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Relaxing the high-frequency gain sign assumption in direct model reference adaptive control[☆]

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ABSTRACT

A new, high performance, solution to the classical problem of direct model reference adaptive control for linear time-invariant systems with *unknown sign of the high frequency gain* is reported in the paper. The proposed algorithm directly estimates this parameter with the only required prior knowledge of a lower bound on its absolute value. To avoid the possible appearance of singularities in the controller calculation a switched projection mechanism is introduced to change, if needed, the sign of the estimate. The recently introduced *dynamic regressor extension and mixing* estimator is used to ensure monotonicity of the estimation error of the high frequency gain, guaranteeing that the switching appears (at most) once and avoiding the possible appearance of chattering—that may happen in classical gradient-based algorithms. Comparative simulations with the Nussbaum gain-based and gradient estimators illustrate the dramatic *performance improvement* of the proposed controller.

1. Introduction

Model reference adaptive control (MRAC) is unquestionably the most widely studied problem in the adaptive literature that has a very long history going back to the 1950's and extending to the present time. The first attempts to solve the MRAC problem followed the classical path of designing an observer, that had to be made adaptive because of the unknown plant parameters, and then feeding back the observed state (see [6]). Very little success was, however, obtained pursuing these lines—essentially because of the difficulty of simultaneously estimating state and parameters. A major breakthrough, essentially due to [2,8], was the introduction of the so-called *direct* control parameterization (see Lemma 1 below), which revealed that the estimation of the plant state could be obviated and only a "good" estimation of the controller parameters was needed to achieve the asymptotic reference model output tracking objective. The intrinsic simplicity of this parametrization motivated the overwhelming majority of the researchers to pursue this line of reasoning and concentrated their efforts into the development of suitable parameter estimators. The interested reader is referred to [12] for a vivid description of the history of MRAC as well as to the existing textbooks [5,14,19] for further information on it.

As it is well-known, the direct control parameterization, referred to as *output-error* parameterization[1] in [19], leads to a *bilinear* regression form, where the parameter that corresponds to the high frequency gain—denoted k_p in the sequel—appears multiplying the controller parameters. This difficulty can be overcome assuming the knowledge of the sign(k_p), under which a globally convergent output-error MRAC may be designed introducing an *overparameterisation* of the regressor and a *normalisation* in, now classical, augmented error-based estimators [5,14,19]. It was shown that these algorithms enjoy the fundamental "self-tuning property", that is, that global tracking is ensured for all reference signals—without imposing the stronger parameter convergence requirement. The use of normalisation and overparameterisation, however, comes with a very high tag for the overall performance of the scheme. Indeed, as thoroughly discussed in [10,16,19], overparameterisation hampers parameter convergence while normalisation "slows down" the adaptation and severely penalizes the parameter convergence rate. As shown in [16], this below par performance can be partially overcome using (the unnormalised) Morse's high order tuners [11], but the additional information of an upper

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[1] The term "output-error" has been used in [7] to refer to a completely different construction used in identification and adaptive control.

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bound on k_p is required and the scheme is significantly more involved.

A major theoretical breakthrough for this problem is due to Nussbaum [15] who, motivated by a conjecture in [9], showed that the sign of k_p is not necessary for stabilization in MRAC. Nussbaum's solution relies on the introduction of a function that changes periodically the sign of the estimator vector field in a "gain scheduling-like" fashion. It is clear that this kind of algorithms is only of theoretical interest since their transient performance is intrinsically bad and practically inadmissible—as it has been repeatedly reported in the literature.

Schemes that require the division by the estimate of k_p in the controller calculation, *e.g.*, the one proposed in Section 4.5.2 of [5] and the other one presented in the paper [4], most incorporate a switched projection to avoid singularities. There are two drawbacks to this approach, on one hand, to the best of the authors' knowledge, no proof of global tracking for this scheme has been reported in the literature without an *unverifiable* assumption of persistency of excitation (PE) of the regressor. On the other hand, there is no guarantee that the switching happens only a finite number of times nor the possible appearance of *chattering* phenomena.

In this paper, a new solution to the problem with improved transient performance is reported, which includes the following modifications:

- (M1) Abandon the bilinear model mentioned above, and adopt instead the overparameterized *linear regression*.
- (M2) Introduce a new factorization of the parameter *estimates* to update directly the controller parameters.
- (M3) Instead of classical gradient estimators we use the recently introduced *dynamic regressor extension and mixing* (DREM) estimator from [1].

The use of a linear parameterization is essential to apply the DREM estimator. Unfortunately, the estimation law still involves the division by an estimate of k_p. Therefore, similarly to the classical schemes, a switched projection of this estimate is added to keep it away from an a priori known band around the zero value. To avoid the undesirable chattering phenomena indicated above we exploit a key feature of DREM: that it ensures *monotonicity* of the estimation error of the parameter k_p, ensuring that the switching appears (at most) once. The monotonicity property holds for all reference signal. However, global tracking can only be ensured for reference signals that satisfy an excitation requirement, which holds true if the aforementioned PE assumption on the regressor of classical schemes is satisfied.

The remainder of the paper is organized as follows. Section 2 formulates the MRAC problem addressed in the paper and briefly reviews the current literature available on this topic. An MRAC, with a gradient-based procedure to estimate the controller parameters using the new factorization mentioned above, is given in Section 3. Section 4 contains our main result, namely, the description of the DREM estimator and its stability properties when applied in a MRAC scheme. Comparative simulations with the classical Nussbaum gain-based and gradient estimators, which illustrate the significant *performance improvement* of the proposed controller, are presented in Section 5. The paper is wrapped-up with concluding remarks in Section 6.

2. The MRAC problem with unknown sign(k_p)

2.1. Problem formulation

We are interested in the classical problem of relaxing the knowledge of the high frequency gain in MRAC of the scalar linear time-invariant (LTI) continuous-time plant

$$D(p)y = k_p N(p)u, \tag{1}$$

where y, u are the plant output and input, respectively, $D(p)$ and $N(p)$ are monic and coprime polynomials of degree n and m, respectively, $p := \frac{d}{dt}$, $\rho := n - m \geq 1$ and $k_p \in \mathbb{R}$ is the high frequency gain. The parameters of $D(p)$ and $N(p)$ are unknown.

We make the following assumptions regarding the plant.

(A.1) $N(p)$ is a Hurwitz polynomial.
(A.2) n and ρ are known.
(A.3) A constant $\underline{k}_p \in \mathbb{R}_+$ verifying

$$|k_p| \geq \underline{k}_p \tag{2}$$

is known.

The MRAC objective is to asymptotically drive to zero the tracking error

$$e = y - \frac{k_m}{D_m(p)} r \tag{3}$$

where $D_m(p)$ is a monic, Hurwitz polynomial of degree ρ, $k_m \in \mathbb{R}$ and r is a bounded reference.

2.2. Remarks on the assumptions

[R1] Assumptions A.1 and A.2, though somehow restrictive, are standard in MRAC (see, for example, [5,14,19]).

[R2] Conspicuous by its absence is the assumption of knowledge of the *sign of the high–frequency gain* of the plant k_p.[2] Relaxing this assumption is the main subject of interest in this note. Instead of its sign we assume that k_p is bounded away from zero—by a known value $\underline{k}_p$—as indicated in (2). Although this is a sensible assumption in all practical scenarios, it will be shown below that the transient behavior is degraded if $\underline{k}_p$ is too small.

[R3] We have assumed $\rho \geq 1$ to simplify the notation in the sequel. As will become clear later, the scheme proposed here—with the adequate technical changes—applies as well to the case $\rho = 0$.

[R4] Without loss of generality we have selected the reference model without zeros, a scenario usually adopted in MRAC designs. The theory can be extended *verbatim* for general reference model transfer functions.

2.3. A key lemma

Instrumental for the development of MRAC is the lemma below, known as the direct control model reference parameterization, first established by [2,8] (see also [5,19] for a modern derivation of the result).

Lemma 1. *Consider the plant (1) and the tracking error (3). There exists a vector $\theta \in \mathbb{R}^{2n}$ such that*

$$e = \frac{k_p}{D_m(p)}(u - \theta^\top \phi) + \epsilon_t, \tag{4}$$

where $\phi \in \mathbb{R}^{2n}$ is the regressor vector given by

$$\phi = \frac{1}{\lambda(p)} \mathrm{col}(u, \dot{u}, \ldots, u^{(n-2)}, y, \dot{y}, \ldots, y^{(n-2)}, \lambda(p)y, \lambda(p)r) \tag{5}$$

with a designer-chosen monic, Hurwitz polynomial $\lambda(p)$ and ϵ_t is an exponentially decaying term due to initial conditions.[3]

[2] In time-domain this is tantamount to the knowledge of the sign of the instantaneous step-response.

[3] These terms will be omitted (without loss of generality) in the sequel.

The MRAC designs are completed proposing a controller of the form

$$u = \hat{\theta}^\top \phi, \tag{6}$$

where $\hat{\theta} \in \mathbb{R}^{2n}$ are the estimates of the parameters θ, which are generated via a parameter adaptation algorithm.

2.4. Existing results

The parameterization (4) contains the product of the unknown parameters k_p and θ. To overcome the difficulty of parameter identification with a bilinear model an overparameterization is introduced, writing (4) as

$$e = \begin{bmatrix} k_p\theta^\top & k_p \end{bmatrix} \begin{bmatrix} -\phi_f \\ u_f \end{bmatrix}, \tag{7}$$

where, to simplify the notation, we defined the filtered signals

$$(\cdot)_f := \frac{1}{D_m(p)}(\cdot).$$

Actually, noting that $\theta_{2n}\phi_{2n} = \frac{k_m}{k_p} r$ and using the definition of the tracking error (3), this linear regression can be simplified to

$$y = \begin{bmatrix} k_p\theta_0^\top & k_p \end{bmatrix} \begin{bmatrix} -(\phi_0)_f \\ u_f \end{bmatrix}, \tag{8}$$

where θ_0 and ϕ_0 contain the first $2n-1$ elements of θ and ϕ, respectively.

As indicated in Section 1, if $\mathrm{sign}(k_p)$ is *known*, it is possible to design an estimator for the parameters $\mathrm{col}(\theta,\ k_p)$ that ensures global tracking for all reference signals. Besides the fact that overparameterization was introduced, this estimator includes a normalization factor, hence it suffers from the all the limitations and performance degradation problems mentioned in Section 1. We refer the interested reader to the classical textbooks [5,14] for several versions of these algorithms.

In Subsection 4.5.2 of [5] it is proposed to use the parameterization (8) to estimate directly the parameters $\mathrm{col}(k_p\theta_0,\ k_p)$ and to recover the controller parameters $\hat{\theta}$ dividing by the estimate $\hat{k}_p$. To avoid singularities, a projection that requires the knowledge of the *sign and a lower bound* of k_p is implemented, *i.e.* Eq. (4.5.18) of [5]. There are two drawbacks to this approach, on one hand, to the best of the authors' knowledge, no proof of global tracking for this scheme has been reported in the literature without an *unverifiable* assumption of PE of the regressor $\mathrm{col}((\phi_0)f,\ u_f)$, which ensures parameter convergence. We underscore the fact that the PE condition is imposed directly on the regressor and, in contrast with MRAC without switchings, cannot be guaranteed assuming sufficient richness of the reference signal. On the other hand, there is no guarantee that the switching happens only a finite number of times nor the possible appearance of *chattering* phenomena.

From the regression form (8) we get

$$\frac{1}{k_p} y = -\theta_0^\top (\phi_0)_f + u_f,$$

hence, recalling that $\theta_{2n} = \frac{k_m}{k_p}$, it can be written in the alternative form

$$u_f = \theta^\top \begin{bmatrix} (\phi_0)_f \\ \frac{1}{k_m} y \end{bmatrix}. \tag{9}$$

This is the so-called, input-error parameterization introduced in Section 3.3.1 of [19]. In [19] an estimator that uses this parameterization and ensures global tracking is proposed. Unfortunately, this algorithm includes a parameter projection for the term $\hat{\theta}_{2n+1}$ that requires, besides the knowledge of $\mathrm{sign}(k_p)$, and upper bound on $|k_p|$. Although there is no division by $\hat{k}_p$ in this algorithm, the projection is essential for the proof. Moreover, it has recently been shown in [3] that, in the absence of the projection, input-error MRAC suffers from an instability mechanism that may give rise to unbounded trajectories—even in the simplest case of first-order plants with $r(t) = 0$.

In summary, with the exception of the highly impractical Nussbaum gain algorithm, all other MRAC schemes require the knowledge of the sign of k_p to prove their stability.

3. Gradient-based MRAC

As indicated in the previous section in Section 4.5.2 of [5] it is proposed to use the linear regression model (7) to estimate directly the parameters $\mathrm{col}(\mu,\ k_p) := \mathrm{col}(k_p\theta,\ k_p)$ using a standard gradient estimator[4]

$$\begin{aligned} \dot{\hat{\mu}} &= -\Gamma_\theta \phi_f (e + \phi_f^\top \hat{\mu} - u_f \hat{k}_p), \\ \dot{\hat{k}}_p &= \gamma_p u_f (e + \phi_f^\top \hat{\mu} - u_f \hat{k}_p) \end{aligned} \tag{10}$$

with normalized adaptation gains $\Gamma_\theta = \Gamma_{\theta 0}/(1+\psi^\top\psi), \Gamma_{\theta 0} \in \mathbb{R}^{2n\times 2n}$ positive-definite and $\gamma_p = \gamma_{p0}/(1+\psi^\top\psi), \gamma_{p0} \in \mathbb{R}_+,\ \psi = \mathrm{col}(-\phi_f, u_f)$, and compute the controller parameters via

$$\hat{\theta} := \frac{\hat{\mu}}{\hat{k}_p}. \tag{11}$$

To be able to avoid the assumption of knowledge of $\mathrm{sign}(k_p)$ we propose in this section to compute the controller parameters directly applying the formula

$$\dot{\hat{\mu}} = \dot{\hat{k}}_p \hat{\theta} + \hat{k}_p \dot{\hat{\theta}}. \tag{12}$$

Assuming *temporarily* that

$$|\hat{k}_p(t)| \geq \underline{k}_p > 0 \tag{13}$$

and doing some basic calculations we obtain from (10) and (12) that

$$\begin{aligned} \dot{\hat{\theta}} &= -(\Gamma_\theta \phi_f + \gamma_p u_f \hat{\theta}) \left(\frac{e}{\hat{k}_p} + \phi_f^\top \hat{\theta} - u_f \right), \\ \dot{\hat{k}}_p &= \gamma_p u_f [e + \hat{k}_p (\phi_f^\top \hat{\theta} - u_f)]. \end{aligned} \tag{14}$$

To enforce the condition (13) a switched projection mechanism is added to the estimator of k_p as follows

$$\hat{k}_p(t_+) = \begin{cases} \hat{k}_p(t) & \text{if } |\hat{k}_p(t)| > \underline{k}_p \\ -\underline{k}_p & \text{if } k_p(t) = \underline{k}_p \text{ and } \dot{\hat{k}}_p(t) < 0 \\ \underline{k}_p & \text{if } k_p(t) = -\underline{k}_p \text{ and } \dot{\hat{k}}_p(t) > 0, \end{cases} \tag{15}$$

with the corresponding resetting of the initial conditions of $\dot{\hat{k}}_p$. Clearly, the modification above implements a "jump" to avoid the interval $[-\underline{k}_p, \underline{k}_p]$.

Unfortunately, the MRAC proposed above suffers from the serious drawback that it is not possible to avoid the possible appearance of *chattering* phenomenon in the band $[-\underline{k}_p, \underline{k}_p]$. Indeed, nothing prevents the estimator from changing the sign of $\dot{\hat{k}}_p$ after the jump. Moreover, there is no guarantee that the number of jumps is finite. To overcome this problem in the section below we propose to replace the standard gradient estimator by a DREM adaptation algorithm proposed in [1].

[4] In the aforementioned reference the equivalent regression model (8) is used.

4. DREM-based MRAC

As discussed in [1] DREM has several advantages over gradient (or least-squares) estimators, including a provable transient performance improvement. The feature of DREM that we exploit in this paper is that it guarantees *monotonicity of each element* of the parameter error. This is a much stronger property than monotonicity of the Euclidean *norm* of the parameter error vector ensured by standard estimators [5,19]. However, this monotonicity property does not preclude the possible appearance of the undesirable chattering phenomena mentioned above. With DREM we will prove that the absolute value of the parameter error $\tilde{k}_p(t)$, is non-increasing ensuring in this way that, at most, there will be one switching in the transient behavior.

4.1. Derivation of the DREM estimator

To apply DREM to the linear regression (7) we rewrite it in compact form as

$$e = \eta^\top \psi \tag{16}$$

with the extended regressor and parameter vectors

$$\psi := \begin{bmatrix} -\phi_f \\ u_f \end{bmatrix}, \ \eta := \begin{bmatrix} \mu \\ k_p \end{bmatrix}. \tag{17}$$

The proposition below is the main result of the paper. To streamline the presentation we introduce $2n+1$ *linear* distinct operators $H_i : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and define the signals

$$(\cdot)_{H_i} = H_i(\cdot), \ i = 1, \ldots, 2n+1. \tag{18}$$

In the sequel the subindex $(\cdot)_i$ will always range in the set $\{1, \ldots, 2n+1\}$ therefore this qualifier will be omitted.

Proposition 1. *[DREM estimator] Consider the linear regression (16) and the operations (18) and define*

$$M_e := \begin{bmatrix} \psi_{H_1}^\top \\ \psi_{H_2}^\top \\ \vdots \\ \psi_{H_{2n+1}}^\top \end{bmatrix}, \ E := \mathrm{adj}\{M_e\} \begin{bmatrix} e_{H_1} \\ e_{H_2} \\ \vdots \\ e_{H_{2n+1}} \end{bmatrix}, \ \Delta := \det\{M_e\}, \tag{19}$$

with adj$\{\cdot\}$ *the adjoint of the matrix. The DREM estimator*

$$\dot{\hat{\eta}}_i = -\gamma_i \Delta (\Delta \hat{\eta}_i - E_i), \ \gamma_i > 0, \tag{20}$$

ensures the following properties.

(i) *[Monotonicity]*

$$|\tilde{\eta}_i(t_2)| \le |\tilde{\eta}_i(t_1)|, \ \forall t_2 \ge t_1 \ge 0. \tag{21}$$

(ii) *[Convergence]*

$$\lim_{t\to\infty} \tilde{\eta}_i(t) = 0 \iff \Delta(t) \notin \mathcal{L}_2 \iff \int_0^\infty \Delta^2(s) ds = \infty. \tag{22}$$

(iii) *[Square integrability]*

$$\Delta \tilde{\eta}_i \in \mathcal{L}_2$$

Proof. Applying H_i to (16) and piling up the signals we get

$$\begin{bmatrix} e_{H_1} \\ e_{H_2} \\ \vdots \\ e_{H_{2n+1}} \end{bmatrix} = M_e \eta.$$

Multiplying by adj$\{M_e\}$ and using the identity

$$\Delta I_{2n+1} := \mathrm{adj}\{M_e\} M_e, \tag{23}$$

with I_s the $s \times s$ identity matrix, yields

$$E_i = \Delta \eta_i. \tag{24}$$

Replacing (24) in (20) one gets

$$\dot{\tilde{\eta}}_i = -\gamma_i \Delta^2 \tilde{\eta}_i. \tag{25}$$

The proof of (i) and (ii) is completed solving these simple scalar differential equations to get

$$\tilde{\eta}_i(t_2) = e^{-\gamma_i \int_{t_1}^{t_2} \Delta^2(\tau) d\tau} \tilde{\eta}_i(t_1), \ t_2 \ge t_1 \ge 0.$$

To establish (iii) evaluate the derivative of the function $\tilde{\eta}_i^2$ along (25) and integrate from zero to infinity. □

It is important to underscore that the identity (23) holds even if M_e is not full-rank.

4.2. Application of DREM to MRAC

To compute the controller parameters we proceed as done in Section 3. That is, define $\hat{\mu} = \hat{k}_p \hat{\theta}$ and apply the formula (12)—notice that $\mu = \mathrm{col}(\eta_1, \ldots, \eta_{2n})$ and $k_p = \eta_{2n+1}$. Assuming *temporarily* (13) and doing some basic calculations we obtain

$$\dot{\hat{\theta}}_j = (\gamma_{2n+1} - \gamma_j) \Delta^2 \hat{\theta}_j + \frac{\Delta}{\hat{k}_p} \left(\gamma_j E_j - \gamma_{2n+1} E_{2n+1} \hat{\theta}_j \right), \ j = 1, \ldots, 2n$$

$$\dot{\hat{k}}_p = -\gamma_{2n+1} \Delta (\Delta \hat{k}_p - E_{2n+1}). \tag{26}$$

As before, to enforce the condition (13) we propose to add a switching mechanism to the estimator of k_p. Namely,

$$\hat{k}_p(t_+) = \begin{cases} \hat{k}_p(t) & \text{if } |\hat{k}_p(t)| > \underline{k}_p \\ -\underline{k}_p & \text{if } k_p(t) = \underline{k}_p \\ \underline{k}_p & \text{if } k_p(t) = -\underline{k}_p. \end{cases} \tag{27}$$

Notice that we have removed the condition on $\dot{\hat{k}}_p$ imposed in (15), which is now unnecessary because $\hat{k}_p(t) = \hat{\eta}_{2n+1}(t)$ is a monotonic function as indicated in (21).

The following corollary of Proposition 1 summarizes the stability properties of the proposed DREM-based MRAC.

Corollary 1. *Consider the plant (1) verifying Assumptions A.1–A.3 and the tracking error given in (3). The adaptive controller (5) and (6), where the parameters are updated with the DREM estimator (18), (19), (26) and (27), ensures the following properties.*

(a) *The following implications are true*

$$\mathrm{sign}(\hat{k}_p(0)) = \mathrm{sign}(k_p) \Rightarrow \text{no switching appears}$$

$$\mathrm{sign}(\hat{k}_p(0)) \ne \mathrm{sign}(k_p) \Rightarrow \text{at most one switching appears.}$$

(b) $\hat{k}_p(t)$ *is a monotonically non-increasing function, more precisely*

$$\tilde{k}_p(t_2) \le \tilde{k}_p(t_1), \ \forall t_2 \ge t_1 \ge 0.$$

(c) *If* $\Delta(t) \notin \mathcal{L}_2$ *then*

$$\lim_{t\to\infty} e(t) = 0, \ \lim_{t\to\infty} \tilde{\theta}_i(t) = 0.$$

4.3. Remarks

[R5] Two simple options for the linear, $\mathcal{L}_\infty$-stable operators H_i used to define the extended regressor matrix M_e (19) are LTI, first order *filters*

$$H_i^{\alpha_i, \beta_i}(p) = \frac{\alpha_i}{p + \beta_i}, \ \alpha_i \ne 0, \beta_i > 0, \tag{28}$$

or *delay* operators of the form

$$H_i^{T_i}(\cdot)(t) := (\cdot)(t - T_i), \ T_i > 0.$$

Note, that the filters have to be chosen distinct.

See [18] for a generalization to linear time-varying filters.

[R6] We underscore the fact that, to ensure that the matrix $M_e(t)$ is not rank deficient for all times, it is necessary to avoid the situation where $\psi_{H_i}(t)=\psi_{H_j}(t),\ \forall t\geq 0$ for $i\neq j$. Hence, the operators H_i should be *different*.

[R7] Unfortunately, we have not been able to prove that the tracking error goes to zero without the assumption that $r(t)$ is such that $\Delta(t)\notin\mathcal{L}_2$. In this respect, the proposed DREM MRAC suffers from the same drawback as the gradient-based MRAC with switching of Section 3 and the one of Section 4.5.2 of [5]. Namely, that there is no proof that the tracking error converges to zero *for all* reference signals without an excitation assumption. This is in contrast with classical MRAC with known sign(k_p) and Nussbaum-based MRAC, which ensure this property.

[R8] We have recently shown that, if $r(t)$ is sufficiently rich, in the classical sense of [19], then $\Delta(t)\notin\mathcal{L}_2$ for *almost all* operators H_i—implying that, under richness conditions, DREM-MRAC ensures parameter (hence, tracking error) convergence with the additional advantage, with respect to gradient estimation, of guaranteeing that the individual parameter errors are monotonic.

[R9] Notice that, even though *all* parameter errors $\tilde{\eta}_i(t)$ are monotonically non-increasing, this is not necessarily the case for $\tilde{\theta}(t)$ because they satisfy $\hat{\theta}_j=\frac{1}{\hat{k}_p}\hat{\eta}_j$ for $j=1,\ldots,2n$, and the division of two monotonic functions is not necessarily monotonic.

[R10] It worth noting that instead of constant adaptation gains $\gamma_j,\ j=1,\ldots,2n+1$ in (26) one can select time-varying functions in the form:

$$\gamma_j=\frac{\gamma_{0j}}{1+\Delta^2}.$$

where $\gamma_{0j},\ j=1,\ldots,2n+1$ are positive constants. Such a choice provides normalization of DREM estimator and, as a result, “almost” uniform convergence of parametric errors. Moreover, the rate of parametric convergence can be regulated by selecting the design parameters γ_{0j}.

Normalization gains are used in the next section with comparative simulations.

5. Simulations

In this section simulation results of three MRAC schemes for the second order plant

$$\left(p^2+d_1p+d_0\right)y=k_pu$$

with unknown parameters $d_1=2,\ d_0=1$ and $k_p=3$ are demonstrated and compared. The first scheme is based on the use of a Nussbaum gain, *cf.* (see Chapter 9 of [14]). The second and the third schemes use the gradient and DREM identifiers described in Sections 3 and 4, respectively.

The objective of MRAC is to ensure boundedness of all signals and asymptotic convergence to zero of the tracking error

$$e=y-\frac{k_m}{D_m(p)}r=y-\frac{6}{p^2+5p+6}r.$$

The reference signal is given as

$$r(t)=3\sin 2t+4\cos t+10,$$

which is sufficiently rich. In the case of MRAC with known sign(k_p), this ensures that the regressor ϕ is persistently exciting (PE) [19], but it is not the case for the switching scheme of Section 3 nor the estimator of Section 4.5.2 of [5]. The error model (4) takes the form

$$e=\frac{k_p}{p^2+5p+6}\left(u-\theta^\top\phi\right),$$

where $\theta=\mathrm{col}(-3,9,-3.67,2)$ and the regressor is given by

$$\phi=\mathrm{col}\left(\frac{1}{\lambda(p)}u,\ \frac{1}{\lambda(p)}y,\ y,\ r\right),$$

where we have selected $\lambda(p)=p+1$. The filtered regressor ϕ_f is then defined as

$$\phi_f=\frac{1}{D_m(p)}\phi=\frac{1}{p^2+5p+6}\phi.$$

The control signal is given by (6) with estimates $\hat{\theta}\in\mathbb{R}^4$, which are generated according to the three aforementioned schemes. Initial conditions for all algorithms are taken as $\hat{\theta}(0)=\mathrm{col}(-10,10,-5,5)$ and $\hat{k}_p(0)=-4$—notice that k_p has the *opposite sign*. The initial conditions for all the other state variables are zero.

5.1. Nussbaum gain MRAC

The algorithm using Nussbaum gain is taken from Section 9.2 of [14], which is a simplified version of the scheme proposed in [13]. It is based on the following modified augmented error scheme

$$\begin{aligned}\varepsilon&=e+N(x)k_N\chi\\ \chi&=\hat{\theta}^\top\phi_f-\frac{k_m}{D_m(p)}\left(\hat{\theta}^\top\phi\right),\end{aligned}$$

where the controller parameters and the gain k_N are given by

$$\begin{aligned}\dot{\hat{\theta}}&=N(x)\frac{\phi_f}{1+\phi_f^\top\phi_f}\varepsilon\\ \dot{k}_N&=-N(x)\frac{\chi}{1+\phi_f^\top\phi_f}\varepsilon,\end{aligned}$$

and the Nussbaum gain $N(x)$ is generated via

$$\begin{aligned}N(x)&=x^2\cos x\\ x&=z+\frac{k_N^2}{2}\\ \dot{z}&=\frac{1}{1+\phi_f^\top\phi_f}\varepsilon^2.\end{aligned}$$

Results of the simulation are shown in Fig. 1. As shown by the figure parameter convergence is very slow and we observe a very large overshoot of e, mainly due to the oscillating behavior of the Nussbaum gain $N(x)$. Furthermore, due to the presence of the second harmonic in the reference signal, there is a (slowly decaying) oscillation in the control signal that induces an oscillation in the tracking error.

5.2. Gradient-based MRAC

In this subsection, we present the simulation results with the gradient identifier (14). The adaptation gains were selected as $\Gamma_{\theta 0}=10I_4$, and $\gamma_{p0}=10$. This selection yields

$$\begin{aligned}\dot{\hat{\theta}}&=-\frac{10}{1+\psi^\top\psi}\left(\phi_f+u_f\hat{\theta}\right)\left(\frac{e}{\hat{k}_p}+\phi_f^\top\hat{\theta}-u_f\right)\\ \dot{\hat{k}}_p&=\frac{10}{1+\psi^\top\psi}u_f\left[e+\hat{k}_p\left(\phi_f^\top\hat{\theta}-u_f\right)\right],\end{aligned}$$

where

$$u_f=\frac{1}{D_m(p)}u=\frac{1}{p^2+5p+6}u,\ \psi=col(-\phi_fu_f).$$

The projection (15) was implemented with $\underline{k}_p=0.1$.

The simulation results are presented in Fig. 2. Compared with the Nussbaum gain estimator the tracking error e peak is significantly—almost ten times—smaller but the convergence of the controller parameters is still very slow and the long term oscillations persist. Moreover, the estimate of the high frequency gain

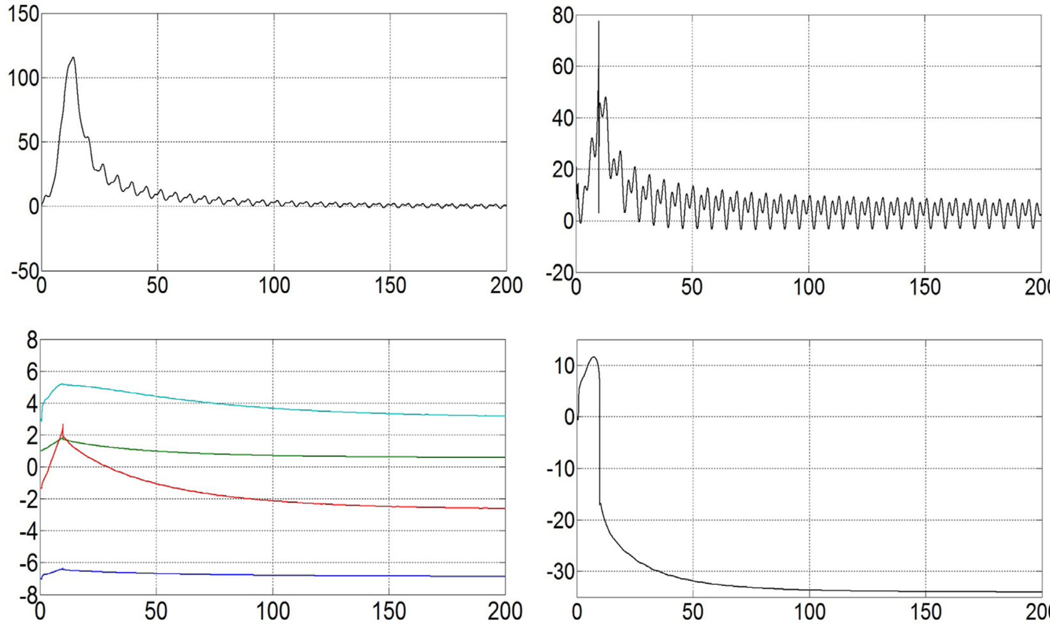

Fig. 1. (Nussbaum gain estimator) Behavior of the tracking error e (upper left corner), the control signal u (upper right corner), the controller parameter errors $\tilde{\theta}$ (lower left corner) and the Nussbaum gain N (lower right corner).

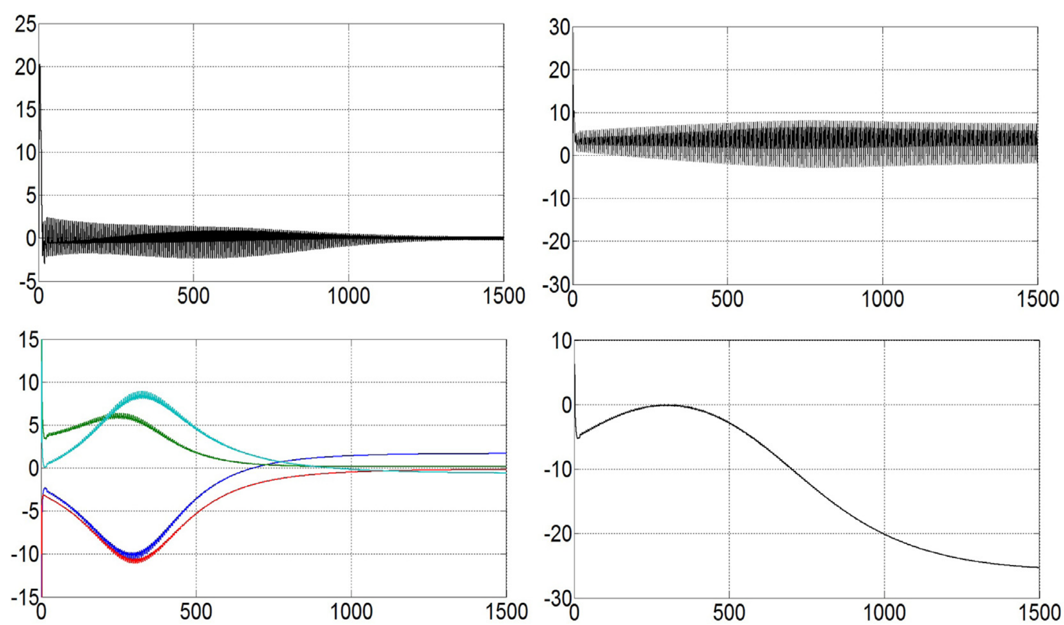

Fig. 2. (Gradient estimator) Behavior of the tracking error e (upper left corner), the control signal u (upper right corner), the controller parameter errors $\tilde{\theta}$ (lower left corner) and the estimation error $\tilde{k}_p$ (lower right corner).

converges very slowly to a value far away from k_p. This simulation evidence illustrates the fact that, in contrast with the case when the sign of k_p is known, $r(t)$ rich *does not* imply PE regressor for the switching MRAC of Section 3.[5]

In Fig. 3, we show a zoom of the first few seconds of the simulation when the estimate $\hat{k}_p$ changes sign, inducing a step change in the controller parameters. The simulation evidence seems to suggest that this step change induces the require excitation to the regressor, because in this time window the estimated parameters move towards their true values—but later drift away from it.

[5] Notice that in Theorem 4.5.3 of [5] parameter convergence is guaranteed under the assumption that the systems *regressor*—not the reference signal—is PE.

5.3. DREM-based MRAC

Implementation of the DREM-based adaptation algorithm (26) is made taking into account Remark [R10] and with the same adaptation gains as the gradient-based estimator, that is $\gamma_{j0} = 10$ for $j = 1, \ldots, 5$. This yields,

$$\dot{\hat{\theta}}_i = 10\frac{\Delta}{\hat{k}_p(1+\Delta^2)}\Big(E_i - E_5\hat{\theta}\Big),\ i = 1, 4$$

$$\dot{\hat{k}}_p = -10\frac{\Delta}{1+\Delta^2}(\Delta\hat{k}_p - E_5),$$

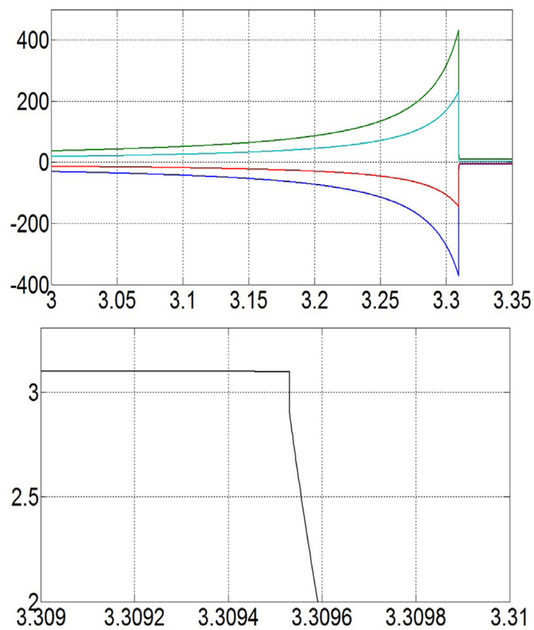

Fig. 3. (Gradient estimator) Zoom of $\tilde{\theta}$ (upper) and $\tilde{k}_p$ (lower) of Fig. 2.

where E, M_e and Δ are defined in (19) with the LTI filters

$$H_1(p) = \frac{1000}{p+1},\ H_2(p) = \frac{2000}{p+2},\ H_3(p) = \frac{4000}{p+4},$$
$$H_4(p) = \frac{8000}{p+8},\ H_5(p) = \frac{16000}{p+16}. \tag{29}$$

As there are no precise rules for the selection of the filters, this was done via trial and error—see [1,18] for some discussion in this respect. The switched projection (27) was also implemented with $\underline{k}_p = 0.1$ as before.

The simulation results depicted in Fig. 4 illustrate the dramatically improved performance achieved with DREM. We draw to the readers attention the *difference in scales*, both in time and in amplitude, of these plots with respect to the previous two cases. The

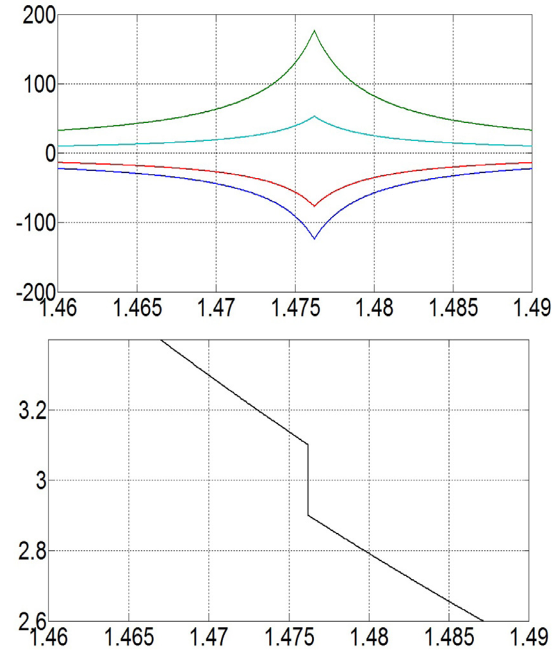

Fig. 5. (DREM estimator) Zoom of $\tilde{\theta}$ (top) and $\tilde{k}_p$ (bottom) of Fig. 4 .

controller parameters converge to their true values immediately after the estimate $\hat{k}_p$ changes sign as seen in Fig. 5, where we show a zoom of the first few seconds of the simulation. Consequently, the tracking error converges to zero in less than 5 s and remains at zero thereafter. This is in total contrast with the previous two estimators where convergence of e took significantly longer and a long oscillation was observed.

6. Concluding remarks and future research

An alternative solution to the problem of MRAC of LTI, minimum phase systems with unknown sign of the high frequency gain has been presented. The proposed scheme replaces the

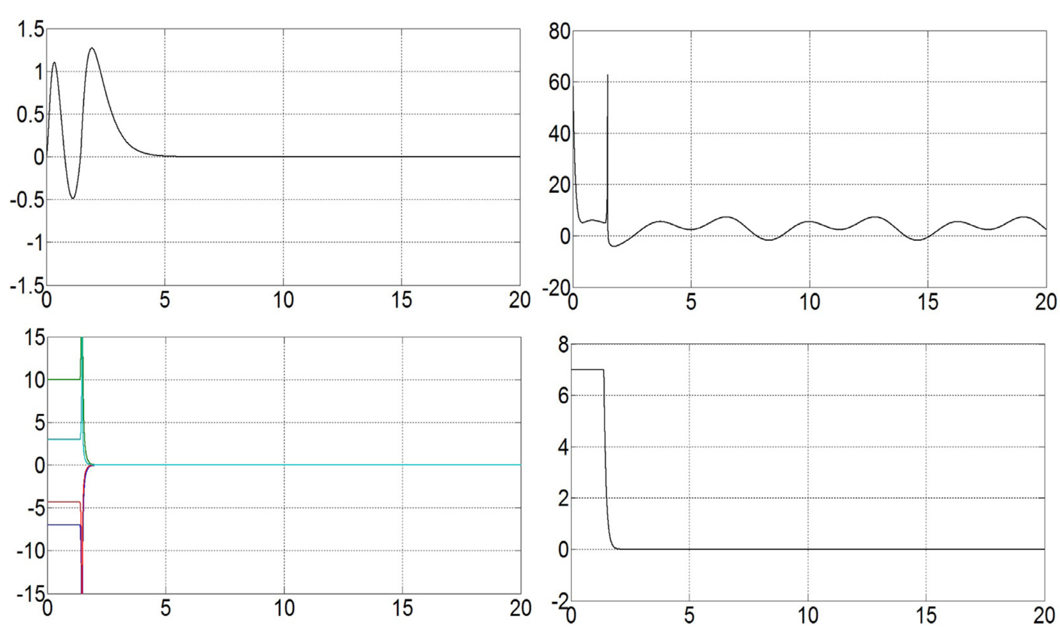

Fig. 4. (DREM estimator) Behavior of the tracking error e (upper left corner), the control signal u (upper right corner), the controller parameter errors $\tilde{\theta}$ (lower left corner) and the estimate error $\tilde{k}_p$ (lower right corner).

latter knowledge by the availability of a lower bound on the absolute value of this gain, required to implement a switched projection of the estimate. The main novelty is the use of a DREM estimator that guarantees, via the monotonicity of the parameter estimation error, that the switching happens at most once. Moreover, it is shown in simulations that, compared with the classical gradient and Nussbaum gain-based schemes, the proposed DREM-MRAC has a dramatically improved transient behavior.

Current research is under way along the following directions:

(i) Prove that DREM-MRAC enjoys the fundamental "self-tuning property" of standard MRAC, that is, that the tracking error converges to zero for all references, independently of the convergence of the estimated parameters. Although extensive simulations show that this is the case, a rigorous theoretical proof is yet to be established.

(ii) Although in the example in the simulation a good choice of the filters in the extended regressor matrix was quite easy to obtain, in general, it is still an open question how to select them in a systematic way. The main criterion for its selection is, of course, to try to guarantee the convergence condition $\Delta(t) \notin \mathcal{L}_2$, but there are no clear indications how this can be done in a systematic way. Some progress for this task is reported in [18], where it is shown that DREM can be recast as a Luenberger functional observer, for which a lot of research has been reported.

(iii) Extend the results to the case of multi-input multi-output systems. In this case, the high frequency gain is a matrix and, in spite of intensive research on the topic, no complete answers have been given on the prior knowledge required to design a provably stable MRAC. See [17] and the recent survey paper [20].

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