Technical Notes and Correspondence

On Lyapunov-Metzler Inequalities and S-Procedure Characterizations for the Stabilization of Switched Linear Systems

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***Abstract*—In this note we present connections between two celebrated tools for the design of stabilising switching laws for continuous-time and discrete-time switched linear systems, namely Lyapunov-Metzler inequalities and S-procedure.**

***Index Terms*—Lyapunov-Metzler inequalities, S-procedure characterization, stabilization, switched linear systems.**

I. Introduction

In this note we study relationships between two celebrated and well-known methods for the design of stabilising switching laws for *switched linear systems* (SLSs) [13], which are given in continuous time by

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{1}$$

with $t \in \mathbb{R}_{\geq 0}$, and in discrete time by

$$x(t+1) = A_{\sigma(t)}x(t) \tag{2}$$

with $t \in \mathbb{N}$. In both cases $x(t) \in \mathbb{R}^n$ denotes the state at time t and $\sigma(t) \in \bar{N} := \{1, 2, \dots, N\}$ indicates which of the subsystems is active at time t. Here, $A_1, A_2, \dots, A_N$ are constant matrices in $\mathbb{R}^{n\times n}$. We are particularly interested in design techniques for the widely used *min-switching* strategies that are state-dependent switching laws given by

$$\sigma(t) \in \arg\min_{j\in\bar{N}} x(t)^\top P_j x(t), \tag{3}$$

where $P_j \in \mathbb{R}^{n\times n}$ is a positive definite matrix associated with the j-th subsystem, $j \in \bar{N}$.

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In the continuous-time setting, a solution to the above design problem was provided in [9] leading to so-called *Lyapunov-Metzler (L-M) inequalities*. These bilinear matrix inequalities (BMIs) involve as free variables the to-be-designed set of positive definite matrices $\{P_1, P_2, \dots, P_N\}$ and a matrix belonging to a subclass of the so-called Metzler matrices (see Definition 1 below). If the L-M inequalities are feasible, a stabilising min-switching strategy as in (3) has been constructed. The idea of L-M inequalities was extended to the discrete-time case in [10].

An alternative method for the design of min-switching laws can be based on the S-procedure. The S-procedure was used for the first time in [14] followed by its theoretical justification in [16]. The term S-procedure (or S-method) was coined in [1], see also, e.g., the survey [15] and the book [2]. The S-procedure is an instrumental tool in control theory and robust optimization analysis, see e.g., [2]–[4]. For stability and performance analysis of piecewise linear (PWL) systems—note that the min-switching law result turns the SLS in a closed-loop system of the PWL form—S-procedure relaxations are heavily used in both continuous time and discrete time since the appearance of the seminal works [5], [12]. In the special case of synthesising a min-switching strategy for a SLS the S-procedure characterizations involve BMIs in which the free variables are formed by sets of scalar quantities and the to-be-designed set of positive definite matrices $\{P_1, P_2, \dots, P_N\}$.

In this note we are interested in investigating the relationships between the L-M inequalities and the S-procedure characterizations for the stabilization of SLSs as in (1) and (2). In particular, our contributions are the following:

1) *Continuous-time setting (Section II):* For a given SLS we show that the existence of a solution to the L-M inequalities is equivalent to the existence of a solution to the S-procedure characterizations. Hence, this establishes for the first time that in the continuous-time case the L-M inequalities are in fact *equivalent* to the corresponding S-procedure characterization.
2) *Discrete-time setting (Section III):* For a given SLS we show that if the L-M inequalities admit a solution, then there exists also a *particular* solution to the S-procedure characterization. In addition, we explicitly show that the S-procedure characterization contains relaxations that are *not* present in the L-M inequalities, making the class of SLSs for which the S-procedure characterization is feasible potentially larger than the class for which the L-M inequalities are feasible.

Notation and Terminology: For a matrix $\Pi \in \mathbb{R}^{n\times m}$ the scalar π_{ij} denotes the element of Π in the i-th row and j-th column for $i \in \bar{n} = \{1, 2, \dots, n\}$ and $j \in \bar{m} = \{1, 2, \dots, m\}$. For a symmetric matrix M we write $M \succ 0$, if M is positive definite. Similarly, we use $M \succeq 0$, $M \prec 0$ and $M \preceq 0$ for M being positive semi-definite, negative definite and negative semi-definite, respectively. We will employ various (sub)classes of the class of Metzler matrices. Recall that a Metzler matrix is a square matrix in which all off-diagonal components are non-negative.

II. Stabilising Continuous-Time Switched Linear Systems

In this section we consider continuous-time SLSs as in (1) with switching laws of the form (3) and $t \in \mathbb{R}_{\geq 0}$.

A. The Lyapunov-Metzler Inequalities

A crucial role in the L-M inequalities is played by the following subset of so-called *Metzler* matrices, see [9].

Definition 1: For $N \in \mathbb{N} \setminus \{0\}$ the set $\mathcal{M}_c^N$ consists of all square matrices $\Pi \in \mathbb{R}^{N\times N}$ satisfying $\pi_{ij} \geq 0$ for all $i, j \in \bar{N}$ with $i \neq j$, and the column sums satisfying $\sum_{i=1}^{N} \pi_{ij} = 0$ for all $j \in \bar{N}$. Stated otherwise,

$$\mathcal{M}_c^N := \{\Pi \in \mathbb{R}^{N\times N} \mid \forall_{i,j\in\bar{N},\, i\neq j}\ \pi_{ij} \geq 0 \text{ and } \forall_{j\in\bar{N}} \sum_{i=1}^{N} \pi_{ij} = 0\}.$$

Theorem 2: [9, Theorem 3] Consider the SLS (1). Suppose that there exist a set of matrices $\{P_1, P_2, \ldots, P_N\} \subset \mathbb{R}^{n\times n}$ and a matrix $\Pi \in \mathcal{M}_c^N$ such that

$$A_j^\top P_j + P_j A_j + \sum_{i\in\bar{N}} \pi_{ij} P_i \prec 0 \text{ for all } j \in \bar{N} \tag{4a}$$

$$P_j \succ 0 \text{ for all } j \in \bar{N} \tag{4b}$$

are satisfied. Then any continuous and piecewise continuously differentiable solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying $\dot{x}(t) = A_{\sigma(t)}x(t)$ and σ satisfying the min-switching rule in (3) for almost all $t \in \mathbb{R}_{\geq 0}$ tends to zero exponentially[1].

The inequalities (4) are called the *Lyapunov-Metzler inequalities*. The above theorem was proven in [9] using

$$V(\xi) := \min_{i\in\bar{N}} \xi^\top P_i \xi \text{ with } \xi \in \mathbb{R}^n \tag{5}$$

as a Lyapunov function.

B. The S-Procedure Characterization

The following result can be obtained by combining the celebrated S-procedure [14], [16] with ideas from the stability analysis of piecewise linear (PWL) systems [12] using a piecewise quadratic Lyapunov function as in (5). Note that the closed-loop system given by (1) and (3) can indeed be written as the PWL system

$$\dot{x}(t) = A_j x(t), \text{ when } x(t) \in \Omega_j \tag{6}$$

with for $j \in \bar{N}$

$$\Omega_j := \{\xi \in \mathbb{R}^n \mid \xi^\top (P_j - P_i)\xi \leq 0 \text{ for all } i \in \bar{N} \setminus \{j\}\}. \tag{7}$$

In particular, we will use the following version of the S-procedure taken from [2, p. 24].

Proposition 3: [2, p. 24] Let $T_0, T_1, \ldots, T_p \in \mathbb{R}^{n\times n}$ be symmetric matrices. If there exist scalars $\tau_1, \tau_2, \ldots, \tau_p \in \mathbb{R}_{\geq 0}$ such that $T_0 - \sum_{i=1}^{p} \tau_i T_i \succ 0$, then for all $\xi \in \mathbb{R}^n \setminus \{0\}$ it holds that

$$\xi^\top T_i \xi \geq 0,\ i = 1, 2, \ldots, p \Rightarrow \xi^\top T_0 \xi > 0.$$

[1]Note that in this theorem global exponential stability is established for standard Carathéodory solutions, but in general not for (all) Filippov solutions [8] possibly including sliding mode behaviour. See Remark 6 below for a more detailed discussion on this matter.

In case $p = 1$ and there exists ξ_0 with $\xi_0 T_1 \xi_0 > 0$, the converse also holds. Similarly, if there exist scalars $\tau_1, \tau_2, \ldots, \tau_p \in \mathbb{R}_{\geq 0}$ such that $T_0 - \sum_{i=1}^{p} \tau_i T_i \succeq 0$, then for all $\xi \in \mathbb{R}^n$ it holds that

$$\xi^\top T_i \xi \geq 0,\ i = 1, 2, \ldots, p \Rightarrow \xi^\top T_0 \xi \geq 0.$$

Combining this S-procedure formulation and a reasoning close to [12, Theorem 1] leads to the following result.

Theorem 4: Consider the SLS (1). Suppose that there exist a set of matrices $\{P_1, P_2, \ldots, P_N\} \subset \mathbb{R}^{n\times n}$ and a set of scalars $\{\alpha_i^j\}_{j,i\in\bar{N},i\neq j}$ such that

$$A_j^\top P_j + P_j A_j \prec \sum_{\substack{i\in\bar{N}\\ i\neq j}} \alpha_i^j (P_j - P_i) \text{ for all } j \in \bar{N} \tag{8a}$$

$$P_j \succ 0 \text{ for all } j \in \bar{N} \text{ and } \alpha_i^j \geq 0 \text{ for all } j, i \in \bar{N}, i \neq j \tag{8b}$$

are satisfied. Then any continuous and piecewise continuously differentiable solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying $\dot{x}(t) = A_{\sigma(t)}x(t)$ and σ satisfying the min-switching rule in (3) for almost all $t \in \mathbb{R}_{\geq 0}$ tends to zero exponentially.

Proof: Recall that the closed-loop system given by (1) and (3) can be written as the PWL system (6) with regions defined by (7). Also note that the Lyapunov function as in (5) is a continuous and piecewise quadratic Lyapunov function given by $V(\xi) = \xi^\top P_j \xi$ when $\xi \in \Omega_j$, $j \in \bar{N}$. Applying the S-procedure result as in Proposition 3 according to the ideas in [12, Theorem 1] to the PWL system (6) based on the Lyapunov function V proves the theorem (note that indeed (8) implies that there exists a positive ε such that $\dot{V}(x(t)) \leq -\varepsilon V(x(t))$ along any solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ as considered in the theorem and for almost all $t \in \mathbb{R}_{\geq 0}$). ∎

We call (8) an S-procedure characterization for the synthesis of stabilising min-switching laws.

C. Connections

Theorem 5: Consider the SLS (1). The following statements are equivalent:

(i) There is a solution to the L-M inequalities (4).

(ii) There is a solution to the S-procedure characterization (8).

Proof: $(\mathbf{i}) \Rightarrow (\mathbf{ii})$: Given that there is a solution to (4), we have

$$A_j^\top P_j + P_j A_j + \pi_{jj} P_j + \sum_{\substack{i\in\bar{N}\\ i\neq j}} \pi_{ij} P_i \prec 0 \text{ for all } j \in \bar{N}. \tag{9}$$

From the properties of Π, the left-hand side of the above inequality is equal to

$$A_j^\top P_j + P_j A_j + \left(-\sum_{\substack{i\in\bar{N}\\ i\neq j}} \pi_{ij}\right) P_j + \sum_{\substack{i\in\bar{N}\\ i\neq j}} \pi_{ij} P_i$$

$$= A_j^\top P_j + P_j A_j - \sum_{\substack{i\in\bar{N}\\ i\neq j}} \pi_{ij} (P_j - P_i). \tag{10}$$

Substituting (10) in (9), we obtain for all $j \in \bar{N}$

$$A_j^\top P_j + P_j A_j \prec \sum_{\substack{i\in\bar{N}\\ i\neq j}} \pi_{ij} (P_j - P_i),$$

which is exactly (8) with $\alpha_i^j = \pi_{ij}$, $i, j \in \bar{N}$, $i \neq j$.

(**ii**) ⇒ (**i**): Since there is a solution to (8), we have

$$A_j^\top P_j + P_j A_j + \left(-\sum_{\substack{i\in\bar N\\ i\neq j}} \alpha_i^j\right) P_j + \sum_{\substack{i\in\bar N\\ i\neq j}} \alpha_i^j P_i \prec 0 \ \text{ for all } \ j\in\bar N.$$

By defining $\alpha_j^j := -\sum_{\substack{i\in\bar N\\ i\neq j}} \alpha_i^j$, the above inequality is equivalent to

$$A_j^\top P_j + P_j A_j + \sum_{i\in\bar N} \alpha_i^j P_i \prec 0 \ \text{ for all } \ j\in\bar N.$$

Consequently, (4) holds with $\pi_{ij} = \alpha_i^j$ for all $i,j\in\bar N$. Note that due to the definition of α_j^j, $j\in\bar N$, we have that $\sum_{i\in\bar N}\pi_{ij} = 0$ for all $j\in\bar N$ and thus $\Pi\in\mathcal{M}_c^N$. ■

In Theorem 5 we identified the *equivalence* between the L-M inequalities (4) and the S-procedure characterization (8).

Remark 6: It is well-known that the L-M inequalities preserve global exponential stability of (1) under the min-switching strategy (3) even when *attractive* sliding modes [8] occur, see [9, Remark 3]. The above is also true for the S-procedure characterization (8), see [13, p. 70] for an extended discussion on this matter. However, there are examples which illustrate that so-called *repulsive* sliding modes [8] can be unstable under the min-switching law indicating that global exponential stability is *not* guaranteed under *all* Filippov solutions. For this reason we formulated Theorems 2 and 4 for standard Carathéodory solutions (in line with [12, Theorem 1]), even though it may be argued that repulsive sliding modes do not appear in practice for such systems. See also [11] for more details on how to guarantee stability of sliding modes in the context of continuous-time PWL systems.

III. Stabilising Discrete-Time Switched Linear Systems

In this section we consider discrete-time SLSs as in (2) with switching laws of the form (3) and $t\in\mathbb{N}$.

A. The Lyapunov-Metzler Inequalities

Also for the discrete-time case a particular class of *Metzler* matrices plays an important role in the L-M inequalities, see [10].

Definition 7: For $N\in\mathbb{N}\setminus\{0\}$ the set $\mathcal{M}_d^N$ consists of all square matrices $\Pi\in\mathbb{R}^{N\times N}$ satisfying $\pi_{ij}\geq 0$ for all $i,j\in\bar N$, and the column sums satisfying $\sum_{i=1}^N \pi_{ij} = 1$ for all $j\in\bar N$. Stated otherwise,

$$\mathcal{M}_d^N := \{\Pi\in\mathbb{R}^{N\times N} \mid \forall_{i,j\in\bar N}\ \pi_{ij}\geq 0 \text{ and } \forall_{j\in\bar N} \sum_{i=1}^N \pi_{ij} = 1\}.$$

Theorem 8: [10, Theorem 3] Consider the SLS (2). Suppose that there exist a set of matrices $\{P_1, P_2,\dots,P_N\}\subset\mathbb{R}^{n\times n}$ and a matrix $\Pi\in\mathcal{M}_d^N$ such that

$$A_j^\top\left(\sum_{i\in\bar N}\pi_{ij}P_i\right)A_j - P_j \prec 0 \text{ for all } j\in\bar N \tag{11a}$$

$$P_j\succ 0 \text{ for all } j\in\bar N \tag{11b}$$

are satisfied. Then the closed-loop SLS given by (2) and (3) is globally exponentially stable.

Similar to the continuous-time case, inequalities (11) are known as the *Lyapunov-Metzler (L-M) inequalities* and the above theorem was proven in [10] by employing the Lyapunov function (5).

B. The S-Procedure Characterization

Based on the reasoning in [5] and using the piecewise quadratic Lyapunov function (5), we can establish that the PWL closed-loop system (2)–(3) is globally exponentially stable. This result will be formalised in the theorem below exploiting the S-procedure formulation as in Proposition 3 and observing that the PWL system (2)–(3) can be written as

$$x(t+1)\in\{A_j x(t)\mid x(t)\in\Omega_j, j\in\bar N\}, \tag{12}$$

where Ω_j is as defined in (7). Note that due to (3), the right-hand side of (12) is setvalued.

Theorem 9: Consider the SLS (2). Suppose that there exist a set of matrices $\{P_1,P_2,\dots,P_N\}\subset\mathbb{R}^{n\times n}$ and two sets $\{\alpha_k^{j\to i}\}_{i,j,k\in\bar N, k\neq j}$ and $\{\beta_\ell^{j\to i}\}_{i,j,\ell\in\bar N,\ell\neq i}$ of scalars such that

$$A_j^\top P_i A_j - P_j \prec \sum_{\substack{k\in\bar N\\ k\neq j}}\alpha_k^{j\to i}(P_j - P_k) + \sum_{\substack{\ell\in\bar N\\ \ell\neq i}}\beta_\ell^{j\to i}A_j^\top(P_i - P_\ell)A_j \quad \text{for all } i,j\in\bar N \tag{13a}$$

$$P_j\succ 0 \text{ for all } j\in\bar N \tag{13b}$$

$$\alpha_k^{j\to i}\geq 0 \text{ for all } i,j,k\in\bar N, k\neq j \tag{13c}$$

$$\beta_\ell^{j\to i}\geq 0 \text{ for all } i,j,\ell\in\bar N,\ell\neq i \tag{13d}$$

are satisfied. Then the closed-loop SLS given by (2) and (3) is globally exponentially stable.

Proof: Due to (13a) being a strict inequality, there exists a $\delta > 0$ such that

$$A_j^\top P_i A_j - (1-\delta)P_j \preceq \sum_{\substack{k\in\bar N\\ k\neq j}}\alpha_k^{j\to i}(P_j - P_k) + \sum_{\substack{\ell\in\bar N\\ \ell\neq i}}\beta_\ell^{j\to i}A_j^\top(P_i - P_\ell)A_j \quad \text{for all } i,j\in\bar N. \tag{14}$$

Based on (14) we will now prove that for all $t\in\mathbb{N}$ and all possible trajectories $x:\mathbb{N}\to\mathbb{R}^n$ of (12) the Lyapunov decrease condition $V(x(t+1))\leq(1-\delta)V(x(t))$ holds. To do so, observe that for each trajectory x at time $t\in\mathbb{N}$ there exist $i,j\in\bar N$ such that $\sigma(t) = j$ and $\sigma(t+1) = i$. Due to the min-switching rule (3) (see also (12)) $\sigma(t) = j$ and $\sigma(t+1) = i$ imply that

$$\begin{aligned} &x(t)^\top(P_k - P_j)x(t)\geq 0 \text{ for all } k\in\bar N\setminus\{j\} \text{ and}\\ &x(t)^\top A_j^\top(P_\ell - P_i)A_j x(t)\geq 0 \text{ for all } \ell\in\bar N\setminus\{i\}.\end{aligned} \tag{15}$$

Moreover, note that for the case $\sigma(t) = j$ and $\sigma(t+1) = i$ it holds that

$$V(x(t+1)) - (1-\delta)V(x(t)) = x^\top(t)(A_j^\top P_i A_j - (1-\delta)P_j)x(t) \tag{16}$$

The fact that (15) implies $V(x(t+1)) - (1-\delta)V(x(t))\leq 0$ follows now from (16), (14), (13c) and (13d) by application of the S-procedure formulation as in Proposition 3, see also [5]. By noting that the function V as in (5) satisfies for some $c_1, c_2 > 0$ that $c_1\|\xi\|^2\leq V(\xi)\leq c_2\|\xi\|^2$ for all $\xi\in\mathbb{R}^n$ due to (13b) and combining this with standard Lyapunov arguments, global exponential stability of (2)–(3) follows. ■

Note that there are N positive definite matrices and $N(N-1)$ free scalar quantities (in matrix $\Pi\in\mathcal{M}_d^N$) involved in a solution to the L-M inequalities (11), while a solution to the S-procedure characterization (13) contains N positive definite matrices and $2N^2(N-1)$ scalar quantities. Unlike the case of continuous-time SLSs, there are now more scalar quantities involved in the S-procedure characterization (13) than in the L-M inequalities (11). This is due to the S-procedure characterization using $N^2(N-1)$

scalar quantities $\{\alpha_k^{j\to i}\}_{i,j,k\in\bar{N},k\neq j}$ corresponding to the regional conditions $x(t)^\top P_k x(t) \geq x(t)^\top P_j x(t)$ when $\sigma(t) = j$, and another $N^2(N-1)$ scalar quantities $\{\beta_\ell^{j\to i}\}_{i,j,\ell\in\bar{N},\ell\neq i}$ corresponding to regional conditions $x(t+1)^\top P_\ell x(t+1) \geq x(t+1)^\top P_i x(t+1)$ when $\sigma(t+1) = i$, see also the proof of Theorem 9. However, although the S-procedure characterization has more free parameters compared to the L-M inequalities, it also has more constraints. Indeed, the S-procedure characterization has $N^2 + N$ matrix inequalities (of size $n \times n$) and $2N^2(N-1)$ scalar inequalities, while the L-M inequalities have $N + 1$ matrix inequalities (of size $n \times n$) and N^2 scalar inequalities.

C. Connections

Theorem 10: Consider the SLS (2). The following statements are equivalent:

(i) There is a solution to the L-M inequalities (11).

(ii) There is a solution to the S-procedure characterization (13) satisfying the following for all $i, j \in \bar{N}$:

a) $\alpha_k^{j\to i} = 0$ when $k \neq j$,

b) $\sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i} \leq 1$,

c) $\beta_\ell^{j\to i} = \beta_\ell^{j\to r}$ for $\ell \in \bar{N} \setminus \{i, r\}$ and $r \in \bar{N}$.

Proof: $(\mathbf{i}) \Rightarrow (\mathbf{ii})$: Pick an arbitrary $i \in \bar{N}$. Given that there is a solution to (11), we have from (11a)

$$A_j^\top \left(\sum_{\ell\in\bar{N}} \pi_{\ell j} P_\ell\right) A_j - P_j + A_j^\top P_i A_j - A_j^\top P_i A_j \prec 0 \ \text{ for all } \ j \in \bar{N}. \tag{17}$$

The left-hand side of the above inequality is equal to

$$A_j^\top \left(\sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \pi_{\ell j} P_\ell - (1 - \pi_{ij}) P_i\right) A_j - P_j + A_j^\top P_i A_j.$$

By the properties of Π, the above expression is equal to

$$A_j^\top \left(\sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \pi_{\ell j} (P_\ell - P_i)\right) A_j - P_j + A_j^\top P_i A_j.$$

Consequently, (17) can be written as

$$A_j^\top P_i A_j - P_j \prec A_j^\top \left(\sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \pi_{\ell j} (P_i - P_\ell)\right) A_j \ \text{ for all } \ j \in \bar{N}.$$

Recall that $i \in \bar{N}$ was chosen arbitrarily. We therefore conclude that (13) holds with $\alpha_k^{j\to i} = 0$ and $\beta_\ell^{j\to i} = \pi_{\ell j}$ for all $i, j, k, \ell \in \bar{N}$, $k \neq j$, $\ell \neq i$. Clearly, $\sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i} = \sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \pi_{\ell j} = 1 - \pi_{ij} \leq 1$, which gives property (b). Moreover, due to the choice of $\beta_\ell^{j\to i} = \pi_{\ell j}$ for all $i, j, \ell \in \bar{N}$, $\ell \neq i$ also (c) holds.

$(\mathbf{ii}) \Rightarrow (\mathbf{i})$: Given that (13) has a solution with (a), (b) and (c) satisfied for all $i, j \in \bar{N}$, we have

$$P_j \succ A_j^\top \left(P_i + \sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i} (P_\ell - P_i)\right) A_j \ \text{ for all } i, j \in \bar{N}. \tag{18}$$

The right-hand side of the above inequality can be written as

$$A_j^\top \left(\Big(1 - \sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i}\Big) P_i + \sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i} P_\ell\right) A_j.$$

By the hypothesis that $\sum_{\ell\in\bar{N},\ell\neq i} \beta_\ell^{j\to i} \leq 1$ and defining $\beta_i^{j\to i} := 1 - \sum_{\ell\in\bar{N},\ell\neq i} \beta_\ell^{j\to i} \geq 0$, the above quantity is

$$A_j^\top \left(\sum_{\ell\in\bar{N}} \beta_\ell^{j\to i} P_\ell\right) A_j.$$

Substituting this in (18), we obtain for all $i, j \in \bar{N}$ that

$$A_j^\top \left(\sum_{\ell\in\bar{N}} \beta_\ell^{j\to i} P_\ell\right) A_j - P_j \prec 0.$$

By taking now $i = 1$, we obtain (11a) with $\pi_{\ell j} = \beta_\ell^{j\to 1}$ for all $j, \ell \in \bar{N}$. Obviously, the resulting Π satisfies $\Pi \in \mathcal{M}_d^N$. This completes the proof. ■

In Theorem 10 we showed that in the context of synthesising min-switching laws (3) stabilising a discrete-time SLS (2), the existence of a solution to the L-M inequalities (11) is equivalent to the existence of a solution to the S-procedure characterization (13) that satisfies $\alpha_k^{j\to i} = 0$ for all $k \in \bar{N}$, $k \neq j$ (and further conditions on the scalars as formulated in (b) and (c) in point (ii) of Theorem 10). Hence, it is clear that the class of SLSs that lead to solutions to the L-M inequalities (11) is included in the class of SLSs that lead to solutions for the S-procedure characterization (13). In fact, it is interesting to observe that the S-procedure relaxation based on the knowledge $x(t)^\top P_k x(t) \geq x(t)^\top P_j x(t)$ when $\sigma(t) = j$ is *not* exploited in the L-M inequalities, even though this is a natural relaxation used in the stability analysis of discrete-time PWL systems, see [5] and the proof of Theorem 10. Only the S-procedure relaxations based on $x(t)^\top A_j^\top P_\ell A_j x(t) \geq x(t)^\top A_j^\top P_i A_j x(t)$, $\ell \in \bar{N}$, and related to $\sigma(t+1) = i$, are partially[2] embedded in the L-M inequalities. Hence, the matrix inequalities (13) in the S-procedure characterization exploit all natural relaxations as also used in [5] for analysing the stability of a PWL system (in our setting given by (12)), while the L-M inequalities (11) only exploit a strict subset of them, which is rather remarkable.

Remark 11: In the current setup of the L-M inequalities there is a direct coupling between the switching law (3) and the adopted Lyapunov function (5), which are both parameterised through the *same* set of $\{P_1, P_2, \ldots, P_N\}$. For comparison reasons we adopted the same Lyapunov function in the S-procedure characterization (13). However, it is possible to further relax the S-procedure characterization in the discrete-time case by using an *arbitrary* piecewise quadratic Lyapunov function of the form $V(x) = x^\top Q_{\sigma(t)} x$ for a set of positive definite matrices $\{Q_1, Q_2, \ldots, Q_N\} \subset \mathbb{R}^{n\times n}$ instead of the Lyapunov function in (5). In fact, in this case we can even relax the positive definiteness requirement on the matrices P_i, Q_i, $i \in \bar{N}$. This would lead to the following conditions that guarantee the global exponential stability of (2) and (3):

There exist two sets of matrices $\{P_1, P_2, \ldots, P_N\} \subset \mathbb{R}^{n\times n}$ and $\{Q_1, Q_2, \ldots, Q_N\} \subset \mathbb{R}^{n\times n}$ and three sets $\{\alpha_k^{j\to i}\}_{i,j,k\in\bar{N},k\neq j}$, $\{\beta_\ell^{j\to i}\}_{i,j,\ell\in\bar{N},\ell\neq i}$, $\{\gamma_k^j\}_{k,j\in\bar{N},k\neq j}$ of scalars such that

$$A_j^\top Q_i A_j - Q_j \prec \sum_{\substack{k\in\bar{N}\\ k\neq j}} \alpha_k^{j\to i} (P_j - P_k) + \sum_{\substack{\ell\in\bar{N}\\ \ell\neq i}} \beta_\ell^{j\to i} A_j^\top \big(P_i - P_\ell\big) A_j \quad \text{for all } i, j \in \bar{N} \tag{19a}$$

[2] *Partial* embedding is meant in the sense that still additional limitations as formulated in (b) and (c) in point (ii) of Theorem 10 are imposed on the scalars, while there is no need for that in the application of the S-procedure.

$$Q_j \succ \sum_{\substack{k \in \bar{N} \\ k \neq j}} \gamma_k^j (P_k - P_j) \text{ for all } j \in \bar{N} \tag{19b}$$

$$\alpha_k^{j \to i} \geq 0 \text{ for all } i, j, k \in \bar{N}, k \neq j \tag{19c}$$

$$\beta_\ell^{j \to i} \geq 0 \text{ for all } i, j, \ell \in \bar{N}, \ell \neq i \tag{19d}$$

$$\gamma_k^j \geq 0 \text{ for all } j, k \in \bar{N}, j \neq k \tag{19e}$$

are satisfied.

Note that in the continuous-time case this relaxation is less useful as the *continuity* requirement on the Lyapunov function (typically required for guaranteeing stability, see also [12]) leads to an intimate relationship between $\{P_1, P_2, \ldots, P_N\}$ and $\{Q_1, Q_2, \ldots, Q_N\}$.

Remark 12: In the recent works [6], [7] also the stabilizability of discrete-time switched linear systems was studied and various novel conditions were presented and analysed. Interestingly, some of the novel conditions were obtained as extensions of the classical L-M inequalities as formulated in (11) (see Section III-B in [7]). Interestingly, for these extended L-M inequalities, which were shown to be equivalent to, amongst others, periodic stabilizability in [6], [7], we could provide similar S-procedure characterizations as in (13) along the lines as discussed in Theorem 9 and the proceeding discussion.

IV. Conclusion

In this note we compared the Lyapunov-Metzler (L-M) inequalities and the S-procedure characterizations for the existence and computation of min-switching signals that stabilise switched linear systems (SLSs) in both the continuous-time and discrete-time setting. In the continuous-time case we established that the existence of solutions to the L-M inequalities is equivalent to the existence of solutions to the S-procedure characterization. Hence, this shows that L-M inequalities conceived in [9] can be directly coupled to older stability guarantees for piecewise linear systems as obtained in, e.g., [12]. For the discrete-time case we showed that solutions to the L-M inequalities can be converted into solutions to a *special* version of the S-procedure characterization. In particular, it is of interest to observe that a natural S-procedure relaxation is not used in the corresponding L-M inequalities (next to additional requirements on the scalar variables for the other relaxations). Such restrictions are not present in the S-procedure characterization, although at present it is an interesting open problem whether the inclusion of this additional relaxation leads to a strictly larger class of SLSs that can be stabilised via min-switching signals.

To the best of the authors' knowledge this is the first time that these two celebrated tools for the design of stabilising switching laws for SLSs are explicitly related to each other. Given the wide use of these tools we believe that these results are useful as they shed some light on their relationships and their potential conservatism.

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References

[1] M. Aizerman and F. Gantmacher, *Absolute Stability of Regulator Systems, Holden-Day Series in Information Systems, Holden-Day*, San Francisco. Originally published as Absolutnaya Ustoichivost' Reguliruyemykh Sistem by The Academy of Sciences of the USSR, Moscow, Russia, 1963, 1964.
[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of SIAM, Philadelphia, PA, 1994.
[3] L. El Ghaoui and S.-I. Niculescu (editors), *Advances in Matrix Inequality Methods in Control. Advances in Design and Control* SIAM, Philadelphia, PA, 2000.
[4] E. Feron, *Nonconvex Quadratic Programming, Semidefinite Relaxations and Randomization Algorithms in Information and Decision Systems, in Advances in Systems Theory*, Kluwer Academic Publishers Norwell, MA, 1999, pp. 255–274.
[5] G. Ferrari-Trecate, F. A. Cuzzola, D. Mignone, and M. Morari, "Analysis of discrete-time piecewise affine and hybrid systems," *Automatica*, vol. 38 (2002), pp. 2139–2146.
[6] M. Fiacchini, A. Girard, and M. Jungers, "On stabilizability conditions for discrete-time switched linear systems," in *Proc. IEEE Conf. Decision Control (CDC)*, 2014, pp. 5469–5474.
[7] M. Fiacchini, A. Girard, and M. Jungers, "On the stabilizability of discrete-time switched linear systems: Novel conditions and comparisons," *IEEE Trans. Autom. Control*, vol. 61 (2016), pp. 1181–1193.
[8] A. Filippov, *Differential Equations With Discontinuous Righthand Sides*, Mathematics and its Applications, Kluwer, Amsterdam, The Netherlands, 1988.
[9] J. C. Geromel and P. Colaneri, "Stability and stabilization of continuous-time switched linear systems," *SIAM J. Control Optim.*, vol. 45 (2006), pp. 1915–1930.
[10] J. C. Geromel and P. Colaneri, "Stability and stabilization of discrete time switched systems," *Int. J. Control*, vol. 79 (2006), pp. 719–728.
[11] W. Heemels and S. Weiland, "Input-to-state stability and interconnections of discontinuous dynamical systems," *Automatica*, vol. 44 (2008), pp. 3079–3086.
[12] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Autom. Control*, vol. 43 (1998), pp. 555–559.
[13] D. Liberzon, *Switching in Systems and Control, Systems & Control: Foundations & Applications*, Birkhäuser Boston Inc., Boston, MA, 2003.
[14] A. I. Lur'e and V. N. Postnikov, *On the Theory of Stability of Control Systems*, Applied Mathematics and Mechanics, in Russian, vol. 8 (1944).
[15] I. Polik and T. Terlaky, "A Survey of the S-lemma," *SIAM Rev.*, vol. 49 (2007), pp. 371–418.
[16] V. A. Yakubovich, *The S-Procedure in Nonlinear Control Theory*, Vestnik Leningrad University, in Russian, (1971), pp. 62–77. English translation in Vestnik Leningrad Univ. 4: 73–93, 1977.