Technical Notes and Correspondence

A Combined Multiple Model Adaptive Control Scheme and Its Application to Nonlinear Systems With Nonlinear Parameterization

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***Abstract*—A combined multiple model adaptive control (CMMAC) scheme, which is a proper combination of the estimator-based MMAC scheme and the unfalsified MMAC scheme, has been proposed with the aim of taking advantage of the strength of each scheme while avoiding their weaknesses. The major novelty of the CMMAC scheme lies in the fact that it monitors not only the adequacy of candidate models in terms of their estimation performances but also the performance of the active candidate controller. As an application of the CMMAC scheme and one example of such new multiple model adaptive controllers, a CMMAC based controller has been designed for a class of nonlinear systems with nonlinear parameterization. Under some sufficient conditions, a strong finite time switching result (which provides a characterization on the maximum number of switching) and the closed-loop stability have been established. A constructive design based on back-stepping is provided for the adaptive control problem of a special class of nonlinearly parameterized systems, which can satisfy all the sufficient conditions to ensure closed-loop stability.**

***Index Terms*—Estimator-based MMAC, multiple model adaptive control (MMAC), nonlinear parameterization, nonlinear systems, unfalsified MMAC.**

I. Introduction

Multiple model adaptive control (MMAC) has attracted a great deal of interest since the late 1980s. There are two important classes of MMAC schemes that make use of a switching logic. One class of such schemes called estimator-based MMAC schemes monitor only the adequacy of the candidate models ([1]–[11]). The MMAC schemes in [3] allow switching to happen at every time instant. However, switching too quickly might also lead to the possibility of unbounded chatter ([2]). In order to prevent switching from happening too fast, hysteresis switching algorithms were proposed in [1], [2], [8], [9], [11] and [10]. Another way to prevent switching from happening too fast is to use so-called dwell-time-switching, which was proposed in [5], [6]. Among the cited results, of particular relevance to this note are those presented in [8], [10]. Because only the adequacy of the candidate models is monitored, a certain notion of detectability is needed to ensure the closed-loop stability. For nonlinear systems with unknown parameters, it is in general a very difficult task to check whether such a detectability condition is satisfied or not. Moreover, although both papers took the disturbances into consideration in the general framework, the closed-loop stability results were actually established only for the ideal case that the systems do not have disturbances. It is not clear how to use the theory in [8], [10] to prove the closed-loop stability when non-decaying disturbances are present. In fact, when non-decaying disturbances are present and the systems under consideration do not have equilibria, it is not clear how to define the notion of detectability.

Another class of MMAC schemes are those proposed in the unfalsified control literature, which monitor only the performance of all candidate controllers, see [12]–[15]. One difficulty of applying this approach to nonlinear adaptive control lies in the fact that it would be very difficult if not impossible to design those *fictitious reference signals* corresponding to all candidate controllers except the active one. Another difficulty of using the unfalsified control design approach is that it requires the notion of cost-detectability. Under the claim that the unfalsified control design does not need any explicit model, it would be an extremely difficult task to check whether cost-detectability is satisfied or not. Even if one has a good mathematical model for the considered system, it would be highly nontrivial to check the cost-detectability if the considered system is a nonlinear system with unknown parameters.

The main purpose of this note is to propose a novel MMAC scheme that can take advantage of the strength of both estimator-based MMAC schemes and unfalsified MMAC schemes but avoid their weaknessness at the same time. As a result, a novel scheme called combined multiple model adaptive control (CMMAC) scheme is developed by combining properly the ideas in estimator-based MMAC schemes and unfalsified MMAC schemes. In the CMMAC scheme, it is proposed to monitor *simultaneously* the adequacy of all candidate models and the performance of *only* the active candidate controller. Unlike the existing estimator-based MMAC schemes which monitor only the adequacy of candidate models, by monitoring simultaneously the performance of the active candidate controller, the notion of detectability in [8], [10] is no longer needed. When compared with existing MMAC schemes in unfalsified control which monitor the performance of *all* candidate controllers, monitoring *only* the performance of the active candidate controller avoids the difficulty in generating *fictitious reference signals*. More importantly, the monitoring of the performance of the active candidate controller together with the monitoring the adequacy of all candidate models removes the need of the concept of cost-detectability.

At the same time, the proposed CMMAC scheme offers a very general framework to design new multiple model adaptive controllers. Theoretically, one can take any existing estimator-based MMAC scheme and any existing unfalsified MMAC scheme to make a new CMMAC scheme.

As an application example of the proposed CMMAC scheme, a CMMAC scheme is designed to solve the adaptive control problems of a class of nonlinear systems with nonlinear parameterization. Nonlinear adaptive control has made great progress since the 1990s after the introduction of the backstepping design approach [16], [17] for linearly parameterized nonlinear systems. More recently, attention has been paid to nonlinear systemswith nonlinear parameterization, see [19]–[25] for some examples. The designed CMMAC scheme offers a new MMAC based strategy to the adaptive control problems of nonlinear systems with nonlinear parameterization, where the existing

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TABLE I
COMPARISON BETWEEN OUR CMMAC SCHEMES WITH TWO EXISTING MMAC SCHEMES

	$CMMAC$	$EMMAC$	$UMMAC$
$Candidate$ $model$ $block$	Yes	Yes	No
$Candidate$ $controller$ $block$	Yes	Yes	Yes
$Monitoring$ $signal$ $block:$ $model$ $adequacy$	Yes	Yes	No
$Monitoring$ $signal$ $block:$ $controller$ $performance$	Yes $(But$ $only$ $active$ $controller)$	No	Yes $(All$ $candidate$ $controllers)$
$Switching$ $logic$ $block$	Yes	Yes	Yes

estimator-based MMAC schemes and unfalsified MMAC schemes will have difficulty.

II. General Structure of CMMAC Schemes

In this section, we shall provide the general structure of our CMMAC schemes and then make a comparison with estimator-based MMAC schemes and unfalsified MMAC schemes.

A. The Structure of CMMAC Schemes

The proposed CMMAC schemes consist of four design blocks, viz., candidate model block, candidate controller block, monitoring signal block and switching logic block. In the design of the candidate model block, one decides the type of candidate models and the number of them. In the candidate controller block, one designs the best possible candidate controllers to achieve certain desired performance. The monitoring signal block is concerned with the design of appropriate signals to monitor the adequacy of the candidate models and the performance of the active controller. Finally, through monitoring the designed monitoring signals, the switch logic block makes a decision on which candidate controller should be used.

B. Comparison With Existing Schemes

A comparison between our CMMAC schemes with estimator-based MMAC (EMMAC) schemes and unfalsified MMAC (UMMAC) schemes is presented in Table I.

Compared with EMMAC schemes, the notion of detectability is no longer needed and thus the difficulty involved in checking the detectability is avoided especially for nonlinear systems with unknown parameters. Compared with UMMAC schemes, the notion of cost-detectability is no longer needed and thus the difficulty involved in checking the cost-detectability is avoided, and moreover, by monitoring only the performance of the active controller, there is no need to generate *fictitious reference signals* (a very difficulty task for nonlinear systems with unknown parameters). Based on the comparisons, one can see that our CMMAC scheme offers a way to take advantage of the strength of both EMMAC and UMMAC schemes but avoid their weaknesses.

III. An Application of the CMMAC Scheme to Nonlinear Systems With Nonlinear Parameterization

A. Systems of Interest and Problem Formulation

In this section, as an application of our general CMMAC scheme, a particular CMMAC scheme will be developed for the following nonlinear system

$$\dot{x}(t) = f(x(t), \theta^*, u(t), d(t)) \tag{1}$$

where $x(t)$, $u(t)$ and $d(t)$ are the system state, control input, and disturbances, respectively; $\theta^* \in R^q$ is an unknown parameter vector; $f(x, \theta^*, u, d)$ is continuous with respect to its arguments.

We make the following assumption regarding the unknown parameter vector θ^*.

- A.1 $\theta^* \in \Theta = \cup_{j=1}^{M} S_j$, where M is a known finite positive integer, and S_j, $j = 1, 2, \ldots, M$ are known and bounded sets and $S_j \cap S_l = \phi$ for $j \neq l$ with ϕ being an empty set. Here, we do not know to which set θ^* belongs although $\theta^* \in \Theta$ is known.

In this note, we are only interested in state feedback controller design and in robust global stability.

B. The Design of a CMMAC Scheme

In this subsection, we shall first provide a detailed design of a CMMAC scheme for system (1) with focus on the candidate controller block and *especially* the monitoring signal block, and then establish a general theory for closed-loop stability.

The condition on the design of candidate controllers is provided in the following assumption.

- A.2 For each set $S_j \subset \Theta$, there exists $\theta_j \in S_j$ and a corresponding continuous control law $u_{\theta_j}(t) = \phi(x(t), \theta_j)$ such that, for any x_0, a) the solution $x(t)$ of $\dot{x} = f(x, \theta, \phi(x, \theta_j), d(t))$ exists and is unique for any $\theta \in \Theta$; b) along the solution $x(t)$ of $\dot{x} = f(x, \theta, \phi(x, \theta_j), d(t))$, $\dot{V}_{\theta_j}(x(t)) \leq -K_{1,\theta_j}(x(t)) + K_{2,\theta_j}(x(t))$ if $\theta \in S_j$, where V_{θ_j} and K_{1,θ_j} are known positive definite continuous functions of $x(t)$ with $V_{\theta_j}(0) = 0$ and $K_{1,\theta_j}(0) = 0$, which satisfy $\lim_{\|x\| \to \infty} V_{\theta_j}(x) = \infty$ and $\lim_{\|x\| \to \infty} K_{1,\theta_j}(x) = \infty$, and $K_{2,\theta_j}(x)$ is known and upper bounded for all x.

Remark 1: By requiring $K_{2,\theta_j}(x)$ and $K_{4,\theta_j}(x, e_{\theta_j})$ to be known and upper bounded, we implicitly require that $\|d\|$ is bounded and an upper bound of $\|d\|$ is available.

Under assumption A.2, it is easy to show the following result.

Lemma 1: For system (1), suppose assumption A.2 is satisfied. Then, there exists a candidate controller $u_{\theta_{j_0}}(t) = \phi(x(t), \theta_{j_0})$ with $\theta_{j_0}, \theta^* \in S_{j_0}$ such that for any x_0, $x(.)$ and $u(.)$ are bounded, and there exists a positive constant δ such that $x(t)$ enters B_δ asymptotically.

In the remainder of this note, the dependence of variables on the time t will drop out whenever it is appropriate. For example, we will write $u_\theta(t) = \phi(x(t), \theta)$ as $u_\theta = \phi(x, \theta)$.

One type of monitoring signal in this note is based on the design of multiple state estimators for system (1), which take the following form:

$$\dot{\hat{x}}_{\theta_j} = H(\hat{x}_{\theta_j} - x) + f_m(x, \theta_j, u) + g(x, \hat{x}_{\theta_j}, \theta_j, u) \tag{2}$$

where $j = 1, \ldots, M$, H is a Hurwitz matrix that can be chosen freely, and $\hat{x}_{\theta_1}(0) = \cdots = \hat{x}_{\theta_M}(0)$. $f_m(x, \theta_j, u)$ is the modeled part of $f(x, \theta_j, u, d(t))$, which is chosen to be close enough to $f(x, \theta_j, u, d(t))$ in a certain sense (for example in a sense of the corresponding condition in Lemma 2 below), and $g(x, \hat{x}_{\theta_j}, \theta_j, u)$ is a design function.

Define estimation error signals as $e_\theta = \hat{x}_\theta - x, \theta = \theta_1, \ldots, \theta_M$, with the designed estimators required to satisfy the following condition.

- A.3 The vector function $g(x, \hat{x}_{\theta_j}, \theta_j, u)$ is designed such that $\dot{V}(e_{\theta_j}) \leq -K_3(e_{\theta_j}) + K_{4,\theta_j}(x, e_{\theta_j})$ when both θ_j and θ^* are in the same set, where V and K_3 are known positive definite continuous functions of e_{θ_j} with $V(0) = 0$ and $K_3(0) = 0$, which satisfy $\lim_{\|e_{\theta_j}\| \to \infty} V(e_{\theta_j}) = \infty$ and $\lim_{\|e_{\theta_j}\| \to \infty} K_3(e_{\theta_j}) = \infty$, and K_{4,θ_j} is known and upper bounded for any x and e_{θ_j} and for any given θ_j.

We provide the following lemma without proof.

Lemma 2: For system (1), suppose that $\|f(x, \theta^*, u, d) - f_m(x, \theta^*, u)\| \leq \rho_m(x, u)$ for any $\theta^* \in \Theta$, and suppose also that, for any $\theta, \theta^* \in \Theta$, there exists a nonnegative continuous function $\rho(x, \theta, u)$ and a continuous and strictly increasing function $\gamma(r)$ with $\gamma(0) = 0$ and $\lim_{r \to \infty} \gamma(r) = \infty$ such that $\|f_m(x, \theta, u) - f_m(x, \theta^*, u)\| \leq \rho(x, \theta, u)\gamma(\|\theta - \theta^*\|)$. Suppose also that M^* is such that for any $\theta^* \in S_j$ and j, $\|\theta_j - \theta^*\| \leq M^*$ and P is a positive definite symmetric matrix such that $H^T P + PH = -I$ with I being an identity matrix. Choose $g(x, \hat{x}_{\theta_j}, \theta_j, u) = -K_g[\rho^2(x, \theta_j, u)\gamma^2(M^*) + \rho_m^2(x, u)]Pe_{\theta_j}$ with K_g a positive design constant. Then the multiple estimators defined by (2) can ensure that assumption A.3 is satisfied.

Remark 2: The condition $\|f_m(x, \theta, u) - f_m(x, \theta^*, u)\| \leq \rho(x, \theta, u)\gamma(\|\theta - \theta^*\|)$ is a rather weak condition since it includes the linear parameterization case as a special case and also the Lipschitz condition as a special case with $\gamma(\|\theta - \theta^*\|) = \|\theta - \theta^*\|$ and $\rho(x, \theta, u)$ a constant. Hence, the Lemma shows that Assumption A.3 can be satisfied under fairly weak conditions through a constructive design of multiple state estimators.

The monitoring signals that will be used to monitor the adequacy of candidate models are defined as

$$\dot{W}_\theta = -\lambda W_\theta + 1 - sgn\left[V(e_\theta(0)) - \int_0^t (K_3(e_\theta) - K_{4,\theta}(x, e_\theta))d\tau - V(e_\theta)\right] \quad (3)$$

where $\theta = \theta_1, \ldots, \theta_M$, λ is a positive constant, and $W_\theta(0) = 0$ for all $\theta = \theta_1, \ldots, \theta_M$. The "sgn" function is defined as follows: $sgn(x) = 1$ if $x \geq 0$; $sgn(x) = -1$ if $x < 0$.

The monitoring signal that will be used to monitor the performance of the active candidate controller is defined as

$$\bar{W}_\theta(t', t) = \int_{t'}^t e^{-\lambda(t-\tau)} \left(1 - sgn\left[V_\theta(x(t')) - \int_{t'}^\tau (K_{1,\theta}(x) - K_{2,\theta}(x))ds - V_\theta(x(\tau))\right]\right) d\tau \quad (4)$$

where $\theta = \theta_1, \ldots, \theta_M$, λ is a positive constant, and $t > t'$.

Remark 3: It should be noted that the above monitoring signals are designed based on certain Lyapunov function inequalities. The idea of using Lyapunov function inequalities for monitoring purposes is not new and has appeared in [26] (which has been brought to our attention by one reviewer). However, as will be shown later in this section, this monitoring technique offers greater advantages than just monitoring directly the estimation error signals (i.e., e_{θ_j}, $j = 1, 2, \ldots, M$) as in [8], [10].

With the candidate controllers, the multiple estimators and the monitoring signals at hand, our CMMAC scheme can now be presented in steps.

1) A CMMAC Scheme:

- Step 1. Choose a dwell-time constant τ_D.
- Step 2. Let $t_0 = 0$ and $k = 0$, and pick a candidate controller in the candidate controller family.
- Step 3. For $t_k \leq t < t_k + \tau_D$, let $\sigma(t) = \sigma(t_k)$ and $u(t) = u_{\theta_{\sigma(t)}}(t)$.
- Step 4. For $t \geq t_k + \tau_D$, monitor all $W-$ signals and *only one* $\bar{W}-$ signal $\bar{W}_{\theta_{\sigma(t_k)}}(t_k, t)$ corresponding to the active candidate controller, let $S_{\min,t} = \{i | W_{\theta_i}(t) = 0\}$. If $\sigma(t_k) \in S_{\min,t}$ and $\bar{W}_{\theta_{\sigma(t_k)}}(t_k, t) = 0$, no new controller is switched on and let $\sigma(t) = \sigma(t_k)$ and $u(t) = u_{\theta_{\sigma(t)}}(t)$. If $\sigma(t_k) \in S_{\min,t}$ but $\bar{W}_{\theta_{\sigma(t_k)}}(t_k, t) > 0$ or $\sigma(t_k)$ does not belong to $S_{\min,t}$, then increment k by 1, let $t_k = t$, pick $\sigma(t_k)$ as any element in $S_{\min,t}$ that is different from $\theta_{\sigma(t_j)}$, $0 \leq j \leq k - 1$, and let a new controller $u(t) = u_{\theta_{\sigma(t)}}(t)$ be switched on at t. Go back to Step 3.

Remark 4: The switching logic in the above CMMAC scheme is actually inspired by the logic proposed in [5]. However, the dwell-time constant τ_D is only introduced to prevent switching too fast.

Let $S_W = \{\theta_i | W_{\theta_i}(t_1) = 0\}$ and $S_{W,\infty} = \{\theta_i | W_{\theta_i}(t) \equiv 0, \ \forall t \in [0, \infty)\}$, and denote the number of elements in S_W as l_W. Then it is not hard to show the following result.

Lemma 3: For system (1) under assumptions $A.1$ to $A.3$, suppose that the multiple estimators are defined by (2), and that $W-$ and $\bar{W}-$signals are defined by (3) and (4), respectively. If the CMMAC scheme is employed, then for any x_0, the maximum number of switching is less than l_W, and there exists a finite time t_f such that $\theta_{\sigma(t)} \equiv \theta_{\sigma(t_f)}$ for all $t \in [t_f, \infty)$ and $\theta_{\sigma(t_f)} \in S_{W,\infty}$.

Remark 5: The lemma provides a characterization on the maximum number of switching, which is equal to $l_W (\leq M - 1)$, the number of elements in S_W. This result is stronger than the usual finite time switching results, which often do not guarantee that the upper bound on the number of switching is smaller than $M - 1$. The result is obtained not only because of assumptions A.1–A.3 but also the use of Lyapunov inequalities for monitoring. For example, if one uses the monitoring signals defined in [8], [10], the same results cannot be reached even if assumptions A.1–A.3 are satisfied.

The main stability result is presented in the following theorem.

Theorem 1: For system (1) that satisfies assumptions A.1 to A.3, suppose that the multiple estimators defined by (2) are designed, and suppose also that the $W-$signals and the $\bar{W}-$signals defined by (3) and (4) are used in the CMMAC scheme to generate a switching controller for system (1). Assume further that no candidate controller $u = \phi(x, \theta_j)$ with $j = 1, \ldots, M$ can make the system $\dot{x} = f(x, \theta^*, \phi(x, \theta_j), d)$ escape in finite time. Then for any x_0, all the closed-loop signals are bounded and there exists a positive constant δ such that $x(t)$ enters B_δ asymptotically.

Proof: For any x_0, it has been shown by Lemma 3 that the maximum number of switching is less than l_W, and there exists a finite time t_f such that $\theta_{\sigma(t)} \equiv \theta_{\sigma(t_f)}$ for all $t \in [t_f, \infty)$ and $\theta_{\sigma(t_f)} \in S_{W,\infty}$.

Suppose that the actual number of switching is l_s. Denote the switching time instants as $t_0 = 0, t_1, \ldots, t_{l_s - 1}, t_{l_s} = t_f$. Then, for any $0 \leq k \leq l_s - 1$, we have either $t_{k+1} - t_k = \tau_D$ or $t_{k+1} - t_k > \tau_D$. If the former is true, then $x(.)$ is bounded on $[t_k, t_{k+1}]$ if $x(t_k)$ is finite because of the non-τ_D escape time assumption. If the latter is true, then, by letting $\sigma(t_k) = \theta_j$, we must have $V_{\theta_j}(x(t)) - V_{\theta_j}(x(t_k)) \leq \int_{t_k}^t (K_{1,\theta_j}(x) - K_{2,\theta_j}(x))d\tau$ on $[t_k, t_{k+1})$, which implies that $\dot{V}_{\theta_j}(x(t)) \leq -K_{1,\theta_j}(x(t)) + K_{2,\theta_j}(x(t))$ for $t \in [t_k, t_{k+1})$. This according to Lemma 1 proves that $x(.)$ is bounded on $[t_k, t_{k+1}]$. In this way, starting from $t = 0$ and for any x_0, the CMMAC scheme and the non-finite time escape assumption will ensure that $x(.)$ is bounded on $[0, t_f]$.

Since $u = \phi(x, \theta_{\sigma(t_f)})$ over $[t_f, \infty)$ and $\theta_{\sigma(t_f)} \in S_{W,\infty}$, it can be shown that $\dot{V}_{\theta_{\sigma(t_f)}}(x(t)) \leq -K_{1,\theta_{\sigma(t_f)}}(x(t)) + K_{2,\theta_{\sigma(t_f)}}(x(t))$ over $[t_f, \infty)$. It is now straightforward that the conclusions of the theorem hold. ¶

Remark 6: Assumptions A.1–A.3 allow the case where $S_{W,\infty}$ has more than one element, which means more than two candidate models are allowed to be indistinguishable through estimator design. In other words, neither the "detectability" condition required in [8], [10] nor the "cost-detectability" condition required by some UMMAC schemes is necessarily satisfied. There are two difficulties with the notion of "detectability". First, it is very difficult to check for the nonlinear systems under consideration. Second, it is not defined for nonlinear systems with non-decaying disturbances, which are the systems under study in this note. Instead of being restrictive as one might think, our assumptions have a big advantage in that notions of "detectability" are avoided.

Remark 7: Since the "detectability" condition required in [10] is not necessarily satisfied or may not be even defined for the considered systems, the MMAC scheme in [10] cannot be used to solve the adaptive control problem of the considered systems under our assumptions. For the UMMAC schemes in [12]–[15], besides the difficulty of checking the "cost-detectability" condition, one does not know how to generate those *fictitious reference signals* required for non-active candidate controllers for the considered nonlinear systems. Therefore, our CMMAC scheme offers solutions for control problems where the existing EMMAC and UMMAC schemes will have difficulty.

IV. Constructive Design of a CMMAC Scheme for a Special Class of Nonlinear Systems

In this section, we shall provide a constructive design of the proposed CMMAC scheme for a special class of nonlinear systems of (1), which take the following form:

$$\begin{aligned}\dot{x}_1 &= x_2 + \xi_1(x_1,\theta^*) + \psi_1(x,d(t))\\ \dot{x}_i &= x_{i+1} + \xi_i(x_1,\dots,x_i,\theta^*) + \psi_i(x,d(t)),\\ &\quad i=2,\dots,n-1\\ \dot{x}_n &= \xi_n(x_1,\dots,x_n,\theta^*) + u + \psi_n(x,d(t))\end{aligned} \tag{5}$$

where θ^* is an unknown parameter vector that belongs to a closed and bounded set $\Theta \in R^q$, $\xi_i(x_1,\dots,x_i,\theta^*)$, $i=1,\dots,n$ are scalar smooth nonlinear functions with $\xi_i(0,\dots,0,\theta^*)=0$, $i=1,\dots,n$, and $\psi_i(x,d(t))$, $i=1,\dots,n$ are unknown.

We need the following assumptions:

- B.1 For each $1\le i\le n$, there exist a known positive smooth function $\rho_i(x_1,\dots,x_i,\theta)$ and a continuous and strictly increasing function $\gamma_i(r)$ with $\gamma_i(0)=0$ and $\lim_{r\to\infty}\gamma_i(r)=\infty$ such that

$$\begin{aligned}&|\xi_i(x_1,\dots,x_i,\theta)-\xi_i(x_1,\dots,x_i,\bar{\theta})|\\ &\qquad\le \rho_i(x_1,\dots,x_i,\theta)\gamma_i(\|\theta-\bar{\theta}\|)\end{aligned}$$

  where $\theta,\bar{\theta}\in\Theta$.
- B.2 For each $1\le i\le n$, there exists a known positive smooth function $\varphi_i(x_1,\dots,x_i)$ such that $|\psi_i(x,d(t))|\le \varphi_i(x_1,\dots,x_i)$.

Since Θ is closed and bounded, it is possible to find finite number of sets, that is, S_j, $j=1,\dots,M$ such that $\Theta=\cup_{j=1}^{M}S_j$ and $\|\theta-\theta_j\|\le M^*$ if $\theta,\theta_j\in S_j$, where M^* can be made sufficiently small by proper partitioning. So, Assumption A.1 can be satisfied.

Define $\xi_{j,i}(x_1,\dots,x_i,\theta_j) = \xi_i(x_1,\dots,x_i,\theta_j)$ and $\rho_{j,i}(x_1,\dots,x_i,\theta_j)=\rho_i(x_1,\dots,x_i,\theta_j)$ for $1\le i\le n, 1\le j\le M$ and write $\xi_{j,i}(x_1,\dots,x_i,\theta_j)$ as $\xi_{j,i}$ for simplicity.

For the considered system (5), by the use of the standard backstepping design procedure in [16], the candidate controllers can be designed as

$$\begin{aligned}u_{\theta_j}(t) &= \phi(x,\theta_j)\\ &= -c_n z_{j,n} - z_{j,n-1} - \xi_{j,n} + \sum_{i=1}^{n-1}\frac{\partial\alpha_{j,n-1}}{\partial x_i}(x_{i+1}+\xi_{j,i})\\ &\quad - k_n\gamma_n^2(M^*)\rho_{j,n}^2 z_{j,n} - l_n\varphi_n^2 z_{j,n}\\ &\quad -\sum_{i=1}^{n-1}\left(\frac{\partial\alpha_{j,n-1}}{\partial x_i}\right)^2(k_n\gamma_i^2(M^*)\rho_{j,i}^2 z_{j,n}+l_n\varphi_i^2 z_{j,n})\end{aligned} \tag{6}$$

where $c_i,k_i,l_i,\ i=1,\dots,n$ are positive design constants, $z_1=z_{j,1}=x_1$ and we have for $i=2,\dots,n$ $z_{j,i}=x_i-\alpha_{j,i-1}(x_1,\dots,x_{i-1},\theta_j)$, and $\alpha_{j,i},\ i=1,\dots,n$ are defined as

$$\begin{aligned}\alpha_{j,1} &= -c_1z_1-\xi_{j,1}-l_1\varphi_1^2z_1-k_1\gamma_1^2(M^*)\rho_{j,1}^2z_1\\ \alpha_{j,i} &= -c_iz_{j,i}-z_{j,i-1}-\xi_{j,i}+\sum_{h=1}^{i-1}\frac{\partial\alpha_{j,i-1}}{\partial x_h}(x_{h+1}+\xi_{j,h})\\ &\quad -k_i\gamma_i^2(M^*)\rho_{j,i}^2z_{j,i}-l_i\varphi_i^2z_{j,i}\\ &\quad -\sum_{h=1}^{i-1}\left(\frac{\partial\alpha_{j,i-1}}{\partial x_h}\right)^2(k_i\gamma_h^2(M^*)\rho_{j,h}^2z_{j,i}+l_i\varphi_h^2z_{j,i}).\end{aligned} \tag{7}$$

Choose $V_j=\sum_{i=1}^{n}(1/2)z_{j,i}^2$. It can be shown that

$$\dot{V}_j\le -\sum_{i=1}^{n}c_iz_{j,1}^2+\beta_{j,n} \tag{8}$$

where $\beta_{j,i},\ i=n,\dots,2,1$ is defined as

$$\begin{aligned}\beta_{j,i} &= \beta_{j,i-1}-k_i\gamma_i^2(M^*)\rho_{j,i}^2z_{j,i}^2+\gamma_i(\|\theta_j-\theta^*\|)\rho_{j,i}|z_{j,i}|\\ &\quad -l_i\varphi_i^2z_{j,i}^2+\varphi_i|z_{j,i}|\\ &\quad -\sum_{h=1}^{i-1}\left(\frac{\partial\alpha_{j,i-1}}{\partial x_h}\right)^2(k_i\gamma_h^2(M^*)\rho_{j,h}^2+l_i\varphi_h^2)z_{j,i}^2\\ &\quad +\sum_{h=1}^{i-1}\left|\frac{\partial\alpha_{j,i-1}}{\partial x_h}\right|(\gamma_h(\|\theta_j-\theta^*\|)\rho_{j,h}+\varphi_h)|z_{j,i}|,\\ \beta_{j,1} &= -k_1\gamma_1^2(M^*)\rho_{j,1}^2z_1^2+|z_1|\rho_{j,1}\gamma_1(\|\theta_j-\theta^*\|)\\ &\quad -l_1\varphi_1^2z_1^2+|z_1|\varphi.\end{aligned} \tag{9}$$

Using the fact $-ax^2+bx\le(b^2/4a)$, $a>0$ repeatedly, it can be shown that

$$\beta_{j,n}\le\sum_{i=1}^{n}\frac{(n-i+1)\gamma_i^2(\|\theta_j-\theta^*\|)}{4k_i\gamma_i^2(M^*)}+\sum_{i=1}^{n}\frac{(n-i+1)}{4l_i}. \tag{10}$$

By replacing $\gamma_i(\|\theta_j-\theta^*\|)$ with $\gamma_i(M^*)$ for $i=1,\dots,n$ in the definition of $\beta_{j,n}$, we get a known and upper bounded function, call it $\bar{\beta}_{j,n}$, depending on x and θ_j. When $\theta^*\in S_j$, we will have $\|\theta_j-\theta^*\|\le M^*$, and thus $\bar{\beta}_{j,n}\le\sum_{i=1}^{n}(n-i+1)/4k_i+\sum_{i=1}^{n}(n-i+1)/4l_i$.

Now let $K_{1,\theta_j}(x(t))=\sum_{i=1}^{n}c_iz_{j,1}^2$ and $K_{2,\theta_j}(x(t))=\bar{\beta}_{j,n}$. Then, it can be shown that that assumption A.2 is satisfied.

We can design multi-estimators as

$$\dot{\hat{x}}_{\theta_j}=H(\hat{x}_{\theta_j}-x)+f_m(x,\theta_j,u)+g(x,\hat{x}_{\theta_j},\theta_j),\ \ j=1,\dots,M \tag{11}$$

where $H=-I$, and $\hat{x}_{\theta_1}(0)=\cdots=\hat{x}_{\theta_M}(0)$.

The function $f_m(x,\theta_j,u)$ is defined as

$$f_m(x,\theta_j,u)=\begin{pmatrix}x_2+\xi_{j,1} & \cdots & x_n+\xi_{j,n-1} & \xi_{j,n}+u\end{pmatrix}^T \tag{12}$$

and $g(x, \hat{x}_{\theta_j}, \theta_j)$ is defined as

$$g = -K_g P e_{\theta_j} \sum_{i=1}^{n} \rho_{j,i}^2 \gamma^2(M^*) - K_g P e_{\theta_j} \sum_{i=1}^{n} \varphi_i^2 \qquad (13)$$

where $e_{\theta_j} = \hat{x}_{\theta_j} - x$, $P = (1/2)I$ since $H = -I$, $\gamma(M^*) = \max\{\gamma_1(M^*), \ldots, \gamma_n(M^*)\}$, and K_g is a positive design constant.

We can obtain the following results based on Theorem 1.

Corollary 1: For system (5) that satisfies Assumptions B1 and B2, suppose that candidate controllers are given by (6) and multi-estimators are designed through (11) to (13). Define the W–and $\bar{W}$–signals as in the previous subsection, then the switching controller generated by the CMMAC scheme ensures that, for any x_0, all the closed-loop signals are bounded and $x(t)$ enters asymptotically into a neighborhood of the origin, whose size can be made as small as desired by choosing the design constants $c_i, k_i, l_i,\ 1 \leq i \leq n$ sufficiently large.

Remark 8: Using (6) and (10), it can be shown that all assumptions A.1–A.3 are satisfied automatically. However, it is not clear how to check the relevant "detectability" condition developed in the literature is satisfied or not for this special class of nonlinear systems. Moreover, for existing UMMAC schemes based on *fictitious reference signals*, it is also not clear how to generate those *fictitious reference signals* for the considered nonlinear systems. This again shows that our assumptions are not necessarily stronger than existing MMAC schemes and thus not that restrictive as they appear.

V. Conclusions

In this note, we have proposed a combined MMAC (CMMAC) scheme and have applied it to solve the adaptive control problem for nonlinear systems with nonlinear parameterization. The CMMAC scheme advocates the novel idea of monitoring *simultaneously* the adequacy of candidate models and the performance of *only* the active candidate controller. In this way, it takes effectively the advantages of existing EMMAC and UMMAC schemes and avoids their weaknesses.

Our results do not address yet the question of what values to select for the nominal parameters θ_j, or how many there should be. The output feedback control problem is certainly of interest and more challenging, which has been addressed by the same authors in [27]. As in all switching-based control schemes such as MMAC and sliding mode control, discontinuous control signal has been used in this note, which might be an issue in some real applications. How to resolve this problem is left as a future topic. Moreover, how to relax the assumption of a valid global model and how to avoid a bad transient are also issues that need to addressed in the future.

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