

Multiple-model adaptive control using set-valued observers

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SUMMARY

A multiple-model adaptive control methodology is proposed that is able to provide stability and performance guarantees, for uncertain linear parameter-varying plants. The identification problem is addressed by taking advantage of recent advances in model falsification using set-valued observers (SVOs). These SVOs provide set-valued estimates of the state of the system, according to its dynamic model. If such estimate is the empty set, the underlying dynamic model is invalidated, and a different controller is connected to the loop. The behavior of the proposed control algorithm is demonstrated in simulation, by resorting to a mass–spring–dashpot system. As a caveat, the computational burden associated with the SVOs can be prohibitive under some circumstances. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In many realistic applications, the model of a system is only known up to some level of accuracy, because of uncertain (and possibly time-varying) parameters and unmodeled dynamics. For some of these applications, a *robust non-adaptive controller* is enough to achieve the desired closed-loop performance, for example, to guarantee a given level of attenuation from the exogenous disturbances inputs to the performance outputs. If, however, the region of uncertainty is large and/or there are stringent performance requirements, such a non-adaptive controller may not exist. To overcome this problem, several solutions are proposed in the literature of adaptive control.

In this paper, we consider an important class of adaptive control architectures, referred to as multiple-model adaptive control (MMAC).[§] In particular, for the sake of simplicity, we are going to address the case where the dynamic model of the system to be controlled has a single parametric uncertainty, $\rho \in [\rho_{\min}, \rho_{\max}]$, despite the fact that the proposed methodology can be extended to the general case in a straightforward manner. Although several switching MMAC methodologies are available to solve this problem, they all share the same principles: in terms of design, the (large) set of parametric uncertainty, Ω , is divided into N (small) subregions, Ω_i , $i = \{1, \dots, N\}$ see Figure 1 and a non-adaptive controller for each of these subregions is synthesized; in terms of implementation, the goal is set to identify to which region the uncertain parameter, ρ , belongs, and then connect to the loop the controller designed for that region.

Several MMAC architectures have been proposed that provide performance and/or stability guarantees, as long as a set of assumptions is met. For instance, [2] uses a parameter estimator to select a

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§For a list of advantages of multiple-model adaptive control, over other adaptive control architectures, the reader is referred, for instance, to [1].

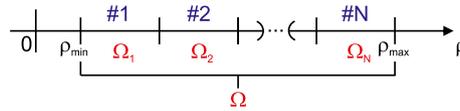


Figure 1. Uncertainty region, Ω , for the parameter ρ , split into N subsets, $\Omega_i, i = \{1, \dots, N\}$.

controller, guaranteeing stability of the closed-loop. Another MMAC approach, referred to as robust MMAC (RMMAC), which was introduced in [3] and references therein, uses a bank of Kalman filters for the identification system and a hypothesis testing strategy to select the controllers. For this case, although simulation results indicate that high levels of performance are obtained, the only guarantees that can be provided are in terms of stability see [4, 5]. In [6], calibrated forecasts are used to guarantee the stability of the closed-loop. The authors in [7] use a Lyapunov-based approach to select controllers, and hence require an in-depth knowledge of the plant. The *unfalsified control* theory see [8–15] and references therein uses an entirely different philosophy and takes advantage of the controlled output error to decide whether the selected controller is delivering the desired performance or not.

The approach adopted in this article builds upon this theory, but using a different model invalidation methodology. Indeed, unlike common MMAC approaches, instead of trying to identify the *correct* region, that is, the region where the uncertain parameter takes value, by hypothesis testing or parameter estimation, we exclude the *wrong* regions. In other words, if the time-evolution of the inputs and outputs of the plant cannot be explained by a model with uncertain parameter ρ , such that $\rho \in \Omega_i$, then region Ω_i cannot be the one which the uncertain parameter belongs to. The invalidation of these uncertainty regions is addressed by using set-valued observers (SVOs), taking advantage of the recent developments presented in [16–18].

In summary, the approach provided in this paper is to use SVOs to decide which non-adaptive controllers should *not* be selected. Similarly to other MMAC architectures, we use a bank of observers - in our case, SVOs - each of which *tuned* for a prespecified region of uncertainty. However, the observers are utilized to *discard* regions, rather than to identify them. Using this strategy, we are able to provide robust stability or, as described in the sequel, even performance guarantees for the closed-loop, even when the model of the plant is uncertain and/or time-varying. These goals are attained by taking advantage of the applicability of the SVOs to linear parameter-varying (LPV) systems - see [19] a class of dynamic systems widely studied in the literature.

The main contribution of the article is, therefore, the development of a novel MMAC technique, which can be applied to linear time-invariant (LTI) and to LPV systems, providing guarantees in terms of performance and stability, respectively. Results in terms of performance are also obtained for LPV systems, under the assumption that the uncertain parameters remain inside a specific region.

The remainder of this article is organized as follows. We start by introducing the notation used in this paper and describing some of the techniques available in the literature for the design of SVOs in Section 2. The MMAC/SVO architecture adopted in this article is presented in Section 3. The controller selection algorithms for time-invariant and time-varying plants are described in Sections 4 and 6, respectively, whereas Section 5 provides stability and guarantees performance for the proposed methods. The theory is illustrated by means of an example, in Section 7. Finally, some conclusions regarding this work are discussed in Section 8.

2. PRELIMINARIES AND NOTATION

The set of positive and strictly positive integers is denoted by \mathbb{Z}_o^+ and \mathbb{Z}^+ , respectively. Moreover, for two vectors $v, w \in \mathbb{R}^m$, the inequality $v \leq w$ is taken element wise, that is,

$$v \leq w \Leftrightarrow v_i \leq w_i, i \in \{1, \dots, m\}.$$

The class of systems considered in this paper, typically referred to as uncertain LPV systems, can be described by

$$\begin{cases} x(k+1) = A(k, \rho(k))x(k) + B(k, \rho(k))u(k) + L(k, \rho(k))d(k) \\ y(k) = C(k, \rho(k))x(k) + N(k, \rho(k))n(k), \end{cases} \quad (1)$$

where $x(0) \in X(0)$, $x(k) \in \mathbb{R}^n$, $d(k) \in \mathbb{R}^{n_d}$, $n(k) \in \mathbb{R}^{n_n}$, $u(k) \in \mathbb{R}^{n_u}$, and $y(k) \in \mathbb{R}^{n_y}$, for $k \in \mathbb{Z}_o^+$. The time-varying vector of parameters, $\rho(k) \in \mathbb{R}^{n_\rho}$, may be uncertain. It is also assumed that

$$|d(k)| := \max_i |d_i(k)| \leq \bar{d},$$

and $|n(k)| := \max_i |n_i(k)| \leq \bar{n}$. At each time, k , the vector of states, is denoted by $x(k)$, and we define $X(0) := \text{Set}(M_0, m_0)$, where

$$\text{Set}(M, m) := \{q : Mq \leq m\} \quad (2)$$

represents a convex polytope. As an additional constraint, it is assumed that the matrices of the dynamics depend affinely on the vector of parameters, as described in the sequel.

Remark 1

The notation $A(k, \rho(k))$ is used instead of $A(k)$ or $A(\rho(k))$ to explicitly state that (in this case) matrix A is time-varying and parameter-varying. The advantages of this notation will become apparent in the following sections. \diamond

2.1. Set-valued observers

The problem of designing SVOs also referred to as set-membership filtering design has been extensively studied in the literature. One of the first algorithms developed to compute (ellipsoidal) set-valued estimates of the state of a system was introduced in [20] and [21]. In [22], an approach to the synthesis problem of SVOs for linear time-varying plants with nonlinear equality constraints is described. A method for active mode observation of switching systems, based on SVOs, has been recently proposed in [23].

An implementable solution to the set-valued estimation of the state of an uncertain LPV system is presented in [24]. In the suggested approach, a set-valued state estimate is provided on each measurement, through the vertices of a polytope, $\mathcal{P}(k)$. However, it is not guaranteed that the *true* state, $x_{\text{true}}(k)$, is contained in $\mathcal{P}(k)$, though the distance between $x_{\text{true}}(k)$ and $\mathcal{P}(k)$ is guaranteed to be bounded.

The solution adopted in this paper is an alternative to the aforementioned approaches, which provides set-valued state estimates that are guaranteed to contain the *true* state of the system, at each sampling time.

Throughout the remainder of this article, we are going to use the approach in [17], to compute set-valued state estimates of dynamic systems that can be modeled by (1). For the sake of completeness, however, the main results of [17] are described in the sequel.

Indeed, let $X(k+1)$ represent the set of possible states at time $k+1$, that is, the state $x(k+1)$ satisfies (1) with $x(k) \in X(k)$ if and only if $x(k+1) \in X(k+1)$. An SVO aims to find $X(k+1)$, based upon (1) and with the additional knowledge that $x(k) \in X(k)$, $x(k-1) \in X(k-1)$, \dots , $x(k-N) \in X(k-N)$, for some finite N . We further require that, for all $x \in X(k+1)$, there exists $x^* \in X(k)$ such that, for $x(k) = x^*$, the observations are compatible with (1). In other words, we want $X(k+1)$ to be the smallest set containing all the solutions to (1). A procedure for discrete time-varying linear systems was introduced in [25], and extensions to uncertain plants were presented in [16] and [17].

As previously mentioned, the computation of $X(k+1)$ based upon $X(k)$ for systems with no model uncertainty can be performed by using the technique described in [25]. Indeed, let the system be described by (1), and assume that the matrices of the dynamics are exactly known, although possibly time-varying. For the sake of simplicity, assume that $N(k, \rho(k)) = I$, and that $A(k, \rho(k)) := A(k)$, $B(k, \rho(k)) := B(k)$, $C(k, \rho(k)) := C(k)$ and $L(k, \rho(k)) := L(k)$. In other

words, assume that the dynamics do not depend on the uncertain vector of parameters, $\rho(k)$. Then, $x(k + 1) \in X(k + 1)$ if and only there exist $x(k)$ and $d(k)$, such that, for the current measurement, $y(k + 1)$, we have

$$P(k) \begin{bmatrix} x(k + 1) \\ x(k) \\ d(k) \end{bmatrix} \leq \begin{bmatrix} B(k)u(k) \\ -B(k)u(k) \\ \bar{d} \\ \bar{d} \\ \tilde{m}(k) \\ m(k - 1) \end{bmatrix} =: p(k) \tag{3}$$

where

$$P(k) := \begin{bmatrix} I & -A(k) & -L(k) \\ -I & A(k) & L(k) \\ 0 & 0 & I \\ 0 & 0 & -I \\ \tilde{M}(k) & 0 & 0 \\ 0 & M(k - 1) & 0 \end{bmatrix}, \quad \tilde{M}(k) = \begin{bmatrix} C(k) \\ -C(k) \end{bmatrix}, \quad \tilde{m}(k) = \begin{bmatrix} \bar{n} + y(k + 1) \\ \bar{n} - y(k + 1) \end{bmatrix},$$

and where $M(k - 1)$ and $m(k - 1)$ are defined such that $X(k) = \text{Set}(M(k - 1), m(k - 1))$. The inequality in (3) provides a description of a set in \mathbb{R}^{2n+n_d} , denoted by

$$\Gamma(k + 1) = \text{Set}(P(k), p(k)).$$

Therefore, it is straightforward to conclude that

$$\hat{x} \in X(k + 1) \Leftrightarrow \exists_{x \in \mathbb{R}^n, d \in \mathbb{R}^{n_d}} : \begin{bmatrix} \hat{x} \\ x \\ d \end{bmatrix} \in \Gamma(k + 1).$$

Hence, the set $X(k + 1)$ can be obtained by projecting $\Gamma(k + 1)$ onto the subspace of the first n coordinates, which can be attained by resorting to the so-called *Fourier–Motzkin elimination method* [25, 26]. Therefore, a description of all the admissible $x(k + 1)$ is obtained that does not depend upon specific $x(k)$ nor $d(k)$.

Notice that (3) can be easily extended to the case where it is desirable to compute $X(k + 1)$ not only based upon $X(k)$, but also upon $X(k - 1), \dots, X(k - N)$, as shown in [27], which may reduce the conservatism of the approach, in case overbounding of the sets is used to decrease computational burden.

For plants with uncertainties, the set $X(k + 1)$ is, in general, non-convex, even if $X(k)$ is convex. Thus, it cannot be represented by a linear inequality as in (2). The approach suggested in [16] is to overbound this set by a convex polytope, $\hat{X}(k + 1)$, therefore adding some conservatism to the solution.

A different method was presented in [17], that requires a smaller computational effort, while reducing the conservatism of the solution. In this scenario, it is assumed that the dynamics in (1) depend affinely upon the uncertain parameters, that is[‡],

$$\begin{aligned} A(k, \rho(k)) &= A_o(k) + \sum_{i=1}^{n_\rho} A_i(k)\rho_i(k), & B(k, \rho(k)) &= B_o(k) + \sum_{i=1}^{n_\rho} B_i(k)\rho_i(k), \\ L(k, \rho(k)) &= L_o(k) + \sum_{i=1}^{n_\rho} L_i(k)\rho_i(k), & C(k, \rho(k)) &= C_o(k) + \sum_{i=1}^{n_\rho} C_i(k)\rho_i(k). \end{aligned}$$

[‡]The uncertainty in $N(k, \rho(k))$ can be treated as a time-varying bound on the noise, which can be addressed in a straightforward manner in (3) by considering a time-varying $\bar{n}(k)$.

Under these constraints, the uncertainty in the dynamics can be addressed by using the methodology described in what follows.

Uncertainty in the B matrix

We start by considering uncertainty solely in the B matrix, that is, we assume that the system can be described by

$$\begin{cases} x(k+1) = A(k)x(k) + L(k)d(k) + \left(B_o(k) + \sum_{j=1}^{n_\rho} \rho_j(k) B_j(k) \right) u(k), \\ y(k) = C(k)x(k) + N(k)n(k), \end{cases} \quad (4)$$

where $x(0) \in X(0)$, $x(k) \in \mathbb{R}^n$, $u(k) \in U \subseteq \mathbb{R}^{n_u}$, $d(k) \in W_d \subseteq \mathbb{R}^{n_d}$, $y(k) \in \mathbb{R}^{n_y}$, $n(k) \in W_n \subseteq \mathbb{R}^{n_n}$, and $\rho(k) \in \mathbb{R}^{n_\rho}$. It is also assumed that $|\rho(k)| \leq 1$. In this case, the uncertain vector, $\rho(k) = [\rho_1(k) \ \cdots \ \rho_{n_\rho}(k)]^T$, represents uncertainty in the input of the plant. Define

$$F_j(k) := F_j(k, u(k)) := B_j(k)u(k), \quad (5)$$

for $j \in \{1, \dots, n_\rho\}$. Then, by substituting (5) in (4), we obtain the following equivalent description for the system

$$\begin{cases} x(k+1) = A(k)x(k) + B_o(k)u(k) + \sum_{j=1}^{n_\rho} F_j(k)\rho_j(k) + L(k)d(k), \\ y(k) = C(k)x(k) + N(k)n(k), \end{cases} \quad (6)$$

where $|\rho_j(k)| \leq 1$. Therefore, each $\rho_j(k)$ can be seen as a bounded exogenous disturbance, acting upon the system. Hence, we recover the formulation in (3), which means that the methodology previously described can be used to obtain $X(k+1)$ based on $X(k)$.

Uncertainty in the C matrix

Consider a dynamic system, S , described by

$$S : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + L(k)d(k), \\ y(k) = \left(C_o(k) + \sum_{j=1}^{n_\rho} \rho_j(k) C_j(k) \right) x(k) + N(k)n(k). \end{cases} \quad (7)$$

In this case, the uncertain vector, $\rho(k)$, represents uncertainty in the output of the plant. Notice that $y(k)$ can be obtained from

$$y(k) = (y^o(k) + N(k)w(k)) + \sum_{j=1}^{n_\rho} (\rho_j(k)y^j(k) + N(k)w(k)), \quad (8)$$

where, for $j \in \{1, \dots, n_\rho\}$, $y^j(\cdot)$ is the output of system \bar{S}_j , described by

$$\bar{S}_j : \begin{cases} x^j(k+1) = A(k)x^j(k) + B(k)u(k) + L(k)d(k), \\ y^j(k) = C_j(k)x^j(k), \end{cases}$$

with $x^j(0) = x(0)$ for all $j \in \{0, \dots, n_\rho\}$, and $w_i(k) = \frac{n_i(k)}{n_\rho+1}$. The block diagram of (8) is depicted in Figure 2.

Because each \bar{S}_j , for $j \in \{0, \dots, n_\rho\}$, is a linear system, and each $\rho_j(k)$, for $j \in \{1, \dots, n_\rho\}$, is an uncertain scalar, we obtain

$$y(k) = (y^o(k) + N(k)w(k)) + \sum_{j=1}^{n_\rho} (\tilde{y}^j(k) + N(k)w(k)), \quad (9)$$

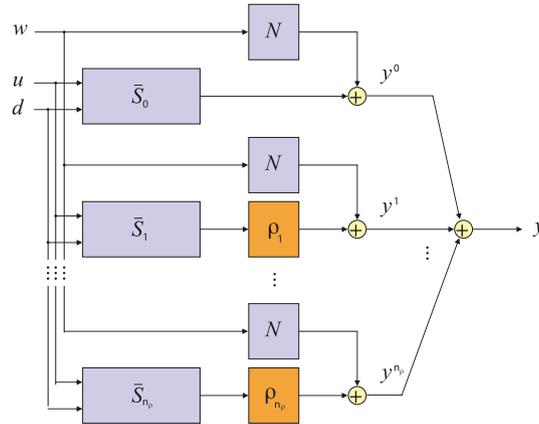


Figure 2. Block diagram of a linear parameter-varying system with uncertainties in the output.

where $\tilde{y}^j(\cdot)$ is the output of system \tilde{S}_j , described by

$$\tilde{S}_j : \begin{cases} \tilde{x}^j(k+1) = A(k)\tilde{x}^j(k) + B(k)\rho_j(k)u(k) + L(k)\rho_j(k)d(k), \\ \tilde{y}^j(k) = C_j(k)\tilde{x}^j(k). \end{cases}$$

Notice that (9) describes the output of an LPV system with uncertain input. Nevertheless, the exogenous disturbances are now multiplied by the uncertainties $\rho_j(k)$, and hence \tilde{S}_j depends upon $\rho_j(\cdot)$ and $d(\cdot)$ in a bilinear fashion. However, this can be avoided by introducing the following relaxation. Because $|\rho_j(k)| \leq 1$, for all $k \in \mathbb{Z}_o^+$, we have that

$$\tilde{d}_j(k) := \rho_j(k)d(k) \Rightarrow |\tilde{d}_j(k)| \leq |d(k)|. \tag{10}$$

Thus, by substituting $\rho_j(k)d(k)$ in (9) by $\tilde{d}_j(k)$ as in (10), we obtain a description of the system, which is linear in the unknown variables, at the cost of some conservatism due to the implication in (10). Once again, the method previously described can be used to compute the set-valued estimate of the state.

Uncertainty in the A matrix

Finally, the problem of designing SVOs for LPV plants with uncertainty in the A matrix is addressed. Let S be described by

$$S : \begin{cases} x(k+1) = \left(A_o(k) + \sum_{j=1}^{n_\rho} \rho_j(k)A_j(k) \right) x(k) + B(k)u(k) + L(k)d(k), \\ y(k) = C(k)x(k) + N(k)n(k), \end{cases} \tag{11}$$

The uncertain vector, $\rho(k)$, represents uncertainty in the dynamics of the plant, and can appear, for instance, in the modeling of several types of physical systems.

For the sake of simplicity, let us consider that $n_\rho = 1$, although the results presented in the sequel can be readily extended for the general case. Further assume that $\text{rank}(A_1(k)) = 1$, for all $k \in \mathbb{Z}_o^+$. Thus, there exist vectors $e_1(k)$ and $f_1(k)$, for each $k \in \mathbb{Z}_o^+$, such that

$$A_1(k) = e_1(k)f_1^T(k).$$

Moreover, define

$$g_1(k) := f_1^T(k)x(k)\rho(k).$$

Then, system S in (11), for $n_\rho = 1$, can be rewritten as

$$S : \begin{cases} x(k+1) = A_o(k)x(k) + e_1(k)g_1(k) + B(k)u(k) + L(k)d(k), \\ y(k) = C(k)x(k) + N(k)n(k), \end{cases} \tag{12}$$

with the additional constraint

$$|g_1(k)| \leq |f_1^T(k)x(k)|. \tag{13}$$

Notice that (12), for arbitrary $g_1(k)$, provides a description of $y(k)$ which is linear in the variables $x(k)$, $g_1(k)$, $u(k)$, $d(k)$ and $n(k)$. Nevertheless, the constraint in (13) is nonlinear regarding $x(k)$. However, for given k , the aforementioned constraint in (13) can be rewritten as

$$\begin{cases} -f_1^T(k)x(k) \leq g_1(k) \leq f_1^T(k)x(k) \\ \text{or} \\ f_1^T(k)x(k) \leq g_1(k) \leq -f_1^T(k)x(k). \end{cases} \tag{14}$$

Therefore, if, at time k , the set of possible states $X(k)$ is a convex polytope, it may happen that at time $k + 1$ the set $X(k + 1)$ will be the union of two convex polytopes. Hence, the description of $X(k + j)$, for $j > 0$, may require an arbitrarily large number of unions of convex polytopes. One way of dealing with this issue is to convexify the estimated set of possible states, by computing the smallest convex set that contains all the possible states. The shortcoming with this approach is that states which are not compatible with (1) may be included, thus adding conservatism to the solution.

If $\text{rank}(A_1(k)) > 1$, then $A_1(k)$ can be written as the sum of several matrices with unitary ranks, as follows

$$A_1(k) = e_{1,1}(k)f_{1,1}^T(k) + \dots + e_{1,m}(k)f_{1,m}^T(k),$$

for some integer $m \geq 0$. Define

$$g_j(k) := f_{1,j}^T(k)x(k)\rho(k),$$

for each $j \in \{1, \dots, m\}$. Then, each $g_j(k)$ can be treated as an independent uncertainty, and the previously described method can be applied. However, this approach can add conservatism to the solution, because we assume no relation between each $g_j(k)$.

A different method to handle uncertainty in the A matrix was presented in [28] and is going to be summarized herein for the sake of completeness. The proposed solution is to overbound set $X(k + 1)$ by a convex one, denoted by $\hat{X}(k + 1)$, which is going to be described as follows.

Let $v_i, i = 1, \dots, 2^{(N+1)n_\rho}$, for some positive scalar N , denote the vertices of the hyper-cube

$$H := \left\{ \delta \in \mathbb{R}^{(N+1)n_\rho} : |\delta| \leq 1 \right\}.$$

Then, we denote by $\hat{X}_{v_i}(k + 1)$ the set of points $x(k + 1)$ that satisfy (11) with $[\rho(k)^T, \dots, \rho(k - N)^T]^T = v_i$ and with $x(k) \in \hat{X}(k), \dots, x(k - N) \in \hat{X}(k - N)$, and that can be obtained with the method in [28], which is a generalization of the previously described SVOs, for $N \in \mathbb{Z}^+$. Further define

$$\hat{X}(k + 1) := \text{co} \left\{ \hat{X}_{v_1}(k + 1), \dots, \hat{X}_{v_{2^{(N+1)n_\rho}}}(k + 1) \right\},$$

where $\text{co}\{p_1, \dots, p_m\}$ is the smallest convex set containing the points p_1, \dots, p_m , also known as the convex hull of p_1, \dots, p_m .

Because, as previously mentioned, $X(k + 1)$ is, in general, non-convex even if $X(k)$ is convex, we are going to use $\hat{X}(k + 1)$ to overbound the set $X(k + 1)$. An illustration for the case $n_\rho = 1, N = 1$, is depicted in Figure 3.

This approach has the valuable property described in the following proposition.

Proposition 1 ([27])

Consider a system described by (11) and assume that $\hat{X}(0) = X(0)$. Then, $X(k) \subseteq \hat{X}(k)$ for all $k \in \{0, 1, 2, \dots\}$.

Remark 2

Notice that Proposition 1 guarantees that the set-valued state estimated of the system, provided by the SVOs, does contain the true state of the plant at each sampling time. \diamond

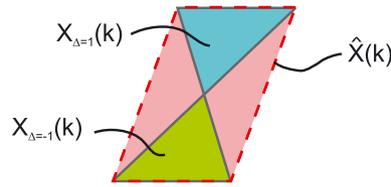


Figure 3. Convex hull, $\hat{X}(k)$, of the sets generated by the solutions to (1) with $n_\rho = 1$, $N = 1$ and for $\rho(k) = 1$ and $\rho(k) = -1$.

In summary, two methods were described, with different properties, to handle uncertainty in the A matrix. In the first case, no conservatism is added whenever the ranks of the uncertainty matrices, A_j , are unitary. Nevertheless, it may require an arbitrarily large number of unions of convex polytopes. The solution proposed to overcome this problem, at the cost of some conservatism, was to compute, at each time k , the convex hull of the admissible sets, and use it as an estimate of $X(k)$. The second method proposed is, in general, less computationally demanding, but also adds some conservatism to the solution.

2.2. Distinguishability of dynamic systems

Let \mathcal{S} denote the set of *plausible* or *admissible* models of the plant to be controlled. We assume that \mathcal{S} is a finite set, with cardinality $N_{\mathcal{S}}$, and that each $S_i \in \mathcal{S}$ can be described by

$$S_i : \begin{cases} x_i(k + 1) = A_i(k, \rho(k))x_i(k) + B_i(k, \rho(k))u(k) + L_i(k, \rho(k))d_i(k), \\ y_i(k) = C_i(k, \rho(k))x_i(k) + N_i(k, \rho(k))n_i(k), \end{cases} \quad (15)$$

for each $i \in \{1, \dots, n_{\mathcal{S}}\}$, and using a nomenclature similar to that of (1). Define

$$\phi_j \in W \times U := (W_d \times W_n) \times U =: \Phi \subseteq \mathbb{R}^{n_u + n_d + n_n}$$

for $j = 0, 1, \dots, k$, and for given k . The sequence $(\phi_0, \phi_1, \dots, \phi_k)$, where $\phi_j = [d_j^T, n_j^T, u_j^T]^T$, denotes the exogenous disturbances, $d_j \in W_d \subseteq \mathbb{R}^{n_d}$, measurements noise, $n_j \in W_n \subseteq \mathbb{R}^{n_n}$, and control input signals, $u_j \in U \subseteq \mathbb{R}^{n_u}$, at time instant j , and $y_i(k)$ is the output of system S_i at time k . The initial state of system S_i is represented by $x_o^i := x_i(0) \in X(0) \subseteq \mathbb{R}^n$. The sets W_d , W_n and U are convex polytopes.

Definition 1 ([18])

Systems S_1 and S_2 , described by (15) with $i = 1$ and $i = 2$, respectively, are said *absolutely* (X_o, U, W) -input distinguishable in N measurements if, for any non-zero

$$(x_o^1, x_o^2, \phi_0, \phi_1, \dots, \phi_N) \in X_o \times X_o \times \overbrace{\Phi \times \dots \times \Phi}^{N \text{ times}},$$

where $\Phi := W \times U = (W_d \times W_n) \times U$, there exists $k \in \{0, 1, \dots, N\}$ such that $y_1(k) \neq y_2(k)$. Moreover, two systems are said *absolutely* (X_o, U, W) -input distinguishable if there exists (bounded) $N \geq 0$ such that they are absolutely (X_o, U, W) -input distinguishable in N measurements.

Unlike other definitions of distinguishability that can be found in the literature [29–31], Definition 1 is important when we want to guarantee that, regardless of the input signals, two systems can be distinguished in a given number of measurements. Sufficient and necessary conditions for evaluating whether a set of linear systems and a set of uncertain LPV system are distinguishable can be found in [18] and [32, Chapter 5], respectively. These references also provide some practical examples illustrating the theory presented. It can be shown that, in general, a persistence type of excitation condition is required on the exogenous disturbances. Moreover, it turns out that this condition can be written as a lower bound on the intensity of the perturbations. We argue that the concept of absolute distinguishability can be used as a tool for the design of model falsification

schemes, in an analogous manner to the use of the concepts of observability and controllability for the synthesis of observers and controllers, respectively.

3. MULTIPLE-MODEL ADAPTIVE CONTROL WITH SET-VALUED OBSERVERS ARCHITECTURE

Figure 4 depicts the basic MMAC architecture adopted in this article, referred to as MMAC/SVO architecture. As previously mentioned, suppose that, for the sake of simplicity, the plant depends upon only one (real-valued) uncertain parameter, $\rho(k)$. It is known, however, that $\rho(k) \in \Omega$, for some set $\Omega \subseteq \mathbb{R}$.

We follow very closely the performance-oriented methodology presented in [3] to design the MMAC/SVO control system. Some guidelines regarding the generalization of this approach for plants with a higher number of parametric uncertainties are also provided.

For starters, we assume that a single and non-adaptive controller is not able to achieve the desired performance for the whole uncertainty region. Otherwise, there would be no reason to use an adaptive controller. Therefore, we need to split this region, Ω , into several smaller regions, say $\Omega_1, \Omega_2, \dots, \Omega_N$, such that $\Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_N = \Omega$. To do so, we first compute the maximum (ideal) performance that we can achieve. This clearly is the case where the exact value of the otherwise uncertain parameter, ρ , is known. To a controller designed for a fixed value of the uncertain parameter, $\rho(k)$, we call fixed non-adaptive robust controller, using the same terminology as in [3].

The design proceeds by defining the required performance for the closed-loop, when the parameter $\rho(k)$ is uncertain. Without loss of generality, we assume that, for each value of the uncertain parameter, $\rho(k)$, we want the performance of the MMAC/SVO not to be smaller than a fixed percentage of the corresponding fixed non-adaptive robust controller. This naturally leads to the splitting of set Ω into smaller subsets, as described in detail in [33].

For each of these subsets $\Omega_i, i = \{1, \dots, N\}$, a controller referred to as local non-adaptive robust controller (LNARC), and denoted by $K_i(\cdot)$, is synthesized. Furthermore, an SVO should also be designed for each of the subsets, using the methodology described in Section 2 and in [17].

Remark 3

We argue that, for any realistic application, these LNARCs should also be robust against plant model errors in addition to parameter ρ . Thus, using a linear fractional transformation representation for the system may be useful if, for instance, mixed- μ controllers are used - see [34, 35]. \diamond

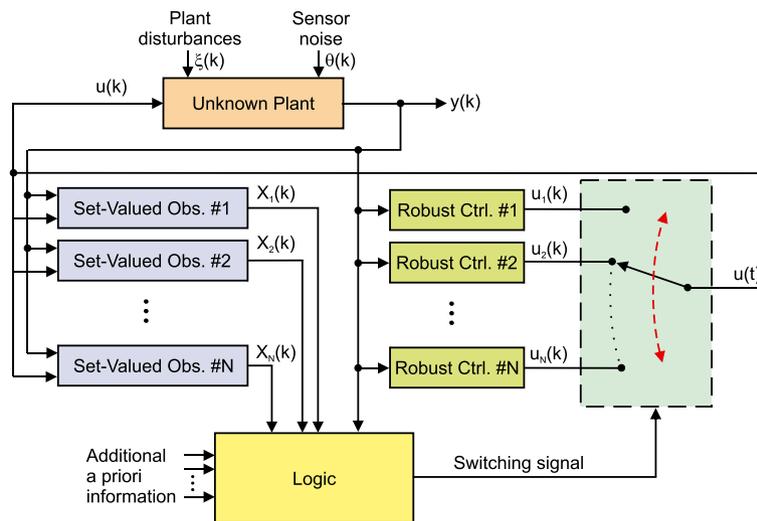


Figure 4. Multiple-model adaptive control with set-valued observers (MMAC/SVO) architecture. X_i is the set-valued state estimate provided by SVO $\#i$.

As shown in [33], this methodology can also be extended to the case where more than one parameter is uncertain, that is, if $\rho(k)$ is a vector, rather than a scalar. However, in these circumstances, the splitting of the set Ω has more than one possible solution, if the same performance-oriented reasoning is adopted. Therefore, additional degrees of freedom in the synthesis of the uncertainty regions Ω_i 's appear, as the number of uncertain parameters increases. As an example, a typical approach for $\rho(k) \in \mathbb{R}^2$ is to consider each Ω_i to be a rectangle, which naturally simplifies the synthesis of such regions.

Remark 4

It is important to stress that a particular methodology such as the one previously described need not be adopted to attain the properties of the MMAC/SVO control architecture. Indeed, as formally explained in the sequel, the only requirement is that, for each region of uncertainty, a stabilizing controller is available. \diamond

4. CONTROLLER SELECTION ALGORITHMS FOR NON-DRIFTING DYNAMICS

Having described the architecture and the design procedure of the MMAC/SVO, two algorithms are proposed to select the appropriate controller at each measurement. In reference to Figure 4, this section is devoted to the description of the behavior of the block entitled *Logic*, for linear systems with *non-drifting dynamics*, defined as follows.

Definition 2

The dynamics of a discrete time linear system, described by (1), are said *non-drifting in Ω_i* , if

$$\rho(k) \in \Omega_i, \forall k \in \mathbb{Z}_0^+.$$

Several approaches can be used to tackle the decision problem at hand. For the sake of clarity, the simple solution depicted in Figure 5(a) is adopted for SVO-based model falsification. This strategy takes into account the fact that, if $\rho(k) \in \Omega_i$, then SVO #*i* does never fail, that is, the set-valued state estimate of the *i*th SVO, $\hat{X}_i(k)$, is never empty. On the other hand, if $\rho(k) \notin \Omega_i$, then it can happen that, for some t_0 , we have $\hat{X}_i(k) = \emptyset$, for all $k \geq t_0$.

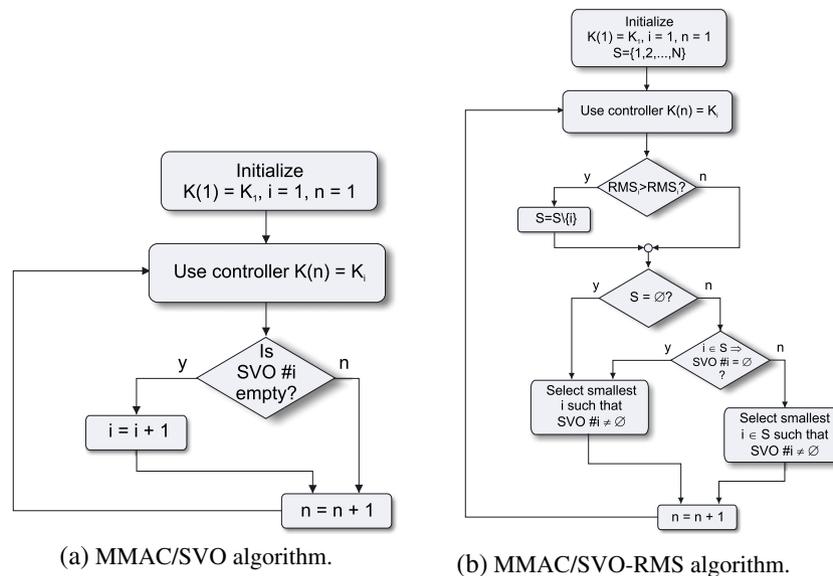


Figure 5. Algorithms for the *Logic* block of the multiple-model adaptive control with set-valued observers (MMAC/SVO) architecture. MMAC/SVO-RMS, MMAC/SVO-root mean square.

In summary, the main strategy in the algorithm is to start by using any controller in the initial set of plausible controllers and then remove from the loop controllers whose corresponding models of the plant have been disqualified. For the sake of simplicity, the controllers are selected in a sequential fashion, in this case, that is, if model #1 is invalidated, we switch to controller #2, whereas if model #2 is invalidated, we switch to controller #3, and so on.

Remark 5

This prerouted controller selection scheme has the disadvantage of potentially generating large transients during the switching between controllers. However, the MMAC/SVO can rely on other decision-making algorithms to choose between the non-invalidated controllers. As described in the following subsection, the only requirement is that the selected controller is not the one synthesized for a previously *falsified* (or invalidated) model. \diamond

4.1. Root mean square analysis

Because the MMAC/SVO can be seen as a *worst case* approach, a controller is not invalidated unless the input/output sequences cannot be explained by the dynamics of the closed-loop system. Hence, an enhancement of the MMAC/SVO architecture is now proposed, based on the fact that, on the one hand, we cannot ‘exclude’ unfalsified models, and that, on the other hand, if a given controller is unlikely to be the *correct* one, then one should try another one first, if available. Having this in mind, it should be noticed that, as an example, the root mean square (RMS) of the output error of the closed-loop system, provides an additional method to assess whether or not we are using the *correct* controller at each time and is given by

$$\text{RMS}_i^h(k) := \sqrt{\frac{1}{h} \sum_{i=k-h+1}^k \|z^i(k)\|^2} \quad (16)$$

where h is a positive integer referred to as *RMS window*, $z^i(k) \in \mathbb{R}^{n_z}$ is the output error typically a linear combination of the output variables and the control input signals of the closed-loop system with controller $K_i(\cdot)$, and $\|z^i(k)\|$ denotes the Euclidean norm of vector $z^i(k)$.

Because we can compute a priori the maximum RMS of the closed-loop system, given a certain bound on the intensity of the disturbances, this maximum RMS value can be used to aid in the invalidation of dynamic models. More specifically, if controller $K_i(\cdot)$ is connected to the loop, and the RMS of the output of the system is greater than the upper bound on the RMS for region Ω_i which was computed *offline* then clearly model # i is not the *correct* one (or the bounds on the disturbances were violated), if the steady state was already attained.

The main advantage of this method is that we potentially discard regions of uncertainty faster than by simply using the SVOs. However, as a shortcoming, the RMS values computed are only valid for steady state, which may not be straightforward to assess in practice.

To guarantee the properties of the MMAC/SVO architecture in Figure 4, a model can only be excluded from the set of eligible ones if it is invalidated by the SVOs. Nevertheless, whenever this set has more than a single element, we are free to select a controller that has not violated the aforementioned RMS condition. The resulting algorithm is depicted in Figure 5(b) where $\overline{\text{RMS}}_i$ is a priori bound on the RMS value of the output error.^{||}

The set \mathbf{S} contains the indexes of all the controllers that have not violated the a priori closed-loop RMS assumption, whenever connected to the loop. If this set is empty, then we can select any controller for which the corresponding model has not been falsified by the SVOs. Otherwise, we can select controller $K_i(\cdot)$, as long as $i \in \mathbf{S}$ and $X_i(k) \neq \emptyset$. If none of the indexes in \mathbf{S} satisfies $X_i(k) \neq \emptyset$, then any LNARC # i , for which $X_i(k) \neq \emptyset$, can be selected.

^{||}The superscript h was omitted for the sake of simplicity.

5. GUARANTEED STABILITY AND PERFORMANCE

Preliminary results on the closed-loop stability and performance guarantees provided by the MMAC/SVO architecture were presented in [16]. However, the assumptions in [16], required to guarantee performance, are seldom easy to be checked in practice. Because these assumptions essentially amount for the eventual selection of the appropriate controller, it is natural to use the concept of *absolute input distinguishability* - see Section 2 and [17, 18, 32] - to pose these requirements, as described in the sequel.

Despite of that, the suggested MMAC/SVO approach guarantees a stable closed-loop under much less restrictive assumptions. Indeed, we first show that, for plants with no *drifting* parameters, the MMAC/SVO is able to provide stability guarantees, by demonstrating that a stabilizing controller is eventually connected to the loop.

The proof for stability is based upon the fact that the SVOs are non-conservative, that is, if $\hat{X}_i(k) \neq \emptyset$, then the output of the plant, $y(\cdot)$, can be explained by the previous and current inputs and outputs, and for some $\rho(\cdot)$, with $\rho(k) \in \Omega_i$ for all $k \in \mathbb{Z}_o^+$. This statement will be explained in a more formal manner in the sequel. For the sake of simplicity, a plant with only one uncertain parameter is used, although the results can be extended to the case where $\rho(k)$ is a vector, instead of a scalar. The following notion of stability is considered throughout the remainder of this paper.

Definition 3

Consider a dynamic system with input $u(\cdot)$ and output $y(\cdot)$. Then, it is said *input/output stable* if, for all $k \in \mathbb{Z}_o^+$, $|u(k)| < \infty \Rightarrow |y(k)| < \infty$.

Consider a system described by (1). Further, consider a partitioning of the uncertainty set, Ω , as described in the previous section ($\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_N$). Moreover, let us posit the following assumptions.

Assumption 1

For each of the uncertainty subsets, $\Omega_i, i = \{1, \dots, N\}$, there is at least one LNARC, referred to as $K_i(\cdot)$, such the closed-loop of model (1) with controller $K_i(\cdot)$ is input/output stable, for $\rho(k) \in \Omega_i, \forall k \in \mathbb{Z}_o^+$.

Remark 6

Assumption 1 does not constrain the plant to be time-invariant. In fact, the only requirement is that the parameters do not *drift* from one region of uncertainty to another. \diamond

Let $\tilde{X}_i(k) := \{x : y(k) = C(k, \rho(k))x + n, |n| \leq \bar{n}, \rho(k) \in \Omega_i\}$, and

$\bar{X}_i(k) = \{x : x = A(k, \rho(k))w + B(k, \rho(k))u(k) + L(k, \rho(k))d, w \in X_i(k - 1), |d| \leq 1, \rho(k) \in \Omega_i\}$.

Assumption 2

The solution of SVO # i is given by

$$X_i(k) = \bar{X}_i(k) \cap \tilde{X}_i(k).$$

In words, the solutions of the SVOs are non-conservative.

Remark 7

As stressed in Section 2, the conservatism of the SVOs depends on the structure of the dynamic model of the plant and on the computational power available. Therefore, the validity of Assumption 2 may not always be straightforward to verify in practice. \diamond

Assumption 3

There exists $i^* \in \{1, \dots, N\}$ such that

$$x(k) \in X_{i^*}(k), \text{ for all } k \in \mathbb{Z}_o^+.$$

Notice that Assumption 3 guarantees that the *true* plant model belongs to the family of *legal* models of at least one of the SVOs.

Assumption 4

The closed-loop system with any of the N eligible controllers does not have a finite escape time.

We stress that Assumption 4 is automatically satisfied if, for instance, all the N controllers and the plant are LTI systems see [4].

Theorem 1

Suppose Assumptions 1–4 are satisfied. Then, the closed-loop system with the MMAC/SVO scheme is input/output stable.

Proof

We first show that the number of switchings is finite. Then, by contradiction, we prove that the closed-loop system is input/output stable.

If $X_i(k) = \emptyset$, then $y(k)$ cannot be explained by the uncertain plant model used by SVO # i . Thus, we switch to a different controller. According to Assumption 3, at least for one value of $j \in \{1, \dots, N\}$, $x(k) \in X_j(k)$, $\forall k$. Hence, the number of switchings is finite and smaller than N . In other words, for some large enough t_o , the controller selected at time instant $k \geq t_o$ is always the same.

Next, suppose that $|y(k)| \rightarrow \infty$ as $k \rightarrow \infty$. Let $K_j(\cdot)$ be the controller selected for $k \geq t_o$. According to Assumption 2, there is a sequence $(d(\cdot), n(\cdot))$, with $|n(k)| \leq \bar{n}$ and $|d(k)| \leq \bar{d}$, for all $k \in \mathbb{Z}_o^+$, such that $y(\cdot)$ can be obtained with model (1) with $\rho(k) \in \Omega_j$ for all $k \in \mathbb{Z}_o^+$. However, according to Assumption 1, controller $K_j(\cdot)$ is able to asymptotically stabilize any plant with $\rho(k) \in \Omega_j$ for all $k \in \mathbb{Z}_o^+$. Because $|d(k)|$ and $|n(k)|$ are bounded for all $k \in \mathbb{Z}_o^+$, and according to Assumption 4, there cannot exist a sequence $(d(\cdot), n(\cdot))$ such that $|y(k)| \rightarrow \infty$, which is a contradiction. \square

5.1. Performance guarantees

Finally, performance guarantees are provided for the closed-loop system using the MMAC/SVO scheme. For the sake of clarity, we introduce the following notation to represent the closed-loop system with a given controller. Let the plant be described by (1), with $\rho(k) \in \Omega_{i^*}$ for all $k \in \mathbb{Z}_o^+$ and some $i^* \in \{1, \dots, N\}$. Denote by $CL_{ij}(\cdot)$ the closed-loop system obtained by interconnecting a plant described by (1), where $\rho(k) \in \Omega_i$, with the LNARC $K_j(\cdot)$. Further, define $\xi(k) := [d^T(k) \quad n^T(k)]^T$, and let $z(\cdot)$ denote a performance output of system (1), defined as $z(k) := C_z(k, \rho(k))x(k)$, for $k \in \mathbb{Z}_o^+$. Moreover, suppose that the following assumption is satisfied.

Assumption 5

There exist $\gamma, \lambda, \sigma > 0$, such that, for each $i \in \{1, \dots, N\}$, there is at least one LNARC, referred to as $K_i(\cdot)$, such that, if $K_i(\cdot)$ is interconnected with a dynamic system described by (1) with $\rho(k) \in \Omega_i$, for $k \geq k_o$, then

$$|z(k)| \leq \gamma \sup_{\kappa \in [0, k]} |\xi(\kappa)| + \lambda e^{-\sigma(k-k_o)} |z(k_o)|,$$

and the dynamics of the closed-loop are asymptotically stable.

Now, the only requirement remaining is that the closed-loop system, for the different possible interconnections, is always distinguishable.

Assumption 6

Let $i, j, \ell, m \in \{1, \dots, N\}$, with $i \neq \ell$ and/or $j \neq m$. Then, the systems $CL_{ij}(\cdot)$ and $CL_{\ell m}(\cdot)$ are absolutely (X^*, W) -input distinguishable in N_{CL} measurements.

Hence, we are now in conditions of stating the following theorem, which provides performance guarantees for the closed-loop system:

Theorem 2

Consider a plant described by (1) and that Assumptions 2–6 are satisfied, where X^* in Assumption 6 is defined so that $X_i(0) \subseteq X^*$ and $i \in \{1, \dots, N\}$. Then, the closed-loop system with the MMAC/SVO scheme is asymptotically stable and satisfies, for $k > N_{CL}$,

$$|z(k)| \leq \gamma \sup_{\kappa \in [0, k]} |\xi(\kappa)| + \lambda e^{-\sigma(k - N_{CL})} |z(N_{CL})|.$$

Proof

Similarly, to the proof of Theorem 1, we can show that the number of switchings is finite. In particular, because of Assumption 6, the maximum number of measurements before the last switching is N_{CL} . Hence, if the plant to be controlled is described by (1) with $\rho(k) \in \Omega_{i^*}$ for $k \in \mathbb{Z}_0^+$, the controller K_{i^*} is selected at time $k^* \leq N_{CL}$, which, using Assumption 5, concludes the proof. \square

6. LINEAR PARAMETER-VARYING SYSTEMS WITH DRIFTING PARAMETERS AND TIME-VARYING BOUNDS ON THE DISTURBANCES

Notwithstanding the fact that the architecture proposed in Section 3, jointly with the algorithm in Section 4, can be applied to time-varying plants, it does not allow for the dynamics of the system to *drift* from one region of uncertainty to another. Moreover, the proposed algorithms are not robust to variations or large uncertainty on the bounds of the disturbances. In fact, if the bound on the disturbances is increased from one measurement to another, it may happen that the *correct* model of the plant gets disqualified. Therefore, we now extend the previous results to time-varying plants whose dynamics may not remain in the same region for all times $k \in \mathbb{Z}_0^+$, and whose bounds on the disturbances may also be time-varying.

As described in [36], in the case of time-varying plants, a model shall never be *disqualified* ‘forever’. Indeed, if the dynamics of the plant drifted at a given time instant, then a previously discarded controller might be the appropriate one to be used from that moment on.

To proceed with the development of the algorithm for time-varying plants, we posit the following assumption.

Assumption 7

There exists $T_{min} > 0$ such that, if $\rho(k) \in \Omega_j$, then there exist k_1 and k_2 with

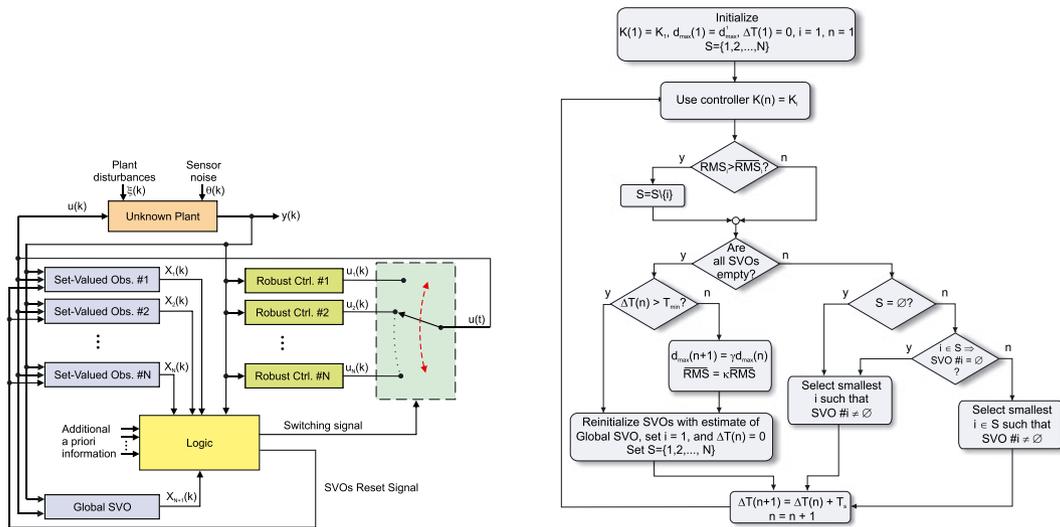
- (i) $|k_2 - k_1| \geq T_{min}$.
- (ii) $k_1 \leq k \leq k_2$.
- (iii) $\rho(\kappa) \in \Omega_j$ for all $\kappa \in [k_1, k_2]$.

We propose the architecture depicted in Figure 6(a) as an extension of the MMAC/SVO scheme for time-varying plants. The main idea is to have an SVO, referred to as *Global SVO*, which is able to provide a set-valued state estimate for all the admissible time-varying uncertainties of the plant. Therefore, unless none of the N families of models which assume that the uncertain parameters are time-varying is able to describe the dynamics of the actual plant, the Global SVO does never provide an empty set-valued estimate of the state.

To account for time-variations on the bounds of the disturbances acting upon the plant, we increase the bounds used by the SVOs, whenever all the models have been falsified twice in less than a given amount of time. This idea is based on Assumption 7 and on the results in [36]. Moreover, although we established that the (tight) bound on the disturbances is unknown, we assume that an (probably very conservative) overbound is known a priori:

Assumption 8

The disturbances are bounded by $|d(k)| \leq \hat{d}, \forall k \in \mathbb{Z}_0^+$, for some known finite constant \hat{d} .



(a) MMAC/SVO architecture for time-varying systems. (b) Algorithm for the Logic block of the MMAC/SVO-RMS architecture for time-varying systems with unknown bound on the disturbances.

Figure 6. Multiple-model adaptive control with set-valued observers (MMAC/SVO) for time-varying systems. MMAC/SVO-RMS, MMAC-root mean square.

Now, suppose that the Global SVO, depicted in Figure 6(a), is computed assuming a bound on the disturbances given by \hat{d} (with $\hat{d} \geq \bar{d}$). Then, it is straightforward to conclude that the set-valued state estimate provided by this SVO is never going to be empty.

The algorithm controlling the Logic block in Figure 6(a) is based upon the following reasoning and is depicted in Figure 6(b). In the first sampling time, all the SVOs (except the Global SVO) assume that the bound on the disturbances is given by d_{max} , where d_{max} satisfies

$$d_{max} \leq \hat{d}.$$

If all but the Global SVO provide empty set-valued state estimates for the plant, it means that none of the N models is able to describe the observed input/output data. Thus, two scenarios have to be considered: (1) the dynamics of the plant have drifted from one region of uncertainty to another and (2) the tentative bound on the disturbances, d_{max} , is smaller than the actual tightest bound on the disturbances.

To distinguish between these two scenarios, we resort to Assumption 7. If the SVOs are reinitialized twice with the set-valued estimate of the Global SVO, in a time-interval smaller than T_{min} , then we conclude that we have to increase the value of d_{max} . Otherwise, the falsification of the models may be explained by the time-variations of the dynamics of the plant. In both circumstances, all the N SVOs should be reinitialized with the set-valued state estimate of the Global SVO. The remaining parts of the algorithm are similar to the time-invariant case.

In the algorithm depicted in Figure 6(b), $\Delta T(k)$ stores the amount of time because the estimates of the SVOs have been reinitialized with that of the Global SVO, and γ is a (fixed) parameter, with $\gamma > 1$. Finally, T_s denotes the sampling period of the SVOs, and $\kappa > 1$ is a constant by which each RMS_i is multiplied whenever the algorithm detects that the bound on the magnitude of the disturbances is greater than that used during the design phase. As in the time-invariant case, the RMS of the output is used to potentially speed up the falsification of the models, although such an approach is not required to ensure a stable closed-loop system, as described in the sequel.

Remark 8

If the set-valued state estimate of the Global SVO is bounded, then so do the set-valued state estimates of the remaining SVOs. This is straightforward to conclude from the fact that, at each

sampling time, the set-valued state estimate computed by any of the N SVOs is inside the set-valued state estimate of the Global SVO. \diamond

6.1. Stability guarantees

Using similar arguments to those of the proof of Theorem 1, stability for time-varying plants can now be established.

Theorem 3

Suppose Assumptions 1–5, 7, and 8 are satisfied. Then, the closed-loop system with the MMAC/SVO architecture for time-varying plants is input/output stable, for sufficiently large T_{\min} .

Proof

First, let us suppose that, for $k \geq \tilde{k}$, we have $d_{\max}(k) = \hat{d}$, for some $\tilde{k} \geq 0$. Using arguments similar to those of the proof of Theorem 1, it is straightforward to show that, for sufficiently large T_{\min} , and $\rho(k) \in \Omega_i$ for all $k \in [k_1, k_2]$ and with $k_2 - k_1 = T_{\min}$, and according to Assumption 5, the norm of the output, at time k_2 , that is, before the drifting of the parameters, is given by

$$|z(k_2)| \leq \gamma \sup_{\kappa \in [0, k]} |\xi(\kappa)| + \lambda e^{-\sigma(k_2 - k_o)} |z(k_o)| = \gamma \sup_{\kappa \in [0, k]} |\xi(\kappa)| + \lambda e^{-\sigma(T_{\min} + k_1 - k_o)} |z(k_o)|,$$

for sufficiently large k_o , with $k_o \leq k_2$. Hence, because Assumption 8 implies that

$$\sup_{\kappa \in [0, k]} |\xi(\kappa)| < \infty,$$

and Assumption 4 implies that

$$|z(k_o)| < \infty,$$

one concludes that there exists $\psi \in \mathbb{R}$, with $0 \leq \psi < \infty$, such that

$$|z(k_2)| \leq \psi.$$

Now, if k_3 is the subsequent time instant when the parameters drift, it can be easily checked that, for sufficiently large T_{\min} ,

$$|z(k_3)| \leq \psi.$$

Following this procedure, and given that Assumption 4 is satisfied, one concludes that

$$|z(k)| < \infty, \forall k \in \mathbb{Z}_o^+.$$

If $d_{\max} \leq \hat{d}$, then the SVOs may incorrectly invalidate dynamic models. Two scenarios can thus occur: (1) all the models are invalidated twice in a time-interval smaller than T_{\min} and (2) the models take longer to be invalidated. If case (1) occurs repeatedly, we have $d_{\max}(k) \geq \hat{d}$, $k \geq \tilde{k}$, for some $\tilde{k} \geq 0$. Hence, the first part of the proof can be applied. If, however, case (2) is verified, then one concludes that, for some \hat{k} , case (1) does not occur for $k \geq \hat{k}$. Nevertheless, in this case, the same arguments of the proof of Theorem 1 can be used to show that the closed-loop is input/output stable. \square

7. SIMULATION: MASS-SPRING-DASHPOT SYSTEM

In this section, the applicability of the MMAC/SVO methodology to the mass-spring-dashpot (MSD) system depicted in Figure 8 is evaluated. A comparison between the MMAC/SVO and another well-known adaptive control methodology, referred to as RMMAC, is also presented, to provide a more insightful analysis.

As briefly mentioned in Section 1, the RMMAC is a multiple-model approach that computes and uses the posterior probabilities of the uncertain parameters of the process model being in a specific region to switch or blend the outputs of a set of controllers, each of which designed for a given uncertainty region. The estimation part is performed by a bank of Kalman filters, whereas for the control part a set of mixed- μ controllers is used see Figure 7, where the LNARCs are mixed- μ controllers, in the original RMMAC architecture. For further details, the interested reader is referred to [3, 33].

The uncertain parameter considered is the spring stiffness, $\rho(\cdot) := k_1(\cdot)$, which is assumed to be time-varying, but with known bounds on the rate of time-variation. Moreover, it is assumed that

$$\forall t \geq 0, k_1(t) \in K := [0.25 \quad 1.75] \text{ N/m.}$$

The state-space description of the dynamics of the MSD system, excluding the dynamics of the disturbances and omitting the time-dependence of $k_1(\cdot)$, is given by

$$\begin{cases} \dot{x}(t) = A(k_1)x(t) + Bu(t) + Ld(t), \\ y(t) = Cx(t) + n(t), \end{cases}$$

where

$$x^T(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]$$

is the state of the plant, and

$$A(k_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [0 \quad 1 \quad 0 \quad 0].$$

where

$$m_1 = m_2 = 1 \text{ kg}, \quad k_2 = 0.15 \text{ N/m}, \quad b_1 = b_2 = 0.1 \text{ N/(m/s)}.$$

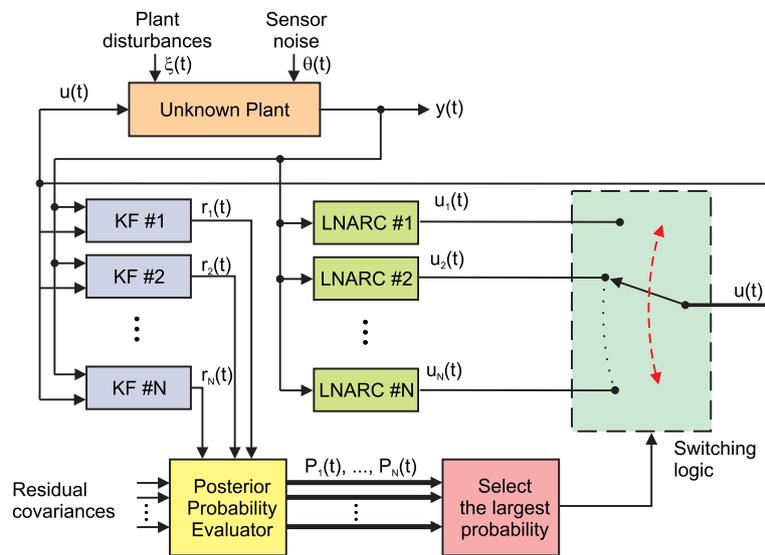


Figure 7. Robust multiple-model adaptive control with bilinear matrix inequality controllers (RMMAC/BMI) architecture.

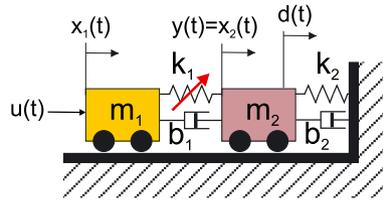


Figure 8. Mass–spring–dashpot (MSD) plant.

The disturbance force, $d(\cdot)$, shown in Figure 8 is generated by driving a low-pass filter, with transfer function $W_d(s)$, with continuous-time bounded Gaussian noise $\xi(\cdot)$, with zero mean, bound of $\Gamma \bar{d}$ and intensity of $\Gamma \Xi$, according to

$$d(s) = \text{sat} \left(\frac{\alpha}{s + \alpha} \xi(s), \Gamma \bar{d} \right) = \text{sat}(W_d(s)\xi(s), \Gamma \bar{d}),$$

where $\Gamma = 1$, $\Xi = 1$, and $\bar{d} = 3$, for the nominal case, and where

$$\text{sat}(x, y) = \begin{cases} x, & \text{if } -y \leq x \leq y, \\ y, & \text{if } x > y, \\ -y, & \text{if } x < -y, \end{cases}$$

for $y \geq 0$. We consider that the sensor noise, $n(\cdot)$, is also obtained from a Gaussian distribution, with zero mean and intensity 10^{-6} , saturated by $\bar{n} = 0.003$. The reader is referred to [33, 37] for further details on the dynamics of the MSD system.

Following the RMMAC/bilinear matrix inequality (BMI) synthesis methodology described in [38] and using the same design choices as the ones described in [3, 33], we obtain $N = 4$ LNARCs which are mixed- μ controllers in the original RMMAC design to achieve at least 70% of the performance, we would have obtained, had we known the value of the uncertain parameter, k_1 . The four regions of uncertainty are summarized in Table I. Then, as explained in [38], the mixed- μ controllers are replaced by the so-called BMI/LPV controllers with similar specifications, but assuming nonzero bounds on the rate of variation of the parameter, k_1 , which results in the RMMAC/BMI adaptive control scheme. In this design, we assume a bound of 0.001 (N/m)/s for the slope (time-variation) of the parameter k_1 .

One SVO for each region of uncertainty was designed, using a sampling time of $T_s = 100$ ms, and considering a bound on the disturbances of

$$|d(k)| \leq d_{\max} := 3 \text{ N}, \forall k \in \mathbb{Z}_o^+.$$

Moreover, a Global SVO was synthesized for the whole region of uncertainty, that is, for $k_1 = [0.25, 1.75]$ N/m, and assuming no constraints for the time-rate of variation of the uncertain parameter, k_1 . Moreover, for the Global SVO, it was considered that the disturbances were bounded by

$$|d(k)| \leq \hat{d} := 30 \text{ N}, \forall k \in \mathbb{Z}_o^+.$$

The MMAC/SVO-RMS algorithm described in the previous section see, in particular, Figure 6(b) was adopted. The simulation results presented in what follows aim at comparing the behavior of the

Table I. Regions of uncertainty for the spring stiffness, k_1 .

Region number	Spring stiffness uncertainty [N/m]
#1	[1.02, 1.75]
#2	[0.64, 1.02]
#3	[0.40, 0.64]
#4	[0.25, 0.40]

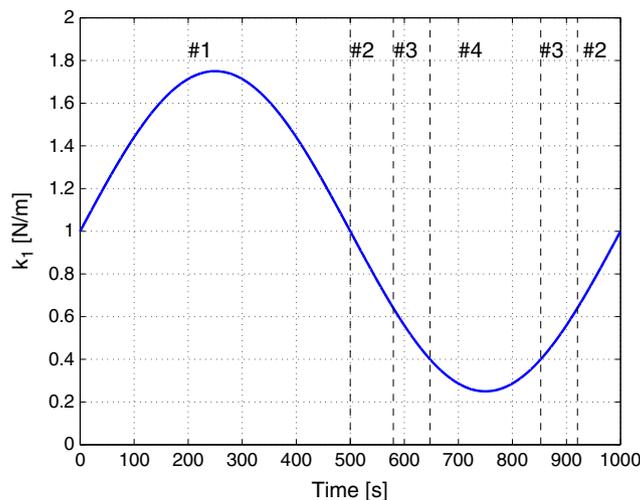


Figure 9. Time-varying spring stiffness, $k_1(\cdot)$, and corresponding uncertainty regions (cf. Table I).

MMAC/SVO with that of the RMMAC/BMI, when applied to the MSD plant previously described. Although several Monte-Carlo simulations have been performed, only a representative set will be considered here for analysis.

Consider that the spring stiffness is time-varying and described by

$$k_1(k) = 1.00 + 0.75 \sin(\omega k T_s) \text{ N/m,}$$

as illustrated in Figure 9, and where

$$\omega = \frac{2\pi}{1000} \text{ rad/s.}$$

Moreover, let Γ follow the time-evolution depicted in Figure 10.

Figure 11 depicts the results obtained for a typical Monte-Carlo run. The MMAC/SVO-RMS scheme exhibits some large transients whenever the local controllers are switched. Nevertheless, contrary to the RMMAC/BMI, it eventually selects the correct controller, as illustrated in Figure 12. The controller selection performed by the Kalman filters was omitted because the decision subsystem of the RMMAC/BMI is not able to converge to any controller for values of Γ greater than

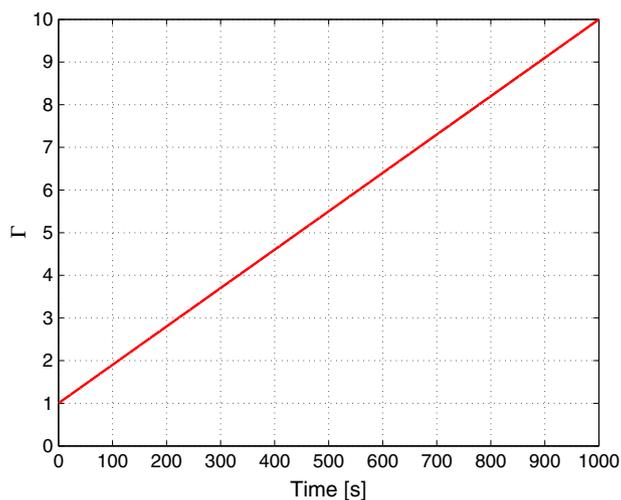


Figure 10. Time-varying magnitude of the disturbances, $\Gamma(\cdot)$.

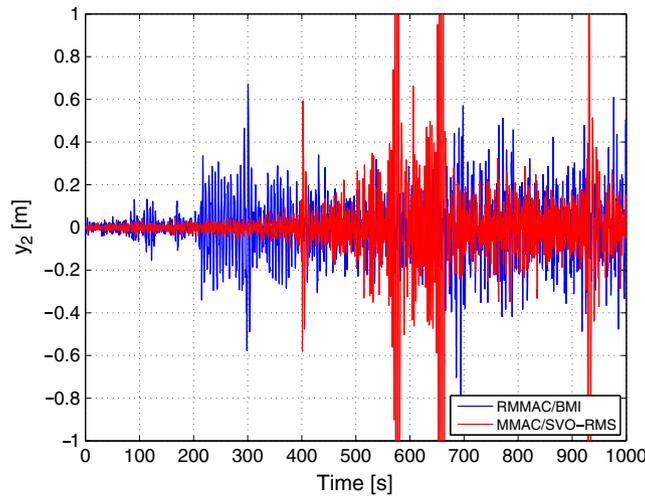


Figure 11. Output of the mass–spring–dashpot (MSD) plant.

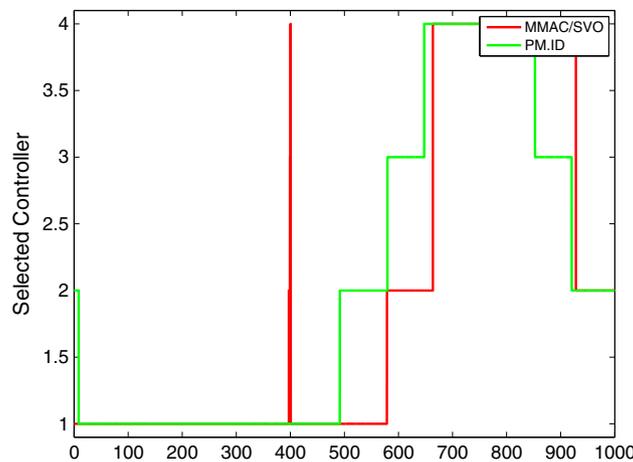


Figure 12. Multiple-model adaptive control with set-valued observers-root mean square (MMAC/SVO-RMS) and perfect model identification (PM.ID) controller selection.

3. Therefore, the RMMAC/BMI keeps switching, every sampling time, between the four available controllers. Figure 12 also depicts the (ideal) *perfect model identification*, that is, the controller that should be selected, according to the current value of the uncertain parameter.

We stress that, in this example, at $t \approx 400$ s, the RMS of the output is larger than expected, according to the a priori information regarding the expected performance of the closed-loop system. Thus, we switch to the next available controller, which is LNARC #2. Only nearly 1 s later, regions of uncertainty #3 and #4 are sequentially invalidated by the SVOs. Because the RMS of the output obtained with the LPV/BMI controller designed for $k_1 \in [0.4, 0.64]$ N/m that is, controller #2 is also larger than expected, because of the increase of the intensity of the disturbances, we switch back to controller #1.

The control signal applied to the plant is depicted in Figure 13. Despite the switching between the controllers, the control signal does not exhibit large transients, nor sudden variations. Nevertheless, methodologies such as the ones presented in [39] can be used, to smooth these transitions.

For further details on the comparison between these two adaptive control approaches, the interested reader is referred to [32, Chapter 8]. A practical application of the MMAC/SVO methodology can also be found in [40].

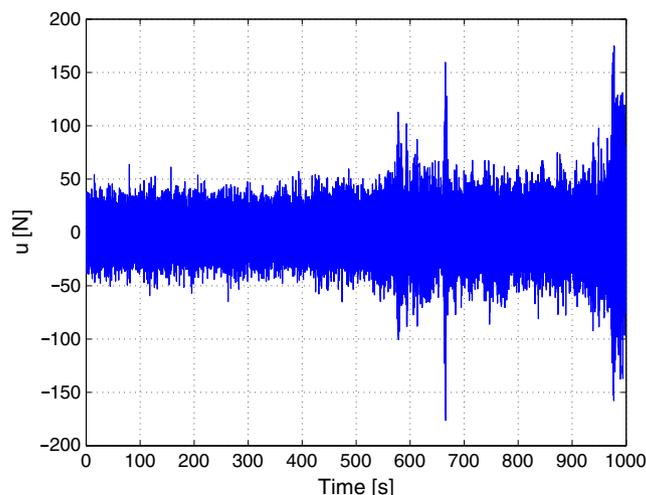


Figure 13. Control signal applied to the mass–spring–dashpot (MSD) plant.

8. CONCLUSIONS

This paper introduced a novel MMAC methodology for LPV systems, that relies on SVOs to invalidate regions of parametric uncertainty. The main advantages of this approach are in terms of guaranteed stability and performance, while posing mild assumptions on the plant to be controlled. The process of model invalidation performed by the SVOs was speeded up by taking into account RMS considerations regarding the closed-loop system.

The applicability of the technique was illustrated by means of a MSD system, showing the benefits in terms of guaranteed stability and attained performance. As a caveat, the computational burden associated to the implementation of the SVOs can pose stringent constraints for higher sampling rates.

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