function is also discussed. The new results provide researchers with a better picture how a general Nussbaum function applies in the scenarios that are more complicated than a scalar constant control coefficient case.

2. Origin of Nussbaum function in parameter adaptive control

A Nussbaum gain used in adaptive control with an unknown control direction was named after the original work by R. D. Nussbaum in Nussbaum (1983). This paper starts with the following theorem cited from the paper. Throughout the paper, $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ the set of non-negative real numbers.

Theorem 2.1 (*Nussbaum, 1983*). *The following system*

$$\begin{aligned}\dot{x} &= ax + \lambda x\sigma(y)h(y),\ \ \lambda \neq 0\\ \dot{y} &= x\sigma(y),\end{aligned} \tag{1}$$

with $\sigma(y) = y^2 + 1$, has the property that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t)$ exists and bounded provided that the function $h(s) : \mathbb{R} \mapsto \mathbb{R}$ is even and differentiable and satisfies the properties:

$$\limsup_{y\to\infty} \int_0^y h(s)ds = \infty, \tag{2}$$

$$\liminf_{y\to\infty} \int_0^y h(s)ds = -\infty. \tag{3}$$

The theorem in Nussbaum (1983) was proved by explicitly "solving" the system with thinking of x as a function of y. The method based on explicit solution of a dynamic system cannot be extended to general situations. An alternative proof is given below using a technical lemma that reveals more essential mechanism and motivates the research in this paper. The proof of the following lemma is similar to that of Lemma 4.3.

Lemma 2.1. *Consider two continuously differentiable functions $V(t) : [0,\infty) \mapsto \mathbb{R}^+$, $y(t) : [0,\infty) \mapsto \mathbb{R}^+$. If*

$$\dot{V}(t) \le (\lambda h(y(t)) + a/\sigma(y(t)))\dot{y}(t) \tag{4}$$

for two constants $a, \lambda \in \mathbb{R}$ and two functions σ and h satisfying the following properties

$$\sigma(s) > 0,\ \ \forall s \in \mathbb{R} \tag{5}$$

$$\limsup_{y\to\infty} \frac{\int_0^y h(s)ds}{\int_0^y 1/\sigma(s)ds} = \infty, \tag{6}$$

$$\liminf_{y\to\infty} \frac{\int_0^y h(s)ds}{\int_0^y 1/\sigma(s)ds} = -\infty, \tag{7}$$

then $V(t)$ and $y(t)$ are bounded over $[0,\infty)$.

An alternative proof of Theorem 2.1: Without loss of generality, we consider the case with $x(0) \ge 0$. It is obvious to see that $x(t) \ge 0$, $\forall t \ge 0$. Define a function $V(x) = x$ whose derivative along the trajectory of the system is

$$\dot{V}(x) = x(\lambda\sigma(y)h(y) + a) = (\lambda h(y) + a/\sigma(y))\dot{y}.$$

The properties (2) and (3) imply (6) and (7), noting the fact that

$$\int_0^{+\infty} 1/\sigma(s)ds = \arctan(+\infty) - \arctan(0) = \pi/2.$$

Then, applying Lemma 2.1 gives that $V(t)$ and $y(t)$ are bounded over $[0,\infty)$. It can be seen that $\ddot{y}(t)$ is bounded. Thus, $\dot{y}(t)$ is uniformly continuous over $[0,\infty)$. When a Nussbaum gain is used in adaptive control, the function $y(t)$ is usually designed with $\dot{y}(t) \ge 0$, $\forall t \ge 0$, e.g., in (1). Therefore, $y(t)$ has a finite limit as $t \to \infty$. By Barbalat's Lemma, one has $\lim_{t\to\infty} \dot{y}(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$. Theorem 2.1 is thus proved. □

3. Motivating examples

Lemma 2.1 was not explicitly given in Nussbaum (1983), but it reveals why a function satisfying the properties (6) and (7) are effective in solving an adaptive control problem with a scalar unknown but constant control coefficient. The function $\sigma(s) = s^2 + 1 > 0$ was used in the original system (Nussbaum, 1983). If we select $\sigma(s) = 1 > 0$, the properties (6) and (7) reduce to those more commonly used in literature, specifically, in the following definition; see, e.g., Ye and Jiang (1998).

Definition 3.1. A continuously differentiable function $h(s) : [0,\infty) \mapsto (-\infty,\infty)$ is called a *Nussbaum function (type A)* if it satisfies

$$\limsup_{y\to\infty} \frac{1}{y}\int_0^y h(s)ds = \infty, \tag{8}$$

$$\liminf_{y\to\infty} \frac{1}{y}\int_0^y h(s)ds = -\infty. \tag{9}$$

Also, with $\sigma(s) = 1$, Lemma 2.1 reduces to the following version that was first proved in Ye and Jiang (1998) and later used in Ge and Wang (2002), Jiang et al. (2004), Xu and Huang (2010) and Ramezani et al. (2016), etc.

Lemma 3.1 (*Ye & Jiang, 1998*). *Consider two continuously differentiable functions $V(t) : [0,\infty) \mapsto \mathbb{R}^+$, $y(t) : [0,\infty) \mapsto \mathbb{R}^+$. If*

$$\dot{V}(t) \le (\lambda h(y(t)) + a)\dot{y}(t) \tag{10}$$

for two constants $a, \lambda \in \mathbb{R}$ and a Nussbaum function (type A) h, then $V(t)$ and $y(t)$ are bounded over $[0,\infty)$.

Example 3.1. We use the following class of nonlinear systems as a test platform to demonstrate the application of Lemma 3.1:

$$\begin{aligned}\dot{z} &= q(z,x,w)\\ \dot{x} &= f(z,x,w) + \lambda u\end{aligned} \tag{11}$$

where $z \in \mathbb{R}^n$ and $x \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the input, and $w \in \mathbb{W}$ represents time-varying uncertainties with $\mathbb{W}$ a compact subset of $\mathbb{R}^l$. The functions q and f are continuously differentiable with $q(0,0,w) = 0$ and $f(0,0,w) = 0$ for all $w \in \mathbb{W}$. The stabilization problem for this class of nonlinear systems, even of a more complicated lower triangular form, has been widely studied in Jiang and Mareels (1997), Lin and Gong (2003), Willems and Byrnes (1984) and Chen and Huang (2015) and many others. When the constant λ represents an unknown control direction, one solution is revisited as follows.

Let h be a Nussbaum function (type A) and consider a controller

$$\begin{aligned}u &= h(y)\rho(x)x\\ \dot{y} &= \epsilon\rho(x)x^2,\ \ \epsilon > 0.\end{aligned} \tag{12}$$

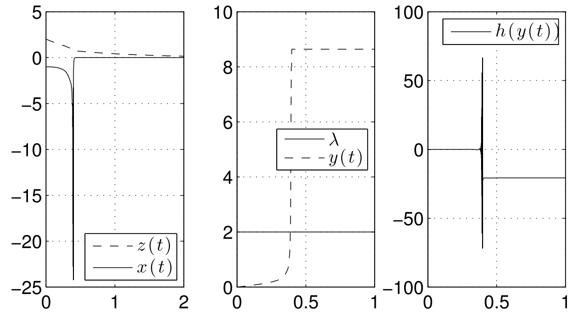

Fig. 1. Stable profile of the closed-loop system under the controller with a Nussbaum function (type A) (the control direction $\lambda = 2$).

With a certain minimum phase assumption and proper selection of $\rho(x)$, we can explicitly construct a Lyapunov function $V(z, x)$ such that

$$\dot{V}(z, x) \le (\lambda h(y) + a)\epsilon\rho(x)x^2 = (\lambda h(y) + a)\dot{y} \tag{13}$$

for some $a > 0$.

From Lemma 3.1, $V(z(t), x(t))$ and $y(t)$ are bounded over $[0, \infty)$, so are $z(t)$ and $x(t)$. Also, it can be seen that $\ddot{y}(t)$ is bounded. Thus, $\dot{y}(t)$ is uniformly continuous over $[0, \infty)$. By Barbalat's Lemma, one has $\lim_{t\to\infty} \dot{y}(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. So, the global stabilization problem for the system (11) is solved by the controller (12).

Numerical simulation is conducted for

$$q(z, x, w) = -z + w_3 x$$
$$f(z, x, w) = w_1 z \cos x + w_2 x^3$$

with uncertainty $w = \text{col}(w_1, w_2, w_3)$. The controller (12) is with $\rho(x) = x^2 + 5$, $\epsilon = 0.1$, and a Nussbaum function (type A) $h(s) = \sin(3\pi s)s^2$. The performance is shown in Fig. 1 for $\lambda = 2$. The gain $y(t)$ increases to a sufficiently large finite number, with which the plant state asymptotically approaches the equilibrium point. The profile for the case with $\lambda = -2$ is similar and thus not repeated. □

Lemma 3.1 holds for a constant λ that represents an unknown control direction. It is interesting to ask whether the same result still works for a time-varying control direction, i.e., $\lambda(t)$. It motivates the following conjecture.

Conjecture. *Consider two continuously differentiable functions $V(t) : [0, \infty) \mapsto \mathbb{R}^+$, $y(t) : [0, \infty) \mapsto \mathbb{R}^+$. Let $\lambda(t) : [0, \infty) \mapsto [\underline{\lambda}, \bar{\lambda}]$ for two constants $\underline{\lambda}$ and $\bar{\lambda}$ satisfying $\underline{\lambda}\bar{\lambda} > 0$. If*

$$\dot{V}(t) \le (\lambda(t)h(y(t)) + a)\dot{y}(t)$$
$$\dot{y}(t) \ge 0, \ \forall t \ge 0 \tag{14}$$

for a constant a and a Nussbaum function (type A) h, then $V(t)$ and $y(t)$ are bounded over $[0, \infty)$.

The conjecture was proved in Ge et al. (2004) and Ye (1999) and later used in many other papers but only for a particular Nussbaum function of the form $h(s) = \sin(s)\exp(s^2)$. It remains open whether it always holds for a general Nussbaum function. The answer is No as shown by a counter example (Example 5.4) after the properties of a Nussbaum function are deeply investigated, following some preliminary attempt in Chen and Huang (2015). Here, we first show some numerical observation in the example below.

Example 3.2. Consider the same numerical example studied in Example 3.1 but with a time-varying control direction function $\lambda(t)$. Following the same arguments in Example 3.1, the global stabilization problem for the system (11) is solved by the same controller (12) with the same Nussbaum function, if the conjecture is true. However, we can find a function $\lambda(t)$ such that the closed-loop system is unstable under the proposed controller. This contradiction shows that the conjecture is not always true. The unstable performance is shown in Fig. 2 where $\lambda(t) = 2 + 0.1\sin(3\pi y(t)) \in [1.9, 2.1]$ is used. □

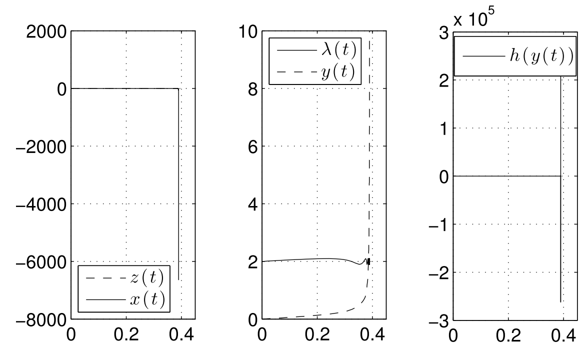

Fig. 2. Unstable profile of the closed-loop system under the controller with a Nussbaum function (type A) (the control direction $\lambda(t)$ is time-varying).

4. New types of Nussbaum functions

As observed from Example 3.2 (and a rigorous counter example given later), the conjecture is false, that is, a Nussbaum function (type A) does not always apply for a time-varying control direction. In literature, only the special Nussbaum function of the form $h(s) = \sin(s)\exp(s^2)$ was proved to be applicable based on the explicit calculation on the function. In this section, we aim to reveal the additional properties held by $\sin(s)\exp(s^2)$ that validates the aforementioned conjecture.

For a function $h(s) : [0, \infty) \mapsto (-\infty, \infty)$, denote its positive and negative truncated functions by $h^+(s)$ and $h^-(s)$, i.e.,

$$h^+(s) = \max\{0, h(s)\}, \ \ h^-(s) = \max\{0, -h(s)\}.$$

Obviously, the truncated functions satisfy the following properties

$$h^+(s) \ge 0$$
$$h^-(s) \ge 0$$
$$h(s) = h^+(s) - h^-(s).$$

Based on these functions, a Nussbaum function (type A) can be rewritten in an equivalent way. In particular, it is easy to check that Eqs. (8) and (9) are equivalent to (15) and (16), respectively.

Definition 4.1. A continuously differentiable function $h(s) : [0, \infty) \mapsto (-\infty, \infty)$ is called a *Nussbaum function (type A)* if it satisfies

$$\limsup_{y\to\infty} \frac{\int_0^y h^+(s)ds - \int_0^y h^-(s)ds}{y} = \infty, \tag{15}$$

$$\limsup_{y\to\infty} \frac{\int_0^y h^-(s)ds - \int_0^y h^+(s)ds}{y} = \infty. \tag{16}$$

Next, we introduce another type of Nussbaum function.

Definition 4.2. A continuously differentiable function $h(s) : [0, \infty) \mapsto (-\infty, \infty)$ is called a *Nussbaum function (type B-L)*, if, for a constant $L > 1$, it satisfies

$$\lim_{y\to\infty} \frac{\int_0^y h^+(s)ds}{y} = \infty, \ \ \limsup_{y\to\infty} \frac{\int_0^y h^+(s)ds}{\int_0^y h^-(s)ds} \ge L, \tag{17}$$

$$\lim_{y\to\infty} \frac{\int_0^y h^-(s)ds}{y} = \infty, \ \ \limsup_{y\to\infty} \frac{\int_0^y h^-(s)ds}{\int_0^y h^+(s)ds} \ge L. \tag{18}$$

Obviously, if L can be selected as infinity, a Nussbaum function (type B-∞) is simply called a Nussbaum function (type B) defined as follows.

Definition 4.3. A continuously differentiable function $h(s):[0,\infty)\mapsto(-\infty,\infty)$ is called a *Nussbaum function (type B-∞, or shortly, type B)* if it satisfies

$$\lim_{y\to\infty}\frac{\int_0^y h^+(s)ds}{y}=\infty,\ \limsup_{y\to\infty}\frac{\int_0^y h^+(s)ds}{\int_0^y h^-(s)ds}=\infty, \tag{19}$$

$$\lim_{y\to\infty}\frac{\int_0^y h^-(s)ds}{y}=\infty,\ \limsup_{y\to\infty}\frac{\int_0^y h^-(s)ds}{\int_0^y h^+(s)ds}=\infty. \tag{20}$$

A legal fraction (with non-zero denominator) assumption is implicitly made in the above two definitions which excludes the trivial function $h(s)\equiv 0$.

The following lemma shows that a Nussbaum function (type B-L or type B) is an enhanced version of a Nussbaum function (type A).

Lemma 4.1. *A Nussbaum function (type B-L or type B) is a Nussbaum function (type A).*

Proof. From the property (17), there exists a sequence $y_1<y_2<\cdots$, with $\lim_{i\to\infty}y_i=\infty$, such that,

$$\lim_{i\to\infty}\frac{\int_0^{y_i} h^+(s)ds}{y_i}=\infty \tag{21}$$

and

$$\lim_{i\to\infty}\frac{\int_0^{y_i} h^+(s)ds}{\int_0^{y_i} h^-(s)ds}\geq L>1. \tag{22}$$

From (22), one has

$$\lim_{i\to\infty}\frac{\int_0^{y_i} h^+(s)ds-\int_0^{y_i} h^-(s)ds}{\int_0^{y_i} h^+(s)ds}\geq 1-\frac{1}{L}>0,$$

which, together with (21), implies

$$\lim_{i\to\infty}\frac{\int_0^{y_i} h^+(s)ds-\int_0^{y_i} h^-(s)ds}{y_i}=\infty. \tag{23}$$

Hence, Eq. (15) is proved. The proof for Eq. (16) is similar. Also, the proof for $L=\infty$ follows the same arguments. □

A Nussbaum function (type B-L) has an interesting property stated in the following lemma. This property will play an important role in developing the main lemmas later.

Lemma 4.2. *Suppose a function h with $h(s)=h^+(s)-h^-(s)$ is a Nussbaum function (type B-L). Let $\hat{h}(s)=\alpha h^+(s)-\beta h^-(s)$ for two constants α and β satisfying $\alpha\beta>0$. Then, $\hat{h}(s)$ is also a Nussbaum function (type B-$\hat{L}$) if*

$$\hat{L}=\min\left\{\frac{\alpha}{\beta},\frac{\beta}{\alpha}\right\}L>1.$$

In particular, the statement holds for $L=\infty$ and $\hat{L}=\infty$.

Proof. We only prove the case with $\alpha,\beta>0$, so we can denote $\hat{h}(s)=\hat{h}^+(s)-\hat{h}^-(s)$ where $\hat{h}^+(s)=\alpha h^+(s)$ and $\hat{h}^-(s)=\beta h^-(s)$. As $h(s)$ is a Nussbaum function (type B-L), it satisfies (17) and (18).

It is straightforward to verify

$$\lim_{y\to\infty}\frac{\int_0^y \hat{h}^+(s)ds}{y}=\infty,\ \lim_{y\to\infty}\frac{\int_0^y \hat{h}^-(s)ds}{y}=\infty$$

from (17) and (18), respectively. From the property (17), one also has

$$\limsup_{y\to\infty}\frac{\int_0^y \hat{h}^+(s)ds}{\int_0^y \hat{h}^-(s)ds}=\limsup_{y\to\infty}\frac{\alpha\int_0^y h^+(s)ds}{\beta\int_0^y h^-(s)ds}\geq\frac{\alpha}{\beta}L\geq\hat{L};$$

and similarly, from the property (18),

$$\limsup_{y\to\infty}\frac{\int_0^y \hat{h}^-(s)ds}{\int_0^y \hat{h}^+(s)ds}\geq\frac{\beta}{\alpha}L\geq\hat{L}.$$

As a result, $\hat{h}(s)$ is a Nussbaum function (type B-$\hat{L}$). □

One of the main results of this paper is given in the following lemma. It was pointed out that the aforementioned conjecture does not hold. However, the following lemma shows that the conjecture is always true when the Nussbaum function (type A) is enhanced to a Nussbaum function (type B-L). As a result, this new lemma will play a fundamental role in dealing with time-varying unknown control direction.

Lemma 4.3. *Consider two continuously differentiable functions $V(t):[0,\infty)\mapsto\mathbb{R}^+$, $y(t):[0,\infty)\mapsto\mathbb{R}^+$. Let $\lambda(t):[0,\infty)\mapsto[\underline{\lambda},\bar{\lambda}]$ for two constants $\underline{\lambda}$ and $\bar{\lambda}$ satisfying $\underline{\lambda}\bar{\lambda}>0$. If*

$$\dot{V}(t)\leq(\lambda(t)h(y(t))+a)\dot{y}(t)$$
$$\dot{y}(t)\geq 0,\ \forall t\geq 0 \tag{24}$$

for a constant a and a Nussbaum function (type B-L) function h with

$$L>\max\left\{\frac{\underline{\lambda}}{\bar{\lambda}},\frac{\bar{\lambda}}{\underline{\lambda}}\right\}, \tag{25}$$

then $V(t)$ and $y(t)$ are bounded over $[0,\infty)$. In particular, the statement holds for $L=\infty$.

Proof. Let

$$\hat{h}(s)=\bar{\lambda}h^+(s)-\underline{\lambda}h^-(s).$$

For $\bar{\lambda}\underline{\lambda}>0$, by Lemma 4.2, $\hat{h}(s)$ is a Nussbaum function (type B-$\hat{L}$) for

$$\hat{L}=\min\left\{\frac{\underline{\lambda}}{\bar{\lambda}},\frac{\bar{\lambda}}{\underline{\lambda}}\right\}L>\min\left\{\frac{\underline{\lambda}}{\bar{\lambda}},\frac{\bar{\lambda}}{\underline{\lambda}}\right\}\max\left\{\frac{\underline{\lambda}}{\bar{\lambda}},\frac{\bar{\lambda}}{\underline{\lambda}}\right\}=1.$$

By Lemma 4.1, $\hat{h}(s)$ is also a Nussbaum function (type A).

It is noted that, for all $\tau\geq 0$,

$$\lambda(\tau)h(y(\tau))=\lambda(\tau)h^+(y(\tau))-\lambda(\tau)h^-(y(\tau))$$
$$\leq\bar{\lambda}h^+(y(\tau))-\underline{\lambda}h^-(y(\tau))=\hat{h}(y(\tau)).$$

Integrating the first inequality of (24) gives, for all $t\geq 0$,

$$\begin{aligned}0\leq V(t)&\leq\int_0^t(\lambda(\tau)h(y(\tau))+a)\dot{y}(\tau)d\tau+V(0)\\&=\int_0^t\lambda(\tau)h(y(\tau))\dot{y}(\tau)d\tau+\int_0^t a\dot{y}(\tau)d\tau+V(0)\\&\leq\int_{y(0)}^{y(t)}\hat{h}(s)ds+\int_0^t a\dot{y}(\tau)d\tau+V(0)\\&=\int_0^{y(t)}\hat{h}(s)ds-\int_0^{y(0)}\hat{h}(s)ds+ay(t)-ay(0)+V(0).\end{aligned} \tag{26}$$

Denote a constant $c_0=\int_0^{y(0)}\hat{h}(s)ds+ay(0)-V(0)$. One has

$$\int_0^{y(t)}\hat{h}(s)ds+ay(t)\geq c_0,\ \forall t\geq 0. \tag{27}$$

As $\hat{h}(s)$ is a Nussbaum function (type A), by (9), there exists $y^*>1$ such that

$$\frac{1}{y^*}\int_0^{y^*}\hat{h}(s)ds<-|c_0|-a.$$

If $y(t)$ is not bounded over $[0,\infty)$, there exists $t^*>0$ such that $y(t^*)=y^*$ and hence

$$\frac{1}{y(t^*)}\int_0^{y(t^*)}\hat{h}(s)ds<-|c_0|-a<\frac{c_0}{y(t^*)}-a.$$

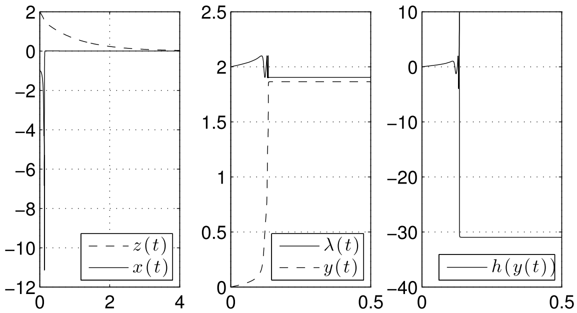

Fig. 3. Stable profile of the closed-loop system under the controller with a Nussbaum function (type B) (the control direction $\lambda(t)$ is time-varying).

As a result,

$$\int_0^{y(t^*)} \hat{h}(s)ds + ay(t^*) < c_0$$

which contradicts (27). As $y(t)$ is bounded over $[0, \infty)$, so is $V(t)$ directly from (26). □

Example 4.1. An unstable phenomenon was observed for a Nussbaum function (type A) in Example 3.2. For the same example with the Nussbaum function (type A) replaced by a Nussbaum function (type B), the global stability of the closed-loop system is now guaranteed by Lemma 4.3. In the numerical simulation, a Nussbaum function (type B) $h(s) = \sin(3\pi s)\exp(s^2)$ is used with the stable performance plotted in Fig. 3. We choose a Nussbaum function (type B) with $L = \infty$, as the boundaries of $\lambda(t)$ and a finite L according to (25) are not assumed known. □

Next, we develop a multivariable version of Lemma 4.3. This version is critically important for handling systems of multiple control inputs in, e.g., multi-agent systems. The result was proved in Chen et al. (2014) and Ding (2015) only for two special Nussbaum functions. It is proved for the first time that the lemma holds for a general Nussbaum function (type B-L).

Lemma 4.4. *Consider continuously differentiable functions $V(t) : [0, \infty) \mapsto \mathbb{R}^+$ and $y_i(t) : [0, \infty) \mapsto \mathbb{R}^+$, $i = 1, \ldots, n$. Let $\lambda_i(t) : [0, \infty) \mapsto [\underline{\lambda}, \bar{\lambda}]$, $i = 1, \ldots, n$, for two constants $\underline{\lambda}$ and $\bar{\lambda}$ satisfying $\underline{\lambda}\bar{\lambda} > 0$. If*

$$\dot{V}(t) \le \sum_{i=1}^{n} (\lambda_i(t)h(y_i(t)) + a_i)\dot{y}_i(t)$$

$$\dot{y}_i(t) \ge 0, \ \forall t \ge 0, \ i = 1, \ldots, n \tag{28}$$

for some constants a_i's and a Nussbaum function (type B-L) h with

$$L > n \max\left\{\frac{\underline{\lambda}}{\bar{\lambda}}, \frac{\bar{\lambda}}{\underline{\lambda}}\right\}, \tag{29}$$

then $V(t)$ and $y_i(t)$, $i = 1, \ldots, n$, are bounded over $[0, \infty)$. In particular, the statement holds for $L = \infty$.

Remark 4.1. When the boundaries $|\bar{\lambda}|$ and $|\underline{\lambda}|$ of the control coefficients are known, a Nussbaum function (type B-L) with L satisfying (29) always works. This explains why $N_0(k) = \cosh(\lambda k)\sin(k)$ as a Nussbaum function (type B-L with L depending on λ) works in Chen et al. (2014) for known boundaries. When the boundaries of the control coefficients are unknown, a Nussbaum function (type B) is sufficient. Also, this explains why $N(k) = e^{k^2/2}(k^2 + 2)\sin(k)$ as a Nussbaum function (type B) works in Ding (2015).

5. Discussions on Nussbaum functions

In the previous section, we have introduced a new type of Nussbaum function. In particular, we used a Nussbaum function (type A) $h(s) = \sin(s)s^2$ and a Nussbaum function (type B) $h(s) = \sin(s)\exp(s^2)$ in the examples. More discussions on these two types of functions are given in this section. A Nussbaum function changes its sign alternatively with a growing magnitude. So, it is typically represented by $h(s) = \theta(s)\xi(s)$ where $\theta(s)$ is a sign changing function, e.g., $\theta(s) = \sin(s)$, and $\xi(s)$ generates a growing magnitude. The two types of Nussbaum functions are classified by the growing rate of $\xi(s)$ as elaborated below through both sufficient and necessary conditions.

Lemma 5.1. *Consider a continuously differentiable function*

$$h(s) = \theta(s)\xi(s)$$

where $\xi(s)$ is an increasing function with $\xi(s) > 0, \ s > 0$ and $\lim_{s\to\infty} \xi(s) = \infty$ and $\theta(s)$ a $2T$-periodic function with $\theta(s) > 0, \ s \in (0, T)$ and $\theta(s) < 0, \ s \in (T, 2T)$. Denote $k_i = iT$ for an integer $i \ge 1$ and let

$$H_i^+ = \int_{k_{2i-2}}^{k_{2i-1}} \theta(s)\xi(s)ds > 0,$$

$$H_i^- = -\int_{k_{2i-1}}^{k_{2i}} \theta(s)\xi(s)ds > 0.$$

(i) If

$$\lim_{i\to\infty} H_i^+ - H_{i-1}^- = \infty \ \text{ and } \ \lim_{i\to\infty} H_i^- - H_i^+ = \infty, \tag{30}$$

then h(s) is a Nussbaum function (type A); if

$$\limsup_{i\to\infty} H_i^+ - H_{i-1}^- < \infty \ \text{ or } \ \limsup_{i\to\infty} H_i^- - H_i^+ < \infty, \tag{31}$$

then h(s) is not a Nussbaum function (type A).

(ii) If, for $L > 1$,

$$\lim_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} \ge L \ \text{ and } \ \lim_{i\to\infty} \frac{H_i^-}{H_i^+} \ge L, \tag{32}$$

then h(s) is a Nussbaum function (type B-L); if

$$\limsup_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} < L \ \text{ or } \ \limsup_{i\to\infty} \frac{H_i^-}{H_i^+} < L, \tag{33}$$

then h(s) is not a Nussbaum function (type B-L). In particular, the statement holds for $L = \infty$.

Proof. It is noted that

$$\int_0^{k_{2i-1}} h^+(s)ds = H_i^+ + \cdots + H_1^+$$

$$\int_0^{k_{2i-1}} h^-(s)ds = H_{i-1}^- + \cdots + H_1^-.$$

Proof of (i). Under (30), for any constant $K > 0$, there exists ℓ such that $H_i^+ - H_{i-1}^- > K, \ i \ge \ell$. Define

$$\kappa = \frac{(H_{\ell-1}^+ + \cdots + H_1^+) - (H_{\ell-2}^- + \cdots + H_1^-)}{K}.$$

As a result,

$$\frac{(H_\hbar^+ + \cdots + H_1^+) - (H_{\hbar-1}^- + \cdots + H_1^-)}{\hbar}$$

$$\ge \frac{(\hbar - \ell + 1)K + \kappa K}{\hbar} \ge \frac{K}{2}$$

for $\hbar \geq 2(\ell - 1 - \kappa)$. Then,

$$\lim_{i\to\infty} \frac{1}{k_{2i-1}} \int_0^{k_{2i-1}} h(s)ds = \lim_{i\to\infty} \frac{(H_i^+ + \cdots + H_1^+) - (H_{i-1}^- + \cdots + H_1^-)}{(2i-1)T} = \infty,$$

which implies (8). The verification of (9) is similar. So, $h(s)$ is a Nussbaum function (type A).

If $\limsup_{i\to\infty} H_i^+ - H_{i-1}^- < \infty$, there exists K such that $H_i^+ - H_{i-1}^- < K$ for all i. As a result,

$$\begin{aligned}\frac{1}{k_{2i-1}} \int_0^{k_{2i-1}} h(s)ds &= \frac{(H_i^+ + \cdots + H_1^+) - (H_{i-1}^- + \cdots + H_1^-)}{(2i-1)T} \\ &= \frac{(H_i^+ - H_{i-1}^-) + \cdots + (H_2^+ - H_1^-) + H_1^+}{(2i-1)T} \\ &\leq \frac{K(i-1) + H_1^+}{(2i-1)T} = \frac{K}{2T} + \frac{-K + 2H_1^+}{2(2i-1)T} \leq \frac{K}{2T} + \frac{2H_1^+}{2T}\end{aligned}$$

for all $i \geq 1$. It implies that $\limsup_{y\to\infty} \frac{1}{y}\int_0^y h(s)ds < \infty$, i.e., $h(s)$ is not a Nussbaum function (type A). If $\limsup_{i\to\infty} H_i^- - H_i^+ < \infty$, the proof follows the similar arguments.

Proof of (ii). Under (32), for any $\epsilon > 0$, there exists $\ell \geq 2$ such that $\frac{H_i^+}{H_{i-1}^-} > L - \epsilon,\ i \geq \ell$. For $\hbar \geq \ell$, one has

$$\frac{\ell - 2}{\hbar - \ell + 1}(H_{\hbar-1}^- + \cdots + H_{\ell-1}^-) \geq H_{\ell-2}^- + \cdots + H_1^-.$$

As a result,

$$\begin{aligned}\frac{H_\hbar^+ + \cdots + H_1^+}{H_{\hbar-1}^- + \cdots + H_1^-} &= \frac{H_\hbar^+ + \cdots + H_\ell^+ + H_{\ell-1}^+ + \cdots + H_1^+}{H_{\hbar-1}^- + \cdots + H_{\ell-1}^- + H_{\ell-2}^- + \cdots + H_1^-} \\ &\geq \frac{H_\hbar^+ + \cdots + H_\ell^+}{(1 + (\ell-2)/(\hbar - \ell + 1))(H_{\hbar-1}^- + \cdots + H_{\ell-1}^-)} \\ &\geq \frac{L - \epsilon}{1 + (\ell-2)/(\hbar - \ell + 1)} = L - \bar{\epsilon} \geq L - 2\epsilon\end{aligned}$$

for $\hbar \geq L(\ell - 2)/\epsilon + \ell - 1$. In the last inequality, it is noted that

$$\begin{aligned}\bar{\epsilon} &= \frac{L(\ell-2)/(\hbar - \ell + 1) + \epsilon}{1 + (\ell-2)/(\hbar - \ell + 1)} \\ &\leq L(\ell-2)/(\hbar - \ell + 1) + \epsilon \leq 2\epsilon.\end{aligned}$$

As a result,

$$\limsup_{i\to\infty} \frac{\int_0^{k_{2i-1}} h^+(s)ds}{\int_0^{k_{2i-1}} h^-(s)ds} = \limsup_{i\to\infty} \frac{H_i^+ + \cdots + H_1^+}{H_{i-1}^- + \cdots - H_1^-} \geq L.$$

For any $K > 0$, there exists ℓ such that $H_i^+ > K, i \geq \ell$. So, one has

$$\frac{H_\hbar^+ + \cdots + H_1^+}{\hbar} \geq \frac{H_\hbar^+ \cdots + H_\ell^+}{\hbar} \geq \frac{K(\hbar - \ell + 1)}{\hbar} \geq \frac{K}{2}$$

for $\hbar \geq 2(\ell - 1)$. Then,

$$\lim_{y\to\infty} \frac{\int_0^y h^+(s)ds}{y} \geq \lim_{i\to\infty} \frac{\int_0^{k_{2i-1}} h^+(s)ds}{k_{2i-1} + 2T} = \lim_{i\to\infty} \frac{H_i^+ + \cdots + H_1^+}{(2i+1)T} = \infty.$$

From above, one has (17). The verification of (18) is similar. So, $h(s)$ is a Nussbaum function (type B-L).

If $\limsup_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} < L$, there exist ϵ and $\ell \geq 2$ such that $\frac{H_i^+}{H_{i-1}^-} < L - \epsilon$ for all $i \geq \ell$. For $\hbar \geq \ell$, one has

$$\frac{\ell - 1}{\hbar - \ell + 1}(H_\hbar^+ + \cdots + H_\ell^+) \geq H_{\ell-1}^+ + \cdots + H_1^+.$$

As a result,

$$\begin{aligned}\frac{H_\hbar^+ + \cdots + H_1^+}{H_{\hbar-1}^- + \cdots + H_1^-} &= \frac{H_\hbar^+ + \cdots + H_\ell^+ + H_{\ell-1}^+ + \cdots + H_1^+}{H_{\hbar-1}^- + \cdots + H_{\ell-1}^- + H_{\ell-2}^- + \cdots + H_1^-} \\ &\leq \frac{(1 + (\ell-1)/(\hbar - \ell + 1))(H_\hbar^+ + \cdots + H_\ell^+)}{H_{\hbar-1}^- + \cdots + H_{\ell-1}^-} \\ &\leq (1 + (\ell-1)/(\hbar - \ell + 1))(L - \epsilon) = L - \bar{\epsilon} \leq L - \epsilon/2\end{aligned}$$

for $\hbar \geq 2(\ell - 1)(L - \epsilon)/\epsilon + \ell - 1$. In the last inequality, it is noted that

$$\bar{\epsilon} = \epsilon - (\ell - 1)/(\hbar - \ell + 1)(L - \epsilon) \geq \epsilon/2.$$

Then,

$$\limsup_{i\to\infty} \frac{\int_0^{k_{2i-1}} h^+(s)ds}{\int_0^{k_{2i-1}} h^-(s)ds} = \limsup_{i\to\infty} \frac{H_i^+ + \cdots + H_1^+}{H_{i-1}^- + \cdots - H_1^-} < L.$$

So, $h(s)$ is not a Nussbaum function (type B-L). If $\limsup_{i\to\infty} \frac{H_i^-}{H_i^+} < L$, the proof follows the similar arguments. □

Remark 5.1. In Lemma 5.1, the sequences H_i^+ and H_i^- represent the definite integrals (energy) of a Nussbaum function, in two opposite directions. For a Nussbaum function (type A), the sequences *arithmetically* increase with the ratio approaching infinity. For a Nussbaum function (type B-L) or a Nussbaum function (type B), the sequences *geometrically* increase with the ratio $L > 1$ or the ratio approaching infinity, respectively.

Example 5.1. Consider the function

$$h(s) = \sin(s)s^2.$$

Using the identity

$$\int \sin(s)s^2 ds = (2 - s^2)\cos s + 2s \sin s,$$

one has

$$\begin{aligned}H_i^+ &= \int_{k_{2i-2}}^{k_{2i-1}} \sin(s)s^2 ds \\ &= -4 + [(2i-1)^2 + (2i-2)^2]\pi^2, \\ H_{i-1}^- &= -\int_{k_{2i-1}}^{k_{2i}} \sin(s)s^2 ds \\ &= -4 + [(2i-2)^2 + (2i-3)^2]\pi^2.\end{aligned}$$

It is easy to check (30), in particular,

$$\begin{aligned}\lim_{i\to\infty} H_i^+ - H_{i-1}^- &= \lim_{i\to\infty} [(2i-1)^2 - (2i-3)^2]\pi^2 \\ &= \lim_{i\to\infty} 2(4i - 4)\pi^2 = \infty.\end{aligned}$$

So, $h(s)$ is a Nussbaum function (type A). For

$$\lim_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} = 1 \text{ and } \lim_{i\to\infty} \frac{H_i^-}{H_i^+} = 1,$$

it is not a Nussbaum function (type B-L) for any $L > 1$. □

Example 5.2. Consider the function

$h(s) = \sin(s)\exp(s)$.

One has (32), in particular,

$$\lim_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} = e^{\pi}.$$

So, $h(s)$ is a Nussbaum function (type B-e^{π}) but not a Nussbaum function (type B). □

Example 5.3. Consider the function

$h(s) = \sin(s)\exp(s^2)$.

One has (32), in particular,

$$\lim_{i\to\infty} \frac{H_i^+}{H_{i-1}^-} = \lim_{i\to\infty} \frac{\int_{k_{2i-2}}^{k_{2i-1}} \sin(s)\exp(s^2)ds}{-\int_{k_{2i-3}}^{k_{2i-2}} \sin(s)\exp(s^2)ds} =$$

$$\lim_{i\to\infty} \frac{\int_0^{\pi} \sin(s)\exp([(s+k_{2i-3})^2 + 2(s+k_{2i-3})\pi + \pi^2])ds}{\int_0^{\pi} \sin(s)\exp((s+k_{2i-3})^2)ds}$$

$$\geq \lim_{i\to\infty} \exp(2k_{2i-3}\pi) = \lim_{i\to\infty} \exp(2(2i-3)\pi^2) = \infty.$$

So, $h(s)$ is a Nussbaum function (type B). □

Finally, it is ready to give a rigorous counter example that shows that the conjecture is false.

Example 5.4 (*A Counter Example to the Conjecture*)**.** We will show that the conjecture does not hold for the Nussbaum function (type A) $h(s) = \sin(s)s^2$. Let $y(t) = t$, $\dot{y}(t) = 1$, and $a = 0$. Consider the following time-varying control coefficients, with $k_i = i\pi$,

$$\lambda(t) = \begin{cases} 1, & t \in [k_{2i-1}, k_{2i}) \\ 1 + \epsilon \sin(t) & t \in [k_{2i-2}, k_{2i-1}) \end{cases}, \quad i \geq 1,$$

for some $\epsilon > 0$. Define

$$\hat{H}_i^+ = \int_{k_{2i-2}}^{k_{2i-1}} \lambda(s)h(s)ds = H_i^+ + \epsilon \int_{k_{2i-2}}^{k_{2i-1}} \sin^2(s)s^2 ds.$$

Note that

$$\int_{k_{2i-2}}^{k_{2i-1}} \sin^2(s)s^2 ds = \frac{1}{6}[(2i-1)^3 - (2i-2)^3]\pi^3 - \frac{1}{4}\pi^2.$$

By using the calculation of H_i^+ and H_i^- in Example 5.1 and comparing the coefficients of i^2, one has

$$\lim_{i\to\infty} \frac{H_i^-}{\hat{H}_i^+} = \frac{8\pi^2}{8\pi^2 + \epsilon(2\pi^3)} < 1.$$

So, there exists a finite $\ell \geq 1$ such that

$H_i^- < \hat{H}_i^+, \ \forall i > \ell$.

Denote $c_i = (\hat{H}_1^+ + \cdots + \hat{H}_i^+) - (H_1^- + \cdots + H_i^-)$ for $i = 1, \ldots, \ell$, and hence a finite $c = \min\{0, c_1, \ldots, c_\ell\} \leq 0$. Consider a continuously differentiable function

$$V(t) = \int_0^t \lambda(\tau)h(\tau)d\tau - c. \tag{34}$$

It is noted that

$$\dot{V}(t) = \lambda(t)h(t) = (\lambda(t)h(y(t)) + a)\dot{y}(t) \tag{35}$$

that satisfies (14). Next, we will show that $V(t) \geq 0$, that is, $\int_0^t \lambda(\tau)h(\tau)d\tau \geq c$, for all $t \geq 0$.

For $t \in [0, k_1)$, it is obvious that $\int_0^t \lambda(\tau)h(\tau)d\tau \geq 0 \geq c$. For any $t \geq k_1$, there exists $\hbar \geq 1$ such that $t \in [k_{2\hbar-1}, k_{2\hbar+1})$, then

$$\int_0^t \lambda(\tau)h(\tau)d\tau \geq (\hat{H}_1^+ + \cdots + \hat{H}_\hbar^+) - (H_1^- + \cdots + H_\hbar^-).$$

If $\hbar \leq \ell$, it is obvious that $\int_0^t \lambda(\tau)h(\tau)d\tau \geq c$; otherwise,

$$\int_0^t \lambda(\tau)h(\tau)d\tau \geq [(\hat{H}_1^+ + \cdots + \hat{H}_\ell^+) - (H_1^- + \cdots + H_\ell^-)]$$
$$+ [(\hat{H}_{\ell+1}^+ + \cdots + \hat{H}_\hbar^+) - (H_{\ell+1}^- + \cdots + H_\hbar^-)] \geq c.$$

From above, a counter example to the conjecture has been constructed, noting that $y(t) = t$ is unbounded. □

6. Conclusion

A Nussbaum function provides a control gain with an alternatively changing sign and a growing magnitude. Two types of Nussbaum functions have been defined in this paper with different growing rates. It has been shown that the existing Nussbaum function (type A) is not always effective in the scenarios of multivariable and/or time-varying control coefficients with unknown signs. Then, a new Nussbaum function (type B) has been proved to be successful in these scenarios. The essential characteristic of a Nussbaum function (type B) is the geometrically increasing rate of its definite integral rather than arithmetically for a Nussbaum function (type A). It is interesting to apply the new Nussbaum functions on more control problems in the future work.

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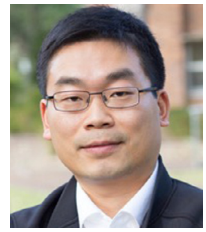

Zhiyong Chen received the B.E. degree from the University of Science and Technology of China, and the M.Phil. and Ph.D. degrees from the Chinese University of Hong Kong, in 2000, 2002 and 2005, respectively. He worked as a research associate at the University of Virginia during 2005–2006. He joined the University of Newcastle, Australia, in 2006, where he is currently a professor. He is also a Changjiang chair professor with Central South University, Changsha, China. His research interests include non-linear systems and control, biological systems, and multi-agent systems. He is/was an associate editor of Automatica, IEEE Transactions on Automatic Control and IEEE Transactions on Cybernetics.