Strict Lyapunov Functions for Model Reference Adaptive Control: Application to Lagrangian Systems

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Abstract—**Model reference adaptive control is a well-understood and popular method used in the case when the plant's constant parameters are unknown and must be identified. The related literature is very rich and there exist many proofs of stability and convergence. Lyapunov functions for such systems, having the property that the total derivative is negative definite, are, however, very scarce. In this note, we use the *Mazenc construction* to design a simple strict Lyapunov function in a rather intuitive manner, based on a first-choice function whose derivative is negative semidefinite. Furthermore, we provide, for the first time in the literature, a Lyapunov function for a popular passivity-based adaptive controller for Lagrangian systems.**

Index Terms—**Adaptive control, identification, Lyapunov, persistency of excitation.**

I. INTRODUCTION

Driven by concrete control problems of physical systems (specifically, mechanical, electrical, and electromechanical), analysis and design of adaptive control systems have been, in turn, a steering force in control theory and practice for decades now. An early significant breakthrough was the certainty-equivalence principle, which states that in the case of parametric uncertainty, one can use the same controller that would be used in the case that all the parameters of the system were known, replacing their values with estimates. Then, an adaptation law is designed to update such estimates. In view of its simplicity, model-reference adaptive control [1] is among the most popular control and estimation techniques, but it also poses significant challenges to stability analysis, due to its inherent nonlinear and time-varying nature; even in the case when the plant to be controlled is linear and autonomous [2].

Two distinct analysis problems naturally appear: that of tracking-errors convergence and that of parameter-estimation errors convergence. For the former, a commonly used method is based on properties of signals in $\mathcal{L}_p$ spaces and the popular *Barbălat's lemma* [3]. Establishing convergence of the parameter estimation errors, however, is a much more challenging task of analysis. The one condition that is invoked under various control schemes is known as persistency of excitation and it was introduced in [4] in the context of identification of discrete-time linear systems. While conserving its conceptual substance, this condition has taken different mathematical forms in the literature, depending on the context: for continuous-time linear systems necessary and sufficient conditions for uniform convergence are presented in [5]–[7]; for non-uniform convergence, in [6]; for nonlinear time-varying systems, in [8] and [9]—see also [10]; for linear parameterized systems in [11]. Its equivalence to uniform complete observability is discussed in detail in [12] and its equivalence to detectability, in the realm of nonlinear time-varying systems, was established in [13]. In the specific context of adaptive control, [10] and [14] provide a thorough study.

Now, because adaptive control systems are inherently nonlinear, a crucial question beyond that of convergence is whether the origin is *uniformly* asymptotically, or exponentially, stable. For linear systems, the latter may be established upon the scrutiny of the state transition matrix [15], or using the concept of uniform complete observability and output injection techniques [12]; see also [16], where an explicit stability bound is computed. For nonlinear systems, however, exponential stability is impossible [17]; one can "only" achieve uniform global asymptotic stability [18].

Certainly, the most *direct* method of stability analysis is Lyapunov's second. Yet, constructing a strict Lyapunov function for adaptive control systems has eluded the research community, at least until [19] where, to the best of our knowledge, the first strict Lyapunov function for nonlinear systems reminiscent of model-reference-adaptive control was proposed. Mazenc *et al.* [19] relies on the neat *Mazenc construction*, which was introduced in [20] and is described in great detail in [21]. Roughly speaking, the starting point is a Lyapunov function whose derivative is negative in part of the state variables only; then, a function of this nonstrict Lyapunov function is constructed. See also [22] and [23].

In this note, we employ the Mazenc construction on a narrower class of systems than the one considered in [19] and [22], but the systems that we study here appear in model-reference-adaptive control of nonlinear systems and other passivity-based control techniques. For the purpose of illustration, we also propose, as far as we know for the first time, a strict Lyapunov function for the popular Slotine-and-Li adaptive tracking controller for Lagrangian systems originally proposed in [24], which continues to inspire solutions to more contemporary problems, such as mutual synchronization of Lagrangian systems—see e.g., [25]–[28].

The rest of this note is organized as follows. In the next section, we describe the class of systems that we consider. In Section II-C, we present our main results, the utility of which is illustrated through a concrete example in Section III. Finally, we conclude with some remarks in Section IV.

II. PROBLEM SETTING AND ITS SOLUTION

A. Motivating Example

We start by emphasizing the well-known fact that, even when the plant is linear autonomous, model-reference-adaptive-control systems

are nonlinear. To that end, let us consider a feedback-linearizable system, in normal form, i.e.

$$\dot{x}_i = x_{i+1} \qquad i \in [1, n-1] \tag{1a}$$

$$\dot{x}_n = \Phi(x)^\top \theta + g(x)u \tag{1b}$$

where $x := [x_1 \cdots x_n]^\top$, $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a regressor function and $\theta \in \mathbb{R}^m$ is a vector of unknown lumped parameters. The control goal is to design u such that this system behaves as the reference model

$$\dot{x}_i^* = x_{i+1}^* \qquad i \in [1, n-1] \tag{2a}$$

$$\dot{x}_n^* = f(x^*) \tag{2b}$$

or, in other words, to steer $x(t) \to x^*(t)$, where $x^*(t)$ is solution of (2). The feedback-linearizing control input that achieves this goal is $u := g(x)^{-1} \left[f(x^*) - \Phi(x)^\top \theta - Ke \right]$ with $e := x - x^*$ and $K := \begin{bmatrix} \kappa_1 \cdots \kappa_n \end{bmatrix}$. In the case, that the parameters θ are unknown, we use the certainty-equivalence adaptive controller

$$u = g(x)^{-1} \left[f(x^*) - \Phi(x)^\top \hat{\theta} - Ke \right] \tag{3a}$$

$$\dot{\hat{\theta}} = \gamma \Phi(x) \mathrm{B}^\top P e \tag{3b}$$

where B is part of a controllable pair (A, B), A is Hurwitz by design, and $P = P^\top > 0$ satisfies $A^\top P + PA = -Q$, given an arbitrary positive-definite symmetric matrix Q. Then, defining the parameter-estimation errors $\tilde{\theta} := \hat{\theta} - \theta$, we see that the closed-loop system takes the form

$$\dot{e} = Ae + B(t, e)\tilde{\theta} \tag{4a}$$

$$\dot{\tilde{\theta}} = -\gamma B(t, e)^\top P e \tag{4b}$$

with $B(t, e) := -\mathrm{B}\Phi(e + x^*(t))^\top$.

Clearly, the equation $\dot{e} = Ae$ corresponds to that of the tracking errors under the assumption that the parameters are known, which is the starting point in certainty-equivalence-based adaptive control. If B depended only on time, we would recover a well-studied system in textbooks on adaptive control, e.g., [15]. However, even in the simplest case in which (1) is linear, that is, if $\Phi(x)^\top \theta := x^\top \theta$ and $g(x) \equiv$ const., the resulting closed-loop system is *nonlinear time varying*; this is a well-documented fact [2], [10] that is sometimes obliterated in the literature. Thus, we see that the benefits of feedback linearization are lost in the case of uncertainties, whereas the use of other nonlinear control techniques, such as passivity based, do not necessarily render the stability analysis problem significantly more complex. In either case, we are often confronted to analyze systems of the form

$$\dot{x}_1 = F(t, x_1) + \Phi(t, x_1)x_2 \tag{5a}$$

$$\dot{x}_2 = -\sigma \Phi(t, x_1)^\top \frac{\partial V_1}{\partial x_1}^\top, \quad \sigma > 0 \tag{5b}$$

with states $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$, $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a function that is (at least) one continuously differentiable and, for the purpose of existence and uniqueness of solutions, the functions F and $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, and $\frac{\partial V_1}{\partial x_1}$ are locally Lipschitz continuous in x, uniformly in t.

Systems (5), under the following standing hypotheses that we describe in detail below, are our subject of study.

B. Main Assumptions

Assumption 1: There exist $\mathcal{K}_\infty$ functions β_1, $\beta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, positive definite functions ρ_1, $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a positive-definite

and radially unbounded function $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\beta_1(|x_1|) \leq V_1(t, x_1) \leq \beta_2(|x_1|) \tag{6a}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} F(t, x_1) \leq -\gamma_1(|x_1|) \tag{6b}$$

$$\max\left\{ \left| \frac{\partial V_1}{\partial x_1} \right|, |F(t, x_1)| \right\} \leq \rho_1(|x_1|). \tag{6c}$$

The condition (6c) simply imposes that $F(t, 0) \equiv 0$ and $F(t, x_1) \equiv 0$ is uniformly bounded in t; similarly, for $\partial V_1 / \partial x_1$.

More importantly, the class of systems for which inequalities (6a) and (6b) hold, includes marginally stable systems. This follows, by a direct computation, observing that the function

$$V_2(t, x) := V_1(t, x_1) + \frac{1}{2\sigma}|x_2|^2 \tag{7}$$

where $x := [x_1^\top \ x_2^\top]^\top$, is positive definite and radially unbounded and, moreover, its total derivative along the trajectories of (5) yields

$$\dot{V}_2(t, x) \leq -\gamma_1(|x_1|) \tag{8}$$

which is negative semidefinite. Hence, a key feature of the systems (5) under Assumption 1 is the property (8) from which, together with (6a), one can conclude uniform global stability and even infer that $x_1 \to 0$. For instance, if $\gamma_1(s) \propto s^2$, we see that $x_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and $x_2 \in \mathcal{L}_\infty$. In this case, standard arguments that appeal to Barbălat's lemma may be used to conclude that $x_1 \to 0$.

In the following, we construct a strict Lyapunov function for (5) by regarding this system as if composed of a nominal (simpler) part and some vanishing *output-injection* terms that depend on x_1. That is, let

$$K(t, x_1) := [\Phi(t, x_1) - \Phi_\circ(t)] \quad \text{and} \quad \Phi_\circ(t) := \Phi(t, 0). \tag{9}$$

Then, the equations in (5) are equivalent to

$$\dot{x}_1 = F(t, x_1) + \Phi_\circ(t)x_2 + K(t, x_1)x_2 \tag{10a}$$

$$\dot{x}_2 = -\sigma \Phi_\circ(t)^\top \frac{\partial V_1}{\partial x_1}^\top - \sigma K(t, x_1)^\top \frac{\partial V_1}{\partial x_1}^\top. \tag{10b}$$

As previously discussed, in view of Assumption 1, we have $x_1(t) \to 0$ which, in view of (9), implies that $K(t, x_1(t)) \to 0$. Furthermore, since by assumption $\Phi(t, x_1)$ is locally Lipschitz in x_1, uniformly in t and it is uniformly bounded in t, there exist a continuous nondecreasing function $\rho_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$|K(t, x_1)| \leq \rho_2(|x_1|). \tag{11}$$

On the other hand, as we shall see through an example in Section III, $\Phi_\circ$ is reminiscent of a function of the system's dynamics, evaluated along exogenous signals, such as reference trajectories. This is all the more significant since our second main hypothesis is inspired by the fact that persistency of excitation of $\Phi_\circ$, and not of $\Phi(t, x)$ as it might be supposed, is sufficient and necessary for uniform global asymptotic stability [18]. This brings us to our second main hypothesis.

Assumption 2: There exist real positive numbers μ, T, and ϕ_M such that

$$\max\left\{ |\Phi_\circ(t)|_\infty, |\dot{\Phi}_\circ(t)|_\infty \right\} \leq \phi_M \tag{12a}$$

$$\int_t^{t+T} \Phi_\circ(s)^\top \Phi_\circ(s)\,ds \geq \mu I \qquad \forall\, t \geq 0. \tag{12b}$$

C. Main Result

Theorem 1: Consider the system (5) under Assumptions 1 and 2. Then, the origin is uniformly globally asymptotically stable and it admits a strict Lyapunov function $\mathcal{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ defined as follows:

$$\mathcal{V}(t, x) := 2c\sigma V_2(t, x) + V_3(t, x) + \frac{1}{4} V_4(t, x_2) + W(t, x) \quad (13)$$

where $c := \phi_M + \frac{\phi_M^2}{2}$

$$V_3(t, x) := \sum_{i=1}^{3} \alpha_i \big(V_2(t, x) \big), \quad i \in \{1, 2, 3\} \quad (14a)$$

$$V_4(t, x_2) := -\int_t^\infty \big| \Phi_\circ(\tau) x_2 \big|^2 e^{t-\tau} d\tau \quad (14b)$$

$$W(t, x) := -x_1^\top \Phi_\circ(t) x_2 \quad (14c)$$

and $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are positive definite nondecreasing functions such that, for any λ_1 and $\lambda_2 > 0$, it holds that

$$\frac{\partial \alpha_1}{\partial V_2}(V_2(t, x)) \gamma_1(|x_1|) \geq \phi_M \Big[\rho_2(|x_1|) - \rho_2(0) \Big] |x_2|^2 \quad (15)$$

$$\frac{\partial \alpha_2}{\partial V_2} \big(\beta_1(|x_1|) \big) \gamma_1(|x_1|) \geq \frac{\lambda_2}{2} \phi_M^2 |x_1|^2 + \frac{\lambda_1}{2} \rho_1(|x_1|)^2$$
$$+ \gamma_2'(|x_1|) + \rho_3(|x_1|)|x_1| \quad (16)$$

$$\alpha_3 \big(\beta_1(|x_1|) \big) \geq c|x_1|^2 \quad (17)$$

where ρ_1 is defined in Assumption 1, ρ_2 is introduced in (11) and, if necessary, it is redefined to satisfy

$$8 \rho_2(0) \phi_M \leq \mu e^{-T}. \quad (18)$$

In addition, the function $\rho_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as

$$\rho_3(|x_1|) := \sigma \phi_M \big[\rho_2(|x_1|) + \phi_M \big] \rho_1(|x_1|) \quad (19)$$

and, for any positive-definite function $\gamma_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\gamma_2'(|x_1|) := \gamma_2(|x_1|) + 2\lambda_2 \phi_M^2 \rho_3(|x_1|)^2. \quad (20)$$

∎

The inequalities (15)–(17) are all met by construction and $K(t, 0) \equiv 0$ so, ρ_2 may be chosen such that $\rho_2(0) = 0$; this obviates the condition (18). Nonetheless, defining ρ_2 so that $\rho_2(0) > 0$ be arbitrarily small may result convenient to verify inequality (15) *locally*, in cases in which the growth of $\rho_2(s)$ is higher than that of $\gamma_1(s)$ near the origin—see Section III ahead. Furthermore, we stress that, through V_2, α_1 is a function of $|x_2|^2$ and since it is smooth by design, $|x_2|^2$ may be *factored out* of $\partial \alpha_1 / \partial V_2$ using the relation

$$\varphi(v) = v \int_0^1 \frac{\partial \varphi}{\partial v}(sv) ds + \varphi(0)$$

which holds for any smooth function $v \mapsto \varphi$. For instance, if α_1 is constructed as a polynomial of degree p, that is

$$\alpha_1(V_2) := k_1 V_2 + k_2 V_2^2 + \cdots + k_q V_2^p \quad (21)$$

Inequality (15) holds if there exist reals $k_i \geq 0$ such that

$$\gamma_1(|x_1|) \sum_{i=2}^{p} i k_i \frac{i-1}{2\sigma} \beta_1(|x_1|)^{i-2} \geq \phi_M \big[\rho_2(|x_1|) - \rho_2(0) \big] \quad (22)$$

where β_1 is introduced in Assumption 1.

Proof of Theorem 1: First, we prove that $\mathcal{V}$ is positive definite on $\mathbb{R}^{n+m}$ and radially unbounded. Then, we show that its total derivative satisfies

$$\dot{\mathcal{V}} \leq -\frac{1}{4} |\Phi_\circ(t) x_2|^2 - \frac{\mu e^{-T}}{8} |x_2|^2 - \gamma_2(|x_1|) \quad (23)$$

which is negative definite.

To establish the positivity and the radial-unboundedness of $\mathcal{V}$, we first observe that

$$\frac{\phi_M}{2} \big[|x_1|^2 + |x_2|^2 \big] \geq W(t, x) \geq -\frac{\phi_M}{2} \big[|x_1|^2 + |x_2|^2 \big]$$

while the function V_4 defined in (14b) satisfies

$$-\phi_M^2 |x_2|^2 \leq V_4(t, x_2) \leq -\mu e^{-T} |x_2|^2. \quad (24)$$

Thus, after (17), we obtain

$$2\sigma c V_2(t, x) + \alpha_3(V_2(t, x)) + \frac{1}{4} V_4(t, x_2) + W(t, x) \geq \frac{c}{2} |x|^2.$$

Furthermore, $\mathcal{V}(t, x)$ is decrescent because it is bounded from above by

$$\big[\alpha_1 + \alpha_2 + \alpha_3 + 2\sigma c \, \mathrm{Id} \big] \circ \Big(\beta_2(|x_1|) + \frac{1}{2\sigma} |x_2|^2 \Big) + \phi_M |x|^2$$

where $\mathrm{Id}(s) = s$.

We proceed now to evaluate the total derivative of $\mathcal{V}$ along the trajectories of (5). To that end, let us define $\xi := \Phi_\circ(t) x_2$. Then, the total derivative of W along the trajectories of (5) yields

$$\dot{W}(t, x) = -F(t, x_1)^\top \xi - |\xi|^2 - x_2^\top K(t, x_1) \Phi_\circ(t) x_2$$
$$- x_1^\top \dot{\Phi}_\circ(t) x_2 - \sigma x_1^\top \Phi_\circ(t) \Phi(t, x_1)^\top \frac{\partial V_1}{\partial x_1}^\top$$

and, for any $\lambda_1, \lambda_2 > 0$, we have

$$\dot{W}(t, x) \leq \frac{\lambda_1}{2} \rho_1(|x_1|)^2 + \frac{|\xi|^2}{2\lambda_1} - |\xi|^2 + \phi_M |x_2|^2 \rho_2(|x_1|)$$
$$+ \frac{|x_2|^2}{2\lambda_2} + \frac{\lambda_2 \phi_M^2}{2} |x_1|^2$$
$$+ \sigma \phi_M |x_1| \big[\rho_2(|x_1|) + \phi_M \big] \rho_1(|x_1|). \quad (25)$$

In the computation of the latter, we used Young's inequality to bound cross-terms as well as (6c), (11), and (12a).

The total derivative of $V_3(t, x)$ satisfies

$$\dot{V}_3(t, x) \leq -\Big[\frac{\partial \alpha_1}{\partial V_2} + \frac{\partial \alpha_2}{\partial V_2} \Big] \gamma_1(|x_1|) \leq 0 \quad (26)$$

where we used $[\partial \alpha_3 / \partial V_2] \dot{V}_2 \leq 0$, which holds since $\alpha_3(V_2)$ is non-decreasing and $\dot{V}_2 \leq 0$. Putting together (25), (26), and using (16), we see that the total derivative of $V_3 + W$ along the trajectories of (10) yields

$$\dot{V}_3(t, x) + \dot{W}(t, x) \leq -\Big[\frac{\partial \alpha_1}{\partial V_2} \gamma_1(|x_1|) - \phi_M \rho_2(|x_1|)|x_2|^2 \Big]$$
$$- \Big[1 - \frac{1}{2\lambda_1} \Big] |\xi|^2 - \gamma_2'(|x_1|) + \frac{1}{2\lambda_2} |x_2|^2$$

so fixing $\lambda_1 > 1$, we obtain, in view of (15),

$$\dot{V}_3(t, x) + \dot{W}(t, x) \leq -\frac{1}{2} |\xi|^2 - \gamma_2'(|x_1|)$$
$$+ \Big[\frac{1}{2\lambda_2} + \phi_M \rho_2(0) \Big] |x_2|^2.$$

Finally, we evaluate the total derivative of V_4 in (13) and add it to the previous expression. We have

$$\dot{V}_4(t, x_2) = |\xi|^2 + V_4(t, x_2)$$
$$+ 2\sigma x_2^\top \left[\int_t^\infty \Phi_\circ(\tau)^\top \Phi_\circ(\tau) \Phi(t, x) e^{t-\tau} d\tau \right] \frac{\partial V_1}{\partial x_1}^\top$$

so, using (12a), we obtain

$$\dot{V}_4(t, x_2) \leq -\mu e^{-T} |x_2|^2 + |\xi|^2 + \frac{1}{2\lambda_2} |x_2|^2$$
$$+ 2\lambda_2 \sigma^2 \phi_M^4 \left[\phi_M + \rho_2(|x_1|) \right]^2 \rho_1(|x_1|)^2.$$

Thus, setting λ_2 to satisfy

$$\frac{1}{\lambda_2} < \frac{1}{8} \mu e^{-T} - \phi_M \rho_2(0)$$

and in view of (20), we obtain inequality (23). ∎

III. ADAPTIVE CONTROL OF LAGRANGIAN SYSTEMS

We illustrate the utility of Theorem 1 by presenting, as far as we know for the first time, a proof via Lyapunov's direct method of uniform global asymptotic stability for a well-known adaptive controller for Euler–Lagrange systems. These are defined by the equations

$$D_\theta(q)\ddot{q} + C_\theta(q, \dot{q})\dot{q} + g_\theta(q) = u, \qquad q \in \mathbb{R}^n \qquad (27)$$

where $D_\theta : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ defines the inertia matrix, which is positive definite symmetric, $C_\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ defines the Coriolis-and-centrifugal-forces matrix, which satisfies

$$\dot{D}_\theta(q) = C_\theta(q, \dot{q}) + C_\theta(q, \dot{q})^\top. \qquad (28)$$

The function $g_\theta : \mathbb{R}^n \to \mathbb{R}^n$ defines the gravity forces vector. All these functions depend, in addition to the generalized positions q and generalized velocities $\dot{q}$, on lumped constant parameters $\theta \in \mathbb{R}^m$. It is assumed, moreover, that D_θ, C_θ, and g_θ are linear in θ, that is, there exists a function $\Upsilon : \mathbb{R}^{3n} \to \mathbb{R}^{m \times n}$ such that the right-hand side of (27) equals to $\Upsilon(q, \dot{q}, \ddot{q})^\top \theta$. Furthermore, in view of (28), C_θ is linear in $\dot{q}$ and, as it is customary, it is assumed that D_θ and C_θ is uniformly bounded for all $q \in \mathbb{R}^n$. More precisely, there exist d_m, d_M, and $k_c > 0$, such that, for all $q, \dot{q} \in \mathbb{R}^n$, and $\theta \in \mathbb{R}^m$

$$d_m \leq |D_\theta(q)| \leq d_M \qquad (29a)$$
$$|C_\theta(q, \dot{q})| \leq k_c |\dot{q}|. \qquad (29b)$$

The adaptive state-feedback tracking control problem for (27) consists in the following. Given a sufficiently smooth reference trajectory $t \mapsto q_d$, bounded and with bounded derivatives, find an adaptive controller so that $\tilde{q} := q - q_d$ and $\dot{\tilde{q}} := \dot{q} - \dot{q}_d$ tend to zero asymptotically. This problem is solved, at least, since the seminal paper [24], where the (non-uniform) convergence of the tracking errors is established and, under a condition of persistency of excitation, in [29], it is showed that the parameter estimation errors also converge to zero. In [30], a strict Lyapunov function, with which global asymptotic stability is established, is given for the non-adaptive case; see also [31] for a statement on global exponential stability. To the best of our knowledge, the first proof of *uniform* global asymptotic stability in the adaptive case (hence, including uniform convergence of the parameter estimation errors) under the action of the controller proposed in [24] was established in [32], under a condition on persistency of excitation along the *reference* trajectories. Here, we present, as far as we know for the first time, a

proof of uniform global asymptotic stability for the adaptive controller of [24], via Lyapunov's direct method.

We recall that the adaptive controller of [24] is given by

$$u = D_{\hat{\theta}}\ddot{q}_r + C_{\hat{\theta}}(q, \dot{q})q_r + g_{\hat{\theta}}(q) - k_d s \qquad (30a)$$
$$\dot{\hat{\theta}} = -\sigma \Psi(t, \tilde{q}, s)s, \quad \sigma > 0 \qquad (30b)$$

where $s := \dot{q}_r - \dot{q}$, $\dot{q}_r := \dot{q}_d(t) - \lambda \tilde{q}$, and λ, k_d are positive constants. Furthermore, $\hat{\theta}$ is an estimate of the unknown lumped parameters θ and Ψ is the so-called regressor function which is defined by the identity

$$D_{\tilde{\theta}}(q)\ddot{q}_r + C_{\tilde{\theta}}(q, \dot{q})\dot{q}_r + g_{\tilde{\theta}}(q) =: \Psi(t, \tilde{q}, s)^\top \tilde{\theta} \qquad (31)$$

which holds in view of the linearity of D_θ, C_θ, and g_θ with respect to θ. For the purpose of analysis, we stress that the regressor Ψ is a function of $(t, \tilde{q}, s)$; indeed, we observe that even though the terms on left-hand side of (31) are written as functions of q, $\dot{q}$, $\ddot{q}_r$, and $\dot{q}_r$, the latter are, in turn, functions of the closed-loop variables $\tilde{q}$ and s, as well as time, through the reference trajectories $\dot{q}_d(t)$ and its derivatives. That is

$$q = \tilde{q} + \dot{q}_d(t), \qquad \dot{q} = \dot{q}_d(t) - \lambda \tilde{q} - s$$
$$\dot{q}_r = \dot{q}_d(t) - \lambda \tilde{q}, \qquad \ddot{q}_r = \ddot{q}_d(t) - \lambda s + \lambda^2 \tilde{q}.$$

Thus, with an abuse of notation, we write the closed-loop dynamics of (27) with (30) as

$$D_\theta(q)\dot{s} + C_\theta(q, \dot{q})s + k_d s = \Psi(t, \tilde{q}, s)^\top \tilde{\theta} \qquad (32a)$$
$$\dot{\tilde{q}} = -\lambda \tilde{q} + s \qquad (32b)$$
$$\dot{\tilde{\theta}} = -\sigma \Psi(t, \tilde{q}, s)s. \qquad (32c)$$

Now, as it was established in [30], a direct computation shows that the total derivative of the function

$$V(t, \tilde{q}, s) := \frac{1}{2} s^\top D_\theta(q)s + \frac{\lambda k_d}{2} |\tilde{q}|^2 + \frac{1}{2\sigma} |\tilde{\theta}|^2 \qquad (33)$$

along the trajectories of (32) yields

$$\dot{V}(t, \tilde{q}, s) \leq -\frac{1}{2} \left[k_d |s|^2 + \lambda k_d |\tilde{q}|^2 \right].$$

Remark 1: Once more, in the previous expressions, we used the familiar notation $D_\theta(q)$ instead of the more appropriate, but cumbersome one, $D_\theta(\tilde{q} + q_d(t))$. •

This is the starting point to establish a direct proof of the following statement, using Lyapunov's first method—cf. [32].

Proposition 1: Consider the system (27) and assume that the inequalities in (29) hold. For the reference trajectories $t \mapsto q_d$ assume that there exists $\beta_d > 0$, such that

$$\max \left\{ |\ddot{q}_d|_\infty, |\dot{q}_d|_\infty, |q_d|_\infty \right\} \leq \beta_d \qquad (34)$$

and there exist T and $\mu > 0$ such that

$$\int_t^{t+T} \Psi(\tau, 0, 0) \Psi(\tau, 0, 0)^\top d\tau \geq \mu I \quad \forall t \geq 0. \qquad (35)$$

Then, the origin in the state space of the closed-loop system (32), that is $\{(\tilde{q}, s, \tilde{\theta}) = (0, 0, 0)\}$, is uniformly globally asymptotically stable. ∎

Remark 2: We stress that $\Psi(t, 0, 0)$ is a matrix that depends only on the reference trajectories and its derivatives and solves the identity

$$D_{\tilde{\theta}}(q_d)\ddot{q}_d + C_{\tilde{\theta}}(q_d, \dot{q}_d)\dot{q}_d + g_{\tilde{\theta}}(q_d) =: \Psi(t, 0, 0)^\top \tilde{\theta} \qquad (36)$$

in which we dropped the argument (t) from $q_d(t)$ and its derivatives. Moreover, the condition (35) is also necessary for uniform global asymptotic stability [32]. •

The statement of Proposition 1 is a consequence of the following original result.

Proposition 2: The closed-loop system (32), under the conditions of Proposition 1, admits the strict Lyapunov function

$$\mathcal{V}(t, \tilde{q}, s, \tilde{\theta}) := \sum_{j=1}^{3} \delta_j \left[\frac{1}{2} s^\top D_\theta(q(t))s + \frac{\lambda k_d}{2} |\tilde{q}|^2 + \frac{1}{2\sigma} |\tilde{\theta}|^2 \right]^j$$
$$- \int_t^\infty \left| D_\theta(q_d(\tau))^{-1} \Psi(\tau, 0, 0)^\top \tilde{\theta} \right|^2 e^{t-\tau} d\tau$$
$$- s^\top D_\theta(q_d(t))^{-1} \Psi(t, 0, 0)^\top \tilde{\theta} \qquad (37)$$

which is positive definite, radially unbounded, and its derivative satisfies

$$\dot{\mathcal{V}}(t, \tilde{q}, s, \tilde{\theta}) \le -\frac{1}{2} \gamma_1' \left[|s|^2 + |\tilde{q}|^2 \right] - \frac{1}{8} \mu e^{-T} |\tilde{\theta}|^2 \qquad (38)$$

where $\gamma_1' := (k_d/2) \min \{1, \lambda\}$, for sufficiently large values of the constant parameters $\delta_j > 0$. ∎

Proof: The construction of $\mathcal{V}$ follows by verifying the conditions of Theorem 1. The first step is to write the closed-loop equations in the form (5). To that end, let

$$x_1 := [\tilde{q} \ s]^\top, \quad x_2 := \tilde{\theta}.$$

Then, since $D_\theta(q)$ is invertible for all $q \in \mathbb{R}^n$, we may define

$$F(t, x_1) := \begin{bmatrix} -\lambda I & I \\ 0 & [D_\theta^{-1} C_\theta + k_d I] \end{bmatrix} \begin{bmatrix} \tilde{q} \\ s \end{bmatrix} \qquad (39)$$

$$\Phi(t, x_1) = \begin{bmatrix} 0 \\ D_\theta^{-1} \Psi(t, \tilde{q}, s)^\top \end{bmatrix} \qquad (40)$$

in which, to simplify the notation, we dropped the arguments of $D_\theta(\tilde{q} + q_d(t))^{-1}$ and $C_\theta(\tilde{q} + q_d(t), \dot{q}_d(t) - \lambda \tilde{q} - s)$.

The function V_1 in Assumption 1 corresponds to the first two terms of the function V in (33) and corresponds to the function proposed in [30], which we repeat here for convenience

$$V_1(t, x_1) := \frac{1}{2} s^\top D_\theta(q(t))s + \frac{\lambda k_d}{2} |\tilde{q}|^2. \qquad (41)$$

That is

$$\frac{\partial V_1}{\partial x_1}^\top = \begin{bmatrix} \lambda k_d \tilde{q} \\ D_\theta(q(t))s \end{bmatrix}$$

so, (32c) has exactly the form (5b) and, actually, the system (32) has the form of (5).

Now, we verify Assumption 1. On one hand, the inequalities in (6a) follow from (29) with

$$\beta_1(|x_1|) := \beta_1' |x_1|^2, \quad \beta_2(|x_1|) := \beta_2' |x_1|^2 \qquad (42)$$

where

$$\beta_1' := \frac{1}{2} \min \{d_m, \lambda k_d\} \qquad \beta_2' := \frac{1}{2} \max \{d_M, \lambda k_d\}.$$

On the other hand, a direct computation using (28) shows that (6b) holds with $\gamma_1(|x_1|) := \gamma_1' |x_1|^2$ and, after (29a)

$$\left| \frac{\partial V_1}{\partial x_1} \right| \le \rho_1' |x_1|, \quad \rho_1' := \max \{d_M, \lambda k_d\}.$$

In addition, a simple inspection of (39) and the inequalities in (29) shows that $F(t, x_1)$ is of order $|x_1|^2$ for large $|x_1|$ and of order $|x_1|$ near zero. Hence, (6c) holds with

$$\rho_1(|x_1|) := \rho_{11} |x_1| + \rho_{12} |x_1|^2 \qquad (43)$$

where $\rho_{11} \ge \rho_1'$ and ρ_{12} depend on β_d in (34) as well as on d_m and k_c in (29).

Next, we verify Assumption 2. Inequality (12a) holds because $\Phi_\circ$ and $\dot{\Phi}_\circ$ are continuous functionals of q_d, $\dot{q}_d$, and $\ddot{q}_d$, which are bounded—see (34), whereas (12b) is equivalent to (35), since $D_\theta(\cdot)^{-1}$ is full rank.

Now, we proceed to construct three functions α_1, α_2, and α_3, satisfying (15)–(17). The fact that β_1, β_2, ρ_1, and γ_1 are polynomials motivates a similar choice for the functions α_i. To determine their respective orders [so that (15)–(17) hold] it is left to find an appropriate function ρ_2 which, in turn, shall determine ρ_3. For this, we see that all the functions on the left-hand side of (27) are at most of quadratic growth hence, in view of (29) and (34), there exist ρ_{21} and $\rho_{22} > 0$ such that

$$\left| \Psi(t, \tilde{q}, s) - \Psi(t, 0, 0) \right| \le \rho_{21} |x_1| + \rho_{22} |x_1|^2. \qquad (44)$$

This implies that (11) holds with $\rho_2(|x_1|) = \rho_{21} |x_1| + \rho_{22} |x_1|^2$. Nonetheless, another simple inspection shows that with this choice of ρ_2, the condition (15) may not hold locally (i.e., near zero) the left-hand side of this inequality is at least of second order in $|x_1|$ (due to the factor γ_1), whereas the right-hand side contains a first-order term of $|x_1|$. Therefore, we use the inequality $|x_1| \le 1 + |x_1|^2$ to redefine ρ_2 as

$$\rho_2(|x_1|) := \frac{\varepsilon \rho_{21}}{2} + \left[\frac{\rho_{21}}{2\varepsilon} + \rho_{22} \right] |x_1|^2, \quad \varepsilon > 0. \qquad (45)$$

With this new choice of ρ_2 (11) still holds and, moreover, inequality (22) holds with $p = 2$, for sufficiently large k_2; in turn, this implies (15). It is to be noted that the choice of $\varepsilon > 0$ is determined by (18), this lifts any possible restriction on the parameters μ and T, which are given. Thus, we conclude that defining $\alpha_1(V_2)$ as a second-order polynomial of V_2 will do.

Next, we construct α_2 so that inequality (16) hold. Let $\gamma_2(|x_1|) := (1/2)\gamma_1' |x_1|^2$. A simple inspection, using (43) and (45), shows that ρ_3 in (19) is a polynomial of fourth order, hence the right-hand side of (16) is bounded from above by

$$\sum_{j=2}^{5} \varrho_j |x_1|^j, \qquad \varrho_j > 0.$$

Therefore, using the triangle inequality

$$\varrho_j |x_1|^j \le \frac{1}{2} \left[\varrho_j |x_1|^{j-1} + \varrho_j |x_1|^{j+1} \right]$$

for $j = 3$ and $j = 5$, we see that (16) holds if $\alpha_2(V_2)$ is a polynomial of third order that is, if

$$\alpha_2(V_2) := \sum_{j=1}^{3} k_j V_2^j.$$

Indeed, if such is the case

$$\frac{\partial \alpha_2}{\partial V_2} \left(\beta_1' |x_1|^2 \right) \gamma_1' |x_1|^2 \ge \gamma_1' \sum_{j=1}^{3} j k_j \beta_1'^{j-1} |x_1|^{2j}.$$

So, (16) holds provided that k_j are sufficiently large.

The term $\alpha_3(V_2)$ is designed to make $\mathcal{V}$ positive definite and radially unbounded. Inequality (17) trivially holds with $\alpha_3(\beta_1) := (c/\beta_1')\beta_1$. Finally, from (40), we see that V_4 and W correspond to the last two terms on the right-hand side of (37), respectively.

Thus, all the conditions of Theorem 1 hold with the function $\mathcal{V}$ in (37), which is of the form (13), α_1, α_2, and α_3 being polynomials in

V_2 of second, third, and first order, respectively, and the inequality (23) corresponds in this case to (38). ∎

IV. CONCLUSION

We have provided a relatively simple and intuitive strict Lyapunov function for nonlinear time-varying systems that appear in the context of passivity-based adaptive control. We believe that the construction provided here may serve as a building block in the solution of other complex problems, such as adaptive observer-based tracking control. In that regard, we provided a strict Lyapunov function for a well-known controller that is recurrently used in its original or in modified forms, in a variety of more modern control problems, such as formation control and synchronization.

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