Switched Systems:
Stability Analysis and Control Synthesis

Lecture Notes for HYCON-EECI Graduate School on Control

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These lecture notes are based on my book [1]. Please see that book for more information, including references to the literature.

Introduction: hybrid & switched systems and their solutions

1 Classes of hybrid and switched systems

Dynamical systems that are described by an interaction between continuous and discrete dynamics are usually called *hybrid systems*. Continuous dynamics may be represented by a continuous-time control system, such as a linear system $\dot{x} = Ax + Bu$ with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$. As an example of discrete dynamics, one can consider a finite-state automaton, with state q taking values in some finite set $\mathcal{Q}$, where transitions between different discrete states are triggered by suitable values of an input variable v. When the input u to the continuous dynamics is some function of the discrete state q and, similarly, the value of the input v to the discrete dynamics is determined by the value of the continuous state x, a hybrid system arises (see Figure 1).

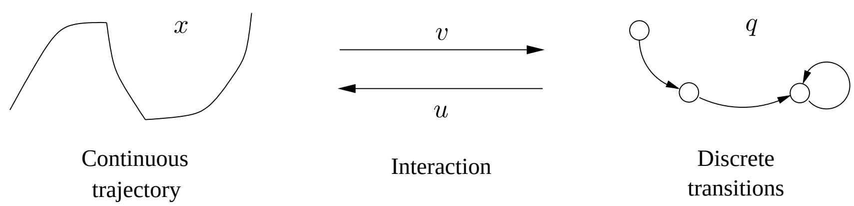

Fig. 1: A hybrid system

Traditionally, control theory has focused either on continuous or on discrete behavior. However, many (if not most) of the dynamical systems encountered in practice are of hybrid nature. The following example is borrowed from [2].

Example 1.1 A very simple model that describes the motion of an automobile might take the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(a, q)$$

where x_1 is the position, x_2 is the velocity, $a \geq 0$ is the acceleration input, and $q \in \{1, 2, 3, 4, 5, -1, 0\}$ is the gear shift position. The function f should be negative and decreasing in a when $q = -1$, negative and independent of a when $q = 0$, and increasing in a, positive for sufficiently large a, and decreasing in q when $q > 0$. In this system, x_1 and x_2 are the continuous states and q is the discrete state. Clearly, the discrete transitions affect the continuous trajectory. In the case of an automatic transmission, the evolution of the continuous state x_2 is in turn used to determine the discrete transitions. In the case of a manual transmission, the discrete transitions are controlled by the driver. It is also natural to consider output variables that depend on both the continuous and the discrete states, such as the engine rotation rate (rpm) which is a function of x_2 and q. □

The field of hybrid systems has a strong interdisciplinary flavor, and different communities have developed different viewpoints. One approach, favored by researchers in computer science, is to concentrate on studying the discrete behavior of the system, while the continuous dynamics are assumed to take a relatively simple form. Basic issues in this context include well-posedness, simulation, and verification. Many researchers in systems and control theory, on the other hand, tend to regard hybrid systems as continuous systems with switching and place a greater emphasis on properties of the continuous state. The main issues then become stability analysis and control synthesis. It is the latter point of view that prevails in these notes.

Thus we are interested in continuous-time systems with (isolated) discrete switching events. We refer to such systems as *switched systems*. A switched system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. This represents a significant departure from hybrid systems, especially at the analysis stage. In switching control design, specifics of the switching mechanism are of greater importance, although typically we will still characterize and exploit only essential properties of the discrete behavior. Having remarked for the purpose of motivation that switched systems can arise from hybrid systems, we henceforth choose switched systems as our focus of study and will generally make no explicit reference to the above connection.

Rather than give a universal formal definition of a switched system, we want to describe several specific categories of systems which will be our main objects of interest. Switching events in switched systems can be classified into

- *State-dependent* versus *time-dependent*;
- *Autonomous (uncontrolled)* versus *controlled*.

Of course, one can have combinations of several types of switching. We now briefly discuss all these possibilities.

1.1 State-dependent switching

Suppose that the continuous state space (e.g., $\mathbb{R}^n$) is partitioned into a finite or infinite number of *operating regions* by means of a family of *switching surfaces*, or *guards*. In each of these regions, a continuous-time dynamical system (described by differential equations, with or without controls) is given. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a *reset map*. In the simplest case, this is a map whose domain is the union of the switching surfaces and whose range is the entire state space, possibly excluding the switching surfaces (more general reset maps can also be considered, as explained below). In summary, the system is specified by

- The family of switching surfaces and the resulting operating regions;
- The family of continuous-time subsystems, one for each operating region;
- The reset map.

In Figure 2, the thick curves denote the switching surfaces, the thin curves with arrows denote the continuous portions of the trajectory, and the dashed lines symbolize the jumps. The instantaneous jumps of the continuous state are sometimes referred to as *impulse effects*. A special case is when such impulse effects are absent, i.e., the reset map is the identity. This means that the state trajectory is continuous everywhere, although it in general loses differentiability when it passes through a switching surface. In most of what follows, we restrict our attention to systems with no impulse effects. However, many of the results and techniques that we will discuss do generalize to systems with impulse effects. Another issue that we are ignoring for the moment is the possibility that some trajectories may "get stuck" on switching surfaces (cf. Section 2.3 below).

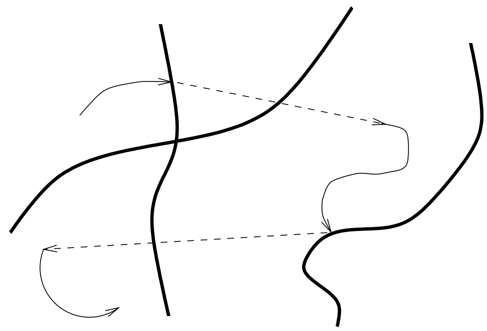

Fig. 2: State-dependent switching

One may argue that the switched system model outlined above (state-dependent switching with no state jumps) is not really hybrid, because even though we can think of the set of operating regions as the discrete state space of the system, this is simply a discontinuous system whose description does not involve discrete dynamics. In other words, its evolution is uniquely determined by the continuous state. The system becomes truly hybrid if the discrete transitions explicitly depend on the value of the discrete state (i.e., the direction from which a switching surface is approached). More complicated state-dependent switching rules are also possible. For example, the operating regions may overlap, and a switching surface may be recognized by the system only in some discrete states. One paradigm that leads to this type of behavior is hysteresis switching, discussed later (see Section 2.4).

1.2 Time-dependent switching

Suppose that we are given a family $f_p,\ p \in \mathcal{P}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, where $\mathcal{P}$ is some index set (typically, $\mathcal{P}$ is a subset of a finite-dimensional linear vector space). This gives rise to a family of systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{1}$$

evolving on $\mathbb{R}^n$. The functions f_p are assumed to be sufficiently regular (at least locally Lipschitz; see Section 2.1 below). The easiest case to think about is when all these systems are linear:

$$f_p(x) = A_p x, \qquad A_p \in \mathbb{R}^{n\times n},\ p \in \mathcal{P} \tag{2}$$

and the index set $\mathcal{P}$ is finite: $\mathcal{P} = \{1, 2, \dots, m\}$.

To define a switched system generated by the above family, we need the notion of a *switching signal*. This is a piecewise constant function $\sigma : [0, \infty) \to \mathcal{P}$. Such a function σ has a finite number of discontinuities—which we call the *switching times*—on every bounded time interval and takes a constant value on every interval between two consecutive switching times. The role of σ is to specify, at each time instant t, the index $\sigma(t) \in \mathcal{P}$ of the *active subsystem*, i.e., the system from the family (1) that is currently being followed. We assume for concreteness that σ is continuous from the right everywhere: $\sigma(t) = \lim_{\tau\to t^+} \sigma(\tau)$ for each $\tau \geq 0$. An example of such a switching signal for the case $\mathcal{P} = \{1, 2\}$ is depicted in Figure 3.

Thus a switched system with time-dependent switching can be described by the equation

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

A particular case is a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)} x(t)$$

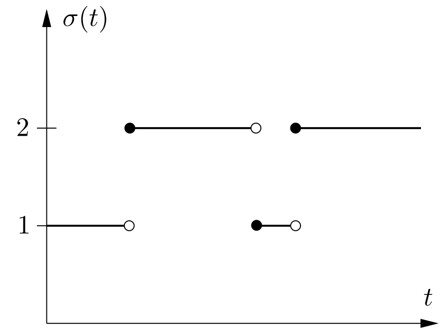

Fig. 3: A switching signal

which arises when all individual subsystems are linear, as in (2). To simplify the notation, we will often omit the time arguments and write

$$\dot{x} = f_\sigma(x) \tag{3}$$

and

$$\dot{x} = A_\sigma x \tag{4}$$

respectively.

Note that it is actually difficult to make a formal distinction between state-dependent and time-dependent switching. If the elements of the index set $\mathcal{P}$ from (1) are in 1-to-1 correspondence with the operating regions discussed in Section 1.1, and if the systems in these regions are those appearing in (1), then every possible trajectory of the system with state-dependent switching is also a solution of the system with time-dependent switching given by (3) for a suitably defined switching signal (but not vice versa). In view of this observation, the latter system can be regarded as a coarser model for the former, which can be used, for example, when the locations of the switching surfaces are unknown. This underscores the importance of developing analysis tools for switched systems like (3).

1.3 Autonomous and controlled switching

By autonomous switching, we mean a situation where we have no direct control over the switching mechanism that triggers the discrete events. This category includes systems with state-dependent switching in which locations of the switching surfaces are predetermined, as well as systems with time-dependent switching in which the rule that defines the switching signal is unknown (or was ignored at the modeling stage). For example, abrupt changes in system dynamics may be caused by unpredictable environmental factors or component failures.

In contrast with the above, in many situations the switching is actually imposed by the designer in order to achieve a desired behavior of the system. In this case, we have direct control over the switching mechanism (which can be state-dependent or time-dependent) and may adjust it as the system evolves. For various reasons, it may be natural to apply discrete control actions, which leads to systems with controlled switching. An important example, which provides motivation and can serve as a unifying framework for studying systems with controlled switching, is that of an *embedded system*, in which computer software interacts with physical devices (see Figure 4).

It is not easy to draw a precise distinction between autonomous and controlled switching, or between state-dependent or time-dependent switching. In a given system, these different types of switching may coexist. For example, if the given process is prone to unpredictable environmental influences or component failures (autonomous switching), then it may be necessary to consider logic-based mechanisms for detecting such events and providing fault-correcting actions (controlled switching).

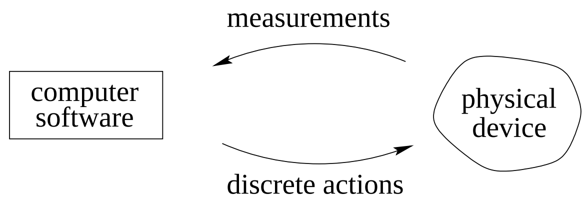

Fig. 4: A computer-controlled system

In the context of the automobile model discussed in Example 1.1, automatic transmission corresponds to autonomous state-dependent switching, whereas manual transmission corresponds to switching being controlled by the driver. In the latter case, state-dependent switching (shifting gears when reaching a certain value of the velocity or rpm) typically makes more sense than time-dependent switching. An exception is parallel parking, which may involve time-periodic switching patterns.

Switched systems with controlled time-dependent switching can be described in a language that is more standard in control theory. Assume that $\mathcal{P}$ is a finite set, say, $\mathcal{P} = \{1, 2, \ldots, m\}$. Then the switched system (3) can be recast as

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \tag{5}$$

where the admissible controls are of the form $u_k = 1$, $u_i = 0$ for all $i \neq k$ (this corresponds to $\sigma = k$). In particular, the switched linear system (4) gives rise to the bilinear system

$$\dot{x} = \sum_{i=1}^{m} A_i x u_i.$$

2 Solutions of switched systems

This section touches upon a few delicate issues that arise in defining solutions of switched systems. In what follows, these issues will mostly be avoided. We begin with some remarks on existence and uniqueness of solutions for systems described by ordinary differential equations.

2.1 Ordinary differential equations

Consider the system

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n. \tag{6}$$

We are looking for a solution $x(\cdot)$ of this system for given initial time t_0 and initial state $x(t_0) = x_0$. It is common to assume that the function f is continuous in t and locally Lipschitz in x uniformly over t. The second condition[1] means that for every pair (t_0, x_0) there exists a constant $L > 0$ such that the inequality

$$|f(t, x) - f(t, y)| \leq L|x - y| \tag{7}$$

holds for all (t, x) and (t, y) in some neighborhood of (t_0, x_0) in $[t_0, \infty) \times \mathbb{R}^n$. (Here and below, we denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^n$.) Under these assumptions, it is well known that the system (6) has a unique solution for every initial condition (t_0, x_0). This solution is defined on some maximal time interval $[t_0, T_{\max})$.

[1] In the time-invariant case, this reduces to the standard Lipschitz condition for a function $f : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.2 To understand why the local Lipschitz condition is necessary, consider the scalar time-invariant system

$$\dot{x} = \sqrt{x}, \qquad x_0 = 0. \tag{8}$$

The functions $x(t) \equiv 0$ and $x(t) = t^2/4$ both satisfy the differential equation (8) and the initial condition $x(0) = 0$. The uniqueness property fails here because the function $f(x) = \sqrt{x}$ is not locally Lipschitz at zero. This fact can be interpreted as follows: due to the rapid growth of $\sqrt{x}$ at zero, it is possible to "break away" from the zero equilibrium. Put differently, there exists a nonzero solution of (8) which, propagated backward in time, reaches the zero equilibrium in finite time. □

The maximal interval $[t_0, T_{\max})$ of existence of a solution may fail to be the entire semi-axis $[t_0, \infty)$. The next example illustrates that a solution may "escape to infinity in finite time." (This will not happen, however, if f is globally Lipschitz in x uniformly in t, i.e., if the Lipschitz condition (7) holds with some Lipschitz constant L for all $x, y \in \mathbb{R}^n$ and all $t \geq t_0$.)

Example 1.3 Consider the scalar time-invariant system

$$\dot{x} = x^2, \qquad x_0 > 0.$$

It is easy to verify that the (unique) solution satisfying $x(0) = x_0$ is given by the formula

$$x(t) = \frac{x_0}{1 - x_0 t}$$

and is only defined on the finite time interval $[0, 1/x_0)$. This is due to the rapid nonlinear growth at infinity of the function $f(x) = x^2$. □

Let us go back to the general situation described by the system (6). Since our view is toward systems with switching, the assumption that the function f is continuous in t is too restrictive. It turns out that for the existence and uniqueness result to hold, it is sufficient to demand that f be piecewise continuous in t. In this case one needs to work with a weaker concept of solution, namely, a continuous function $x(\cdot)$ that satisfies the corresponding integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau.$$

A function with these properties is piecewise differentiable and satisfies the differential equation (6) almost everywhere. Such functions are known as *absolutely continuous* and provide solutions of (6) *in the sense of Carathéodory*. Solutions of the switched system (3) will be interpreted in this way.

2.2 Zeno behavior

We now illustrate, with the help of the bouncing ball example, a peculiar type of behavior that can occur in switched systems.

Example 1.4 Consider a ball bouncing on the floor. Denote by h its height above the floor and by v its velocity (taking the positive velocity direction to be upwards). Normalizing the gravitational constant, we obtain the following equations of motion, valid between the impact times:

$$\begin{aligned} \dot{h} &= v \\ \dot{v} &= -1. \end{aligned} \tag{9}$$

At the time of an impact, i.e., when the ball hits the floor, its velocity changes according to the rule

$$v(t) = -rv(t^-) \tag{10}$$

where $v(t^-)$ is the ball's velocity right before the impact, $v(t)$ is the velocity right after the impact, and $r \in (0,1)$ is the restitution coefficient. This model can be viewed as a state-dependent switched system with impulse effects. Switching events (impacts) are triggered by the condition $h = 0$. They cause instantaneous jumps in the value of the velocity v which is one of the two continuous state variables. Since the continuous dynamics are always the same, all trajectories belong to the same operating region $\{(h,v) : h \geq 0,\, v \in \mathbb{R}\}$.

Integration of (9) gives

$$\begin{aligned} v(t) &= -(t-t_0) + v(t_0) \\ h(t) &= -\frac{(t-t_0)^2}{2} + v(t_0)(t-t_0) + h(t_0). \end{aligned} \tag{11}$$

Let the initial conditions be $t_0 = 0$, $h(0) = 0$, and $v(0) = 1$. By (11), until the first switching time we have

$$\begin{aligned} v(t) &= -t + 1 \\ h(t) &= -\frac{t^2}{2} + t. \end{aligned}$$

The first switch occurs at $t = 2$ since $h(2) = 0$. We have $v(2^-) = -1$, hence $v(2) = r$ in view of (10). Using (11) again with $t_0 = 2$, $h(2) = 0$, and $v(2) = r$, we obtain

$$\begin{aligned} v(t) &= -t + 2 + r \\ h(t) &= -\frac{(t-2)^2}{2} + (t-2)r. \end{aligned}$$

From this it is easy to deduce that the next switch occurs at $t = 2+2r$ and the velocity after this switch is $v(2+2r) = r^2$.

Continuing this analysis, one sees that the switching times form the sequence 2, $2+2r$, $2+2r+2r^2$, $2+2r+2r^2+2r^3$, ... and that the corresponding velocities form the sequence r^2, r^3, r^4, and so on. The interesting conclusion is that the switching times have a finite *accumulation point*, which is the sum of the geometric series

$$\sum_{k=0}^{\infty} 2r^k = \frac{2}{1-r}.$$

At this time the switching events "pile up," i.e., the ball makes infinitely many bounces prior to this time! This is an example of the so-called *Zeno behavior* (see Figure 5).

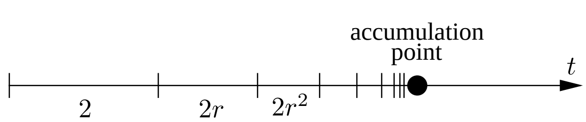

Fig. 5: Zeno behavior

Since both $h(t)$ and $v(t)$ obtained by previous reasoning converge to zero as $t \to \frac{2}{1-r}$, it is natural to extend the solution beyond this time by setting

$$h(t), v(t) := 0, \qquad t \geq \frac{2}{1-r}.$$

Thus the ball stops bouncing, which is a reasonable outcome; of course, in reality this will happen after a finite number of jumps. □

In more complicated hybrid systems, the task of detecting possible Zeno trajectories and extending them beyond their accumulation points is far from trivial. This topic is beyond the scope of these notes. In what follows, we either explicitly rule out Zeno behavior or show that it cannot occur.

2.3 Sliding modes

Consider a switched system with state-dependent switching, described by a single switching surface $\mathcal{S}$ and two subsystems $\dot{x} = f_i(x)$, $i = 1, 2$, one on each side of $\mathcal{S}$. Suppose that there are no impulse effects, so that the state does not jump at the switching events. In Section 1.1 we tacitly assumed that when the continuous trajectory hits $\mathcal{S}$, it crosses over to the other side. This will indeed be true if at the corresponding point $x \in \mathcal{S}$, both vectors $f_1(x)$ and $f_2(x)$ point in the same direction relative to $\mathcal{S}$, as in Figure 6(a); a solution in the sense of Carathéodory is then naturally obtained. However, consider the situation shown in Figure 6(b), where in the vicinity of $\mathcal{S}$ the vector fields f_1 and f_2 both point toward $\mathcal{S}$. In this case, we cannot describe the behavior of the system in the same way as before.

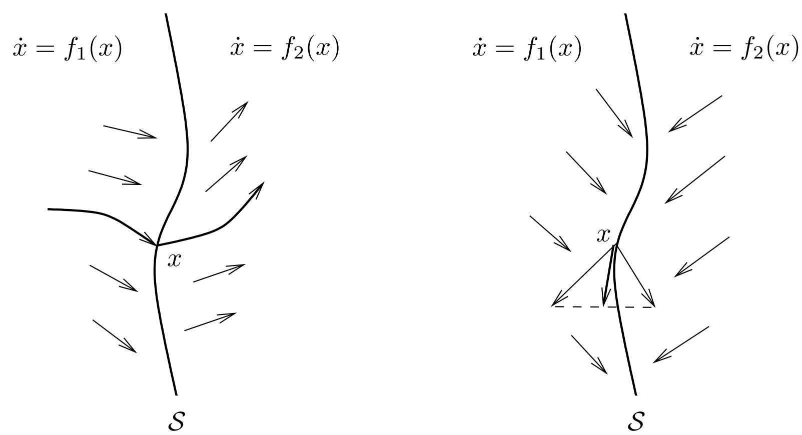

Fig. 6: (a) Crossing a switching surface, (b) a sliding mode

A way to resolve the above difficulty is provided by a different concept of solution, introduced by Filippov to deal precisely with problems of this kind. According to Filippov's definition, one enriches the set of admissible velocities for points $x \in \mathcal{S}$ by including all convex combinations of the vectors $f_1(x)$ and $f_2(x)$. Thus an absolutely continuous function $x(\cdot)$ is a solution of the switched system in the sense of Filippov if it satisfies the *differential inclusion*

$$\dot{x} \in F(x) \tag{12}$$

where F is a multi-valued function defined as follows. For $x \in \mathcal{S}$, we set

$$F(x) = \text{co}\{f_1(x), f_2(x)\} := \{\alpha f_1(x) + (1-\alpha) f_2(x) : \alpha \in [0,1]\}$$

while for $x \notin \mathcal{S}$, we simply set $F(x) = f_1(x)$ or $F(x) = f_2(x)$ depending on which side of S the point x lies on.

It is not hard to see what Filippov solutions look like in the situation shown in Figure 6(b). Once the trajectory hits the switching surface $\mathcal{S}$, it cannot leave it because the vector fields on both sides are pointing toward $\mathcal{S}$. Therefore, the only possible behavior for the solution is to slide on $\mathcal{S}$. We thus obtain what is known as a *sliding mode*. To describe the sliding motion precisely, note that there is a unique convex combination of $f_1(x)$ and $f_2(x)$ that is tangent to $\mathcal{S}$ at the point x. This convex combination determines the instantaneous velocity of the trajectory starting at x; see Figure 6(b). For every $x_0 \in \mathcal{S}$, the resulting solution $x(\cdot)$ is the only absolutely continuous function that satisfies the differential inclusion (12).

From the switched system viewpoint, a sliding mode can be interpreted as infinitely fast switching, or *chattering*. This phenomenon is often undesirable in mathematical models of real systems, because in practice it corresponds to very fast switching which causes excessive equipment wear. On the other hand, we see from the above discussion that a sliding mode yields a behavior that is significantly different from the behavior of each individual subsystem. For this reason, sliding modes are sometimes created on purpose to solve control problems that may be difficult or impossible to solve otherwise.

Example 1.5 Consider the following state-dependent switched linear system in the plane:

$$\dot{x} = \begin{cases} A_1 x & \text{if } \ x_2 \geq x_1 \\ A_2 x & \text{if } \ x_2 < x_1 \end{cases}$$

where

$$A_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

One can show that a sliding motion occurs in the first quadrant if $\lambda < 1$. For $\lambda > -1$ the corresponding trajectory approaches the origin along the switching line (a *stable sliding mode*) while for $\lambda < -1$ it goes away from the origin (an *unstable sliding mode*). □

Exercise 1.1 Prove this.

2.4 Hysteresis switching

We will often be interested in approximating a sliding mode behavior, while avoiding chattering and maintaining the property that two consecutive switching events are always separated by a time interval of positive length. Consider again the system shown in Figure 6(b). Construct two overlapping open regions Ω_1 and Ω_2 by offsetting the switching surface $\mathcal{S}$, as shown in Figure 7(a). In this figure, the original switching surface is shown by a dashed curve, the newly obtained switching surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are shown by the two solid curves, the region Ω_1 is on the left, the region Ω_2 is on the right, and their intersection is the stripe between the new switching surfaces (excluding these surfaces themselves).

We want to follow the subsystem $\dot{x} = f_1(x)$ in the region Ω_1 and the subsystem $\dot{x} = f_2(x)$ in the region Ω_2. Thus switching events occur when the trajectory hits one of the switching surfaces $\mathcal{S}_1$, $\mathcal{S}_2$. This is formalized by introducing a discrete state σ, whose evolution is described as follows. Let $\sigma(0) = 1$ if $x(0) \in \Omega_1$ and $\sigma(0) = 2$ otherwise. For each $t > 0$, if $\sigma(t^-) = i \in \{1, 2\}$ and $x(t) \in \Omega_i$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = 1$ but $x(t) \notin \Omega_1$, let $\sigma(t) = 2$. Similarly, if $\sigma(t^-) = 2$ but $x(t) \notin \Omega_2$, let $\sigma(t) = 1$. Repeating this procedure, we generate a piecewise constant signal σ which is continuous from the right everywhere. Since σ can change its value only after the continuous trajectory has passed through the intersection of Ω_1 and Ω_2, chattering is avoided. A typical solution trajectory is shown in Figure 7(b).

This standard idea, known as *hysteresis switching*, is very useful in control design (we will return to it later). The resulting closed-loop system is a hybrid system, σ being its discrete state. Unlike

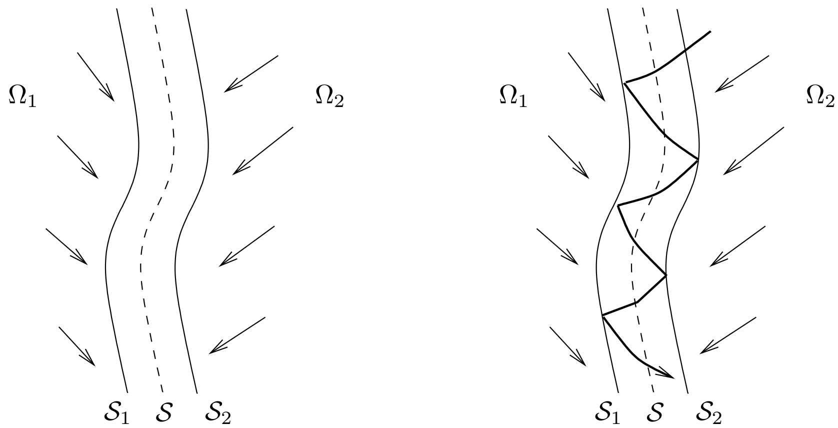

Fig. 7: Hysteresis: (a) switching regions, (b) a typical trajectory

the system with state-dependent switching discussed in Section 1.1, this system is truly hybrid because its discrete part has "memory": the value of σ is not determined by the current value of x alone, but depends also on the previous value of σ. The instantaneous change in x is, in turn, dependent not only on the value of x but also on the value of σ.

Stability of switched systems: motivation and background

We will be investigating stability issues for switched systems of the form (3). For the moment, we concern ourselves with asymptotic stability, although other forms of stability are also of interest. To understand what the basic questions are, consider the situation where $\mathcal{P} = \{1, 2\}$ and $x \in \mathbb{R}^2$, so that we are switching between two systems in the plane. First, suppose that the two individual subsystems are asymptotically stable, with trajectories as shown on the left in Figure 8 (the solid curve and the dotted curve). For different choices of the switching signal, the switched system might be asymptotically stable or unstable (these two possibilities are shown in Figure 8 on the right).

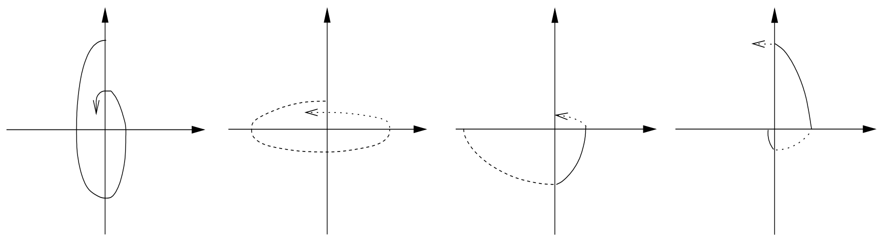

Fig. 8: Switching between stable systems

Similarly, Figure 9 illustrates the case when both individual subsystems are unstable. Again, the switched system may be either asymptotically stable or unstable, depending on a particular switching signal.

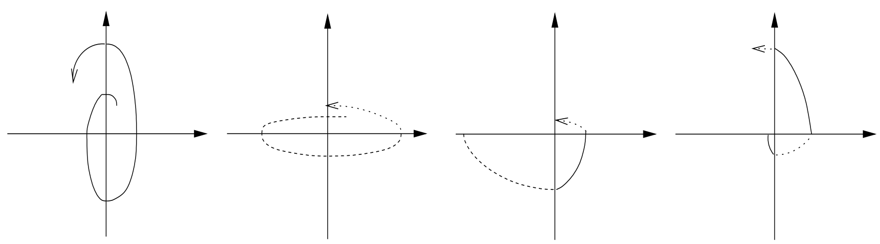

Fig. 9: Switching between unstable systems

From these two examples, the following facts can be deduced:

- Unconstrained switching may destabilize a switched system even if all individual subsystems are stable.[2]
- It may be possible to stabilize a switched system by means of suitably constrained switching even if all individual subsystems are unstable.

Thus we will be studying the following two main problems:

1. Find conditions that guarantee asymptotic stability of a switched system for arbitrary switching signals.
2. If a switched system is not asymptotically stable for arbitrary switching, identify those switching signals for which it is asymptotically stable.

[2] However, there are certain limitations to what types of instability are possible in this case. For example, it is easy to see that trajectories of such a switched system cannot escape to infinity in finite time.

The first problem is relevant when the switching mechanism is either unknown or too complicated to be useful in the stability analysis. When studying the first problem, one is led to investigate possible sources of instability, which in turn provides insight into the more practical second problem.

In the context of the second problem, it is natural to distinguish between two situations. If some or all of the individual subsystems are asymptotically stable, then it is of interest to characterize, as completely as possible, the class of switching signals that preserve asymptotic stability (such switching signals clearly exist; for example, just let $\sigma(t) \equiv p$, where p is the index of some asymptotically stable subsystem). On the other hand, if all individual subsystems are unstable, then the task at hand is to construct at least one stabilizing switching signal, which may actually be quite difficult or even impossible.

The two problems described above are more rigorously formulated and studied below. In what follows, basic familiarity with Lyapunov's stability theory (for general nonlinear systems) is needed, and so we begin by reviewing necessary concepts and results from Lyapunov's stability theory. We restrict our attention to the time-invariant system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{13}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. We also assume that the origin is an (isolated) equilibrium point of the system (13), i.e., $f(0) = 0$, and confine our attention to stability properties of this equilibrium.

3 Stability background

3.1 Stability definitions

Since the system (13) is time-invariant, we let the initial time be $t_0 = 0$ without loss of generality. The origin is said to be a *stable* equilibrium of (13), in the sense of Lyapunov, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that we have

$$|x(0)| \le \delta \quad \Rightarrow \quad |x(t)| \le \varepsilon \qquad \forall\, t \ge 0.$$

In this case we will also simply say that the system (13) is *stable*. A similar convention will apply to other stability concepts introduced below.

The system (13) is called *asymptotically stable* if it is stable and δ can be chosen so that

$$|x(0)| \le \delta \quad \Rightarrow \quad x(t) \to 0 \ \text{ as } \ t \to \infty.$$

The set of all initial states from which the trajectories converge to the origin is called the *region of attraction*. If the above condition holds for all δ, i.e., if the origin is a stable equilibrium and its region of attraction is the entire state space, then the system (13) is called *globally asymptotically stable*.

If the system is not necessarily stable but has the property that all solutions with initial conditions in some neighborhood of the origin converge to the origin, then it is called (locally) *attractive*. We say that the system is *globally attractive* if its solutions converge to the origin from all initial conditions.

The system (13) is called *exponentially stable* if there exist positive constants δ, c, and λ such that all solutions of (13) with $|x(0)| \le \delta$ satisfy the inequality

$$|x(t)| \le c|x(0)|e^{-\lambda t} \qquad \forall\, t \ge 0. \tag{14}$$

If this exponential decay estimate holds for all δ, the system is said to be *globally exponentially stable*. The constant λ in (14) is occasionally referred to as a *stability margin*.

3.2 Function classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of *class* $\mathcal{K}_\infty$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. We will write $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ to indicate that α is a class $\mathcal{K}_\infty$ function and β is a class $\mathcal{KL}$ function, respectively.

As an immediate application of these function classes, we can rewrite the stability definitions of the previous section in a more compact way. Indeed, stability of the system (13) is equivalent to the property that there exist a $\delta > 0$ and a class $\mathcal{K}$ function α such that all solutions with $|x(0)| \leq \delta$ satisfy

$$|x(t)| \leq \alpha(|x(0)|) \qquad \forall\, t \geq 0.$$

Asymptotic stability is equivalent to the existence of a $\delta > 0$ and a class $\mathcal{KL}$ function β such that all solutions with $|x(0)| \leq \delta$ satisfy

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall\, t \geq 0.$$

Global asymptotic stability amounts to the existence of a class $\mathcal{KL}$ function β such that the inequality

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall\, t \geq 0$$

holds for all initial conditions. Exponential stability means that the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some $c, \lambda > 0$.

3.3 Lyapunov's direct (second) method

Consider a $\mathcal{C}^1$ (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$. It is called *positive definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. If $V(x) \to \infty$ as $|x| \to \infty$, then V is said to be *radially unbounded.* If V is both positive definite and radially unbounded, then there exist two class $\mathcal{K}_\infty$ functions α_1, α_2 such that V satisfies

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \forall\, x. \tag{15}$$

We write $\dot{V}$ for the derivative of V along solutions of the system (13), i.e.,

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x).$$

The main result of Lyapunov's stability theory is expressed by the following statement.

Theorem 2.1 (Lyapunov) *Suppose that there exists a positive definite* $\mathcal{C}^1$ *function* $V : \mathbb{R}^n \to \mathbb{R}$ *whose derivative along solutions of the system* (13) *satisfies*

$$\dot{V} \leq 0 \qquad \forall\, x. \tag{16}$$

Then the system (13) *is stable. If the derivative of* V *satisfies*

$$\dot{V} < 0 \qquad \forall\, x \neq 0 \tag{17}$$

then (13) *is asymptotically stable. If in the latter case* V *is also radially unbounded, then* (13) *is globally asymptotically stable.*

We refer to a positive definite $\mathcal{C}^1$ function V as a *weak Lyapunov function* if it satisfies the inequality (16) and a *Lyapunov function* if it satisfies the inequality (17). The conclusions of the theorem remain valid when V is merely continuous and not necessarily $\mathcal{C}^1$, provided that the inequalities (16) and (17) are replaced by the conditions that V is nonincreasing and strictly decreasing along nonzero solutions, respectively (this can be seen from the proof outlined below).

Sketch of proof of Theorem 2.1. First assume that (16) holds. Consider the ball around the origin of a given radius $\varepsilon > 0$. Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$. Denote by δ the radius of some ball around the origin which is inside the set $\{x : V(x) \leq b\}$ (see Figure 10). Since V is nonincreasing along solutions, each solution starting in the smaller ball of radius δ satisfies $V(x(t)) \leq b$, hence it remains inside the bigger ball of radius ε. This proves stability.

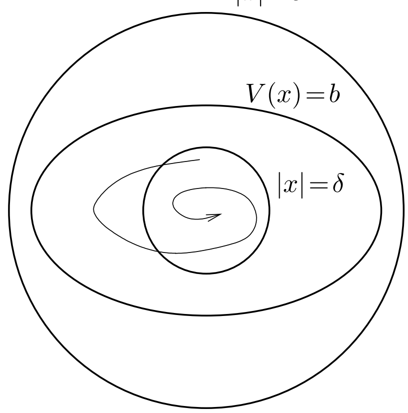

Fig. 10: Proving Lyapunov stability

To prove the second statement of the theorem, take an arbitrary initial condition satisfying $|x(0)| \leq \delta$, where δ is as defined above (for some ε). Since V is positive and decreasing along the corresponding solution, it has a limit $c \geq 0$ as $t \to \infty$. If we can prove that $c = 0$, then we have asymptotic stability (in view of positive definiteness of V and the fact that x stays bounded in norm by ε). Suppose that c is positive. Then the solution cannot enter the set $\{x : V(x) < c\}$. In this case the solution evolves in a compact set that does not contain the origin. For example, we can take this set to be $S := \{x : r \leq |x| \leq \varepsilon\}$ for a sufficiently small $r > 0$. Let $d := \max_{x \in S} \dot{V}(x)$; this number is well defined and negative in view of (17) and compactness of S. We have $\dot{V} \leq d$, hence $V(t) \leq V(0) + dt$. But then V will eventually become smaller than c, which is a contradiction.

The above argument is valid locally around the origin, because the level sets of V may not all be bounded and so δ may stay bounded as we increase ε to infinity. If V is radially unbounded, then all its level sets are bounded. Thus we can have $\delta \to \infty$ as $\varepsilon \to \infty$, and global asymptotic stability follows. □

Exercise 2.1 Assuming that V is radially unbounded and using the functions α_1 and α_2 from the formula (15), write down a possible definition of δ as a function of ε which can be used in the above proof.

Various converse Lyapunov theorems show that the conditions of Theorem 2.1 are also necessary. For example, if the system is asymptotically stable, then there exists a positive definite $\mathcal{C}^1$ function V that satisfies the inequality (17).

Example 2.1 It is well known that for the linear time-invariant system

$$\dot{x} = Ax \tag{18}$$

asymptotic stability, exponential stability, and their global versions are all equivalent and amount to the property that A is a *Hurwitz* matrix, i.e., all eigenvalues of A have negative real parts. Fixing an arbitrary positive definite symmetric matrix Q and finding the unique positive definite symmetric matrix P that satisfies the Lyapunov equation

$$A^T P + PA = -Q$$

one obtains a quadratic Lyapunov function $V(x) = x^T Px$ whose derivative along solutions is $\dot{V} = -x^T Qx$. The explicit formula for P is

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

Indeed, we have

$$A^T P + PA = \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = -Q$$

because A is Hurwitz. □

3.4 LaSalle's invariance principle

With some additional knowledge about the behavior of solutions, it is possible to prove asymptotic stability using a weak Lyapunov function, which satisfies the nonstrict inequality (16). This is facilitated by *LaSalle's invariance principle.*

A set M is called (positively) *invariant* with respect to the given system if all solutions starting in M remain in M for all future times. We now state a version of LaSalle's theorem which is the most useful one for our purposes.

Theorem 2.2 (LaSalle) *Suppose that there exists a positive definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions of the system* (13) *satisfies the inequality* (16)*. Let M be the largest invariant set contained in the set $\{x : \dot{V}(x) = 0\}$. Then the system* (13) *is stable and every solution that remains bounded for $t \geq 0$ approaches M as $t \to \infty$. In particular, if all solutions remain bounded and $M = \{0\}$, then the system* (13) *is globally asymptotically stable.*

To deduce global asymptotic stability with the help of this result, one needs to check two conditions. First, all solutions of the system must be bounded. This property follows automatically from the inequality (16) if V is chosen to be radially unbounded; however, radial unboundedness of V is not necessary when boundedness of solutions can be established by other means.[3] The second condition is that $\dot{V}$ is not identically zero along any nonzero solution. We also remark that if one only wants to prove asymptotic convergence of bounded solutions to zero and is not concerned with Lyapunov stability of the origin, then positive definiteness of V is not needed (this is in contrast with Theorem 2.1).

Example 2.2 Consider the two-dimensional system

$$\ddot{x} + a\dot{x} + f(x) = 0$$

[3] When just local asymptotic stability is of interest, it suffices to note that boundedness of solutions starting sufficiently close to the origin is guaranteed by the first part of Theorem 2.1.

where $a > 0$ and the function f satisfies $f(0) = 0$ and $xf(x) > 0$ for all $x \neq 0$. Systems of this form frequently arise in models of mechanical systems with damping or electrical circuits with nonlinear capacitors or inductors. The equivalent first-order state-space description is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - f(x_1). \end{aligned} \tag{19}$$

Consider the function

$$V(x_1, x_2) := \frac{x_2^2}{2} + F(x_1) \tag{20}$$

where $F(x_1) := \int_0^{x_1} f(x)dx$. Assume that f is such that F is positive definite and radially unbounded (this is true, for example, under the sector condition $k_1 x_1^2 \leq x_1 f(x_1) \leq k_2 x_1^2$, $0 < k_1 < k_2 \leq \infty$). The derivative of the function (20) along solutions of (19) is given by

$$\dot{V} = -ax_2^2 \leq 0.$$

Moreover, $x_2 \equiv 0$ implies that x_1 is constant, and the second equation in (19) then implies that $x_1 \equiv 0$ as well. Therefore, the system (19) is globally asymptotically stable by Theorem 2.2. □

While Lyapunov's stability theorem readily generalizes to time-varying systems, for LaSalle's invariance principle this is not the case. Instead, one usually works with the weaker property that all solutions approach the set $\{x : \dot{V}(x) = 0\}$.

3.5 Lyapunov's indirect (first) method

Lyapunov's indirect method allows one to deduce stability properties of the nonlinear system (13), where f is $\mathcal{C}^1$, from stability properties of its *linearization*, which is the linear system (18) with

$$A := \frac{\partial f}{\partial x}(0). \tag{21}$$

By the mean value theorem, we can write

$$f(x) = Ax + g(x)x$$

where the matrix g is given row-wise by $g_i(x) := \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)$ for some point z_i on the line segment connecting x to the origin, $i = 1, \ldots, n$. Since $\frac{\partial f}{\partial x}$ is continuous, we have $g(x) \to 0$ as $x \to 0$. From this it follows that if the matrix A is Hurwitz (i.e., all its eigenvalues are in the open left half of the complex plane), then a quadratic Lyapunov function for the linearization serves—locally—as a Lyapunov function for the original nonlinear system. Moreover, its rate of decay in a neighborhood of the origin can be bounded from below by a quadratic function, which implies that stability is in fact exponential. This is summarized by the following result.

Theorem 2.3 *If f is $\mathcal{C}^1$ and the matrix* (21) *is Hurwitz, then the system* (13) *is locally exponentially stable.*

It is also known that if the matrix A has at least one eigenvalue with a positive real part, the nonlinear system (13) is not stable. If A has eigenvalues on the imaginary axis but no eigenvalues in the open right half-plane, the linearization test is inconclusive. However, in this *critical* case the system (13) cannot be exponentially stable, since exponential stability of the linearization is not only a sufficient but also a necessary condition for (local) exponential stability of the nonlinear system.

3.6 Input-to-state stability

It is of interest to extend stability concepts to systems with disturbance inputs. In the linear case represented by the system

$$\dot{x} = Ax + Bd$$

it is well known that if the matrix A is Hurwitz, i.e., if the unforced system $\dot{x} = Ax$ is asymptotically stable, then bounded inputs d lead to bounded states while inputs converging to zero produce states converging to zero. Now, consider a nonlinear system of the form

$$\dot{x} = f(x, d) \tag{22}$$

where d is a measurable locally essentially bounded[4] disturbance input. In general, global asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ does not guarantee input-to-state properties of the kind mentioned above. For example, the scalar system

$$\dot{x} = -x + xd \tag{23}$$

has unbounded trajectories under the bounded input $d \equiv 2$. This motivates the following important concept, introduced by Sontag.

The system (22) is called *input-to-state stable* (ISS) with respect to d if for some functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every input d the corresponding solution of (22) satisfies the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|d\|_{[0,t]}) \qquad \forall\, t \ge 0 \tag{24}$$

where $\|d\|_{[0,t]} := \operatorname{ess\,sup}\{|d(s)| : s \in [0, t]\}$ (supremum norm on $[0, t]$ except for a set of measure zero). Since the system (22) is time-invariant, the same property results if we write

$$|x(t)| \le \beta(|x(t_0)|, t - t_0) + \gamma(\|d\|_{[t_0,t]}) \qquad \forall\, t \ge t_0 \ge 0.$$

The ISS property admits the following Lyapunov-like equivalent characterization: the system (22) is ISS if and only if there exists a positive definite radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions α and χ we have

$$\frac{\partial V}{\partial x} f(x, d) \le -\alpha(|x|) + \chi(|d|) \qquad \forall\, x, d.$$

This is in turn equivalent to the following "gain margin" condition:

$$|x| \ge \rho(|d|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, d) \le -\bar{\alpha}(|x|)$$

where $\bar{\alpha}, \rho \in \mathcal{K}_\infty$. Such functions V are called *ISS-Lyapunov functions*.

Exercise 2.2 Prove that if the system (22) is ISS, then $d(t) \to 0$ implies $x(t) \to 0$.

The system (22) is said to be *locally input-to-state stable* (locally ISS) if the bound (24) is valid for solutions with sufficiently small initial conditions and inputs, i.e., if there exists a $\delta > 0$ such that (24) is satisfied whenever $|x(0)| \le \delta$ and $\|u\|_{[0,t]} \le \delta$. It turns out (local) asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ implies local ISS.

[4] The reader not familiar with this terminology may assume that d is piecewise continuous.

GUAS and common Lyapunov functions; commutation relations and stability under arbitrary switching

4 Uniform stability and common Lyapunov functions

4.1 Uniform stability concepts

Given a family of systems (1), we want to study the following question: when is the switched system (3) asymptotically stable for every switching signal σ? We are assuming here that the individual subsystems have the origin as a common equilibrium point: $f_p(0) = 0$ for all $p \in \mathcal{P}$. Clearly, a necessary condition for (asymptotic) stability under arbitrary switching is that all of the individual subsystems are (asymptotically) stable. Indeed, if the pth system is unstable for some $p \in \mathcal{P}$, then the switched system is unstable for $\sigma(t) \equiv p$.

Therefore, throughout this lecture it will be assumed that all individual subsystems are asymptotically stable. Our earlier discussion shows that this condition is not sufficient for asymptotic stability under arbitrary switching. Thus one needs to determine what additional requirements on the systems from (1) must be imposed.

Recalling the equivalence between the switched system (3) for $\mathcal{P} = \{1, 2, \ldots, m\}$ and the control system (5), we see that asymptotic stability of (3) for arbitrary switching corresponds to a lack of controllability of (5). Indeed, it means that for any admissible control input, the resulting solution trajectory must approach the origin.

Instead of just asymptotic stability for each particular switching signal, a somewhat stronger property is desirable, namely, asymptotic or exponential stability that is *uniform* over the set of all switching signals. The relevant stability concepts are the following appropriately modified versions of the standard stability concepts for time-invariant systems.

We will say that the switched system (3) is *uniformly asymptotically stable* if there exist a positive constant δ and a class $\mathcal{KL}$ function β such that for all switching signals σ the solutions of (3) with $|x(0)| \le \delta$ satisfy the inequality

$$|x(t)| \le \beta(|x(0)|, t) \qquad \forall\, t \ge 0. \tag{25}$$

If the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some $c, \lambda > 0$, so that the above inequality takes the form

$$|x(t)| \le c|x(0)|e^{-\lambda t} \qquad \forall\, t \ge 0 \tag{26}$$

then the system (3) is called *uniformly exponentially stable*. If the inequalities (25) and (26) are valid for all switching signals and all initial conditions, we obtain *global uniform asymptotic stability* (GUAS) and *global uniform exponential stability* (GUES), respectively.

Equivalent definitions can be given in terms of ε–δ properties of solutions. The term "uniform" is used here to describe uniformity with respect to switching signals. This is not to be confused with the more common usage which refers to uniformity with respect to the initial time for time-varying systems.

Exercise 3.1 Prove that for the switched linear system (4), GUAS implies GUES. Can you identify a larger class of switched systems for which the same statement holds?

4.2 Common Lyapunov functions

Lyapunov's stability theorem from Section 3.3 has a direct extension which provides a basic tool for studying uniform stability of the switched system (3). This extension is obtained by requiring the existence of a single Lyapunov function whose derivative along solutions of all systems in the family (1) satisfies suitable inequalities. We are particularly interested in obtaining a Lyapunov condition for GUAS. To do this, we must take special care in formulating a counterpart of the inequality (17) which ensures a uniform rate of decay.

Given a positive definite continuously differentiable ($\mathcal{C}^1$) function $V : \mathbb{R}^n \to \mathbb{R}$, we will say that it is a *common Lyapunov function* for the family of systems (1) if there exists a positive definite continuous function $W : \mathbb{R}^n \to \mathbb{R}$ such that we have

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x) \qquad \forall\, x, \quad \forall\, p \in \mathcal{P}. \tag{27}$$

The following result will be used throughout.

Theorem 3.1 *If all systems in the family* (1) *share a radially unbounded common Lyapunov function, then the switched system* (3) *is GUAS.*

Theorem 3.1 is well known and can be derived in the same way as the standard Lyapunov stability theorem (cf. Section 3.3). The main point is that the rate of decrease of V along solutions, given by (27), is not affected by switching, hence asymptotic stability is uniform with respect to σ.

Remark 3.1 In the special case when both V and W are quadratic (or, more generally, are bounded from above and below by monomials of the same degree in $|x|$), it is easy to show that the switched system is GUES. This situation will be discussed in detail later. □

If we replace the inequality (27) by the weaker condition

$$\frac{\partial V}{\partial x} f_p(x) < 0 \qquad \forall\, x \neq 0, \quad \forall\, p \in \mathcal{P} \tag{28}$$

then the result no longer holds. This can be seen from a close examination of the proof of Lyapunov's theorem and also from the following example.

Example 3.1 With reference to (1), let $f_p(x) = -px$ and $\mathcal{P} = (0, 1]$. This gives a family of systems, each of which is globally asymptotically stable and has $V(x) = x^2/2$ as a Lyapunov function. The resulting switched system

$$\dot{x} = -\sigma x$$

has the solutions

$$x(t) = e^{-\int_0^t \sigma(\tau) d\tau} x(0).$$

Thus we see that every switching signal $\sigma \in \mathcal{L}_1$ produces a trajectory that does not converge to zero. This happens because the rate of decay of V along the pth subsystem is given by

$$\frac{\partial V}{\partial x} f_p(x) = -px^2 \tag{29}$$

and this gets smaller for small values of p, so we do not have asymptotic stability if σ goes to zero too fast. Note that V is not a common Lyapunov function according to the above definition, since the right-hand sides in (29) for $0 < p \leq 1$ cannot all be upper-bounded by one negative definite function. □

The property expressed by (28) is sufficient for GUAS if $\mathcal{P}$ is a compact set and f_p depends continuously on p for each fixed x. (Under these conditions we can construct W by taking the maximum of the left-hand side of (28) over p, which is well defined.) This holds trivially if $\mathcal{P}$ is a finite set. For infinite $\mathcal{P}$, such compactness assumptions are usually reasonable and will be imposed in most of what follows.

Note that while we do not have asymptotic stability in the above example, stability in the sense of Lyapunov is always preserved under switching between one-dimensional stable systems. Interesting phenomena such as the one demonstrated by Figure 8 are only possible in dimensions 2 and higher.

Remark 3.2 If $\mathcal{P}$ is not a discrete set, it is also meaningful to consider the time-varying system described by (3) with a piecewise continuous (but not necessarily piecewise constant) signal σ. The existence of a common Lyapunov function implies global uniform asymptotic stability of this more general system; in fact, the same proof remains valid in this case. Although we will not mention it explicitly, many of the results presented below apply to such time-varying systems. □

The continuous differentiability assumption on V can sometimes be relaxed by requiring merely that V be continuous and decrease uniformly along solutions of each system in (1). This amounts to replacing the inequality (27) with its integral version.

4.3 A converse Lyapunov theorem

In the next several lectures, we will be concerned with identifying classes of switched systems that are GUAS. The most common approach to this problem consists of searching for a common Lyapunov function shared by the individual subsystems. The question arises whether the existence of a common Lyapunov function is a more severe requirement than GUAS. A negative answer to this question—and a justification for the common Lyapunov function approach—follows from the converse Lyapunov theorem for switched systems, which says that the GUAS property of a switched system implies the existence of a common Lyapunov function. For such a converse Lyapunov theorem to hold, we need the family of systems (1) to satisfy suitable uniform (with respect to p) boundedness and regularity conditions. It is easy to see—and important to know—that these conditions automatically hold when the index set $\mathcal{P}$ is finite (recall that the functions f_p are always assumed to be locally Lipschitz in x).

Theorem 3.2 *Assume that the switched system* (3) *is GUAS, the set* $\{f_p(x) : p \in \mathcal{P}\}$ *is bounded for each* x*, and the function* $(x, p) \mapsto f_p(x)$ *is locally Lipschitz in* x *uniformly over* p*. Then all systems in the family* (1) *share a radially unbounded smooth common Lyapunov function.*

There is a useful result which we find convenient to state here as a corollary of Theorem 3.2. It says that if the switched system (3) is GUAS, then all "convex combinations" of the individual subsystems from the family (1) must be globally asymptotically stable. These convex combinations are defined by the vector fields

$$f_{p,q,\alpha}(x) := \alpha f_p(x) + (1-\alpha) f_q(x), \qquad p, q \in \mathcal{P}, \quad \alpha \in [0,1].$$

Corollary 3.3 *Under the assumptions of Theorem 3.2, for every* $\alpha \in [0,1]$ *and all* $p, q \in \mathcal{P}$ *the system*

$$\dot{x} = f_{p,q,\alpha}(x) \tag{30}$$

is globally asymptotically stable.

This can be proved by observing that a common Lyapunov function V provided by Theorem 3.2 decreases along solutions of (30). Indeed, from the inequality (27) we easily obtain

$$\frac{\partial V}{\partial x} f_{p,q,\alpha}(x) = \alpha \frac{\partial V}{\partial x} f_p(x) + (1-\alpha)\frac{\partial V}{\partial x} f_q(x) \le -W(x) \qquad \forall\, x. \tag{31}$$

A different justification of Corollary 3.3 comes from the fact that one can mimic the behavior of the convex combination (30) by means of fast switching between the subsystems $\dot{x} = f_p(x)$ and $\dot{x} = f_q(x)$, spending the correct proportion of time (α versus $1-\alpha$) on each one. This can be formalized with the help of the so-called *relaxation theorem* for differential inclusions, which in our context implies that the set of solutions of the switched system (3) is dense in the set of solutions of the "relaxed" switched system generated by the family of systems

$$\{\dot{x} = f_{p,q,\alpha}(x) : p, q \in \mathcal{P},\ \alpha \in [0,1]\}. \tag{32}$$

Therefore, if there exists a convex combination that is not asymptotically stable, then the switched system cannot be GUAS.

Remark 3.3 The formula (31) actually says more, namely, that V is a common Lyapunov function for the enlarged family of systems (32). By Theorem 3.1, the relaxed switched system generated by this family is also GUAS. □

A convex combination of two asymptotically stable vector fields is not necessarily asymptotically stable. As a simple example, consider the two matrices

$$A_1 := \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -0.1 & 2 \\ -1 & -0.1 \end{pmatrix}.$$

These matrices are both Hurwitz, but their average $(A_1 + A_2)/2$ is not. Stability of all convex combinations often serves as an easily checkable necessary condition for GUAS. To see that this condition is not sufficient, consider the two matrices

$$A_1 := \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}.$$

It is easy to check that all convex combinations of these matrices are Hurwitz. Trajectories of the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ look approximately the same as the first two plots in Figure 8 on page 13, and by switching it is possible to obtain unbounded trajectories such as the one shown on the last plot in that figure.

4.4 Switched linear systems

We now discuss how the above notions and results specialize to the switched linear system (4), in which all individual subsystems are linear. First, recall that for a linear time-invariant system $\dot{x} = Ax$, global exponential stability is equivalent to the seemingly weaker property of local attractivity (the latter means that all trajectories starting in some neighborhood of the origin converge to the origin). In fact, the different versions of asymptotic stability all amount to the property that A be a Hurwitz matrix—i.e., the eigenvalues of A lie in the open left half of the complex plane—and are characterized by the existence of a quadratic Lyapunov function

$$V(x) = x^T P x \tag{33}$$

where P is a positive definite symmetric matrix.

Now consider the switched linear system (4). Assume that $\{A_p : p \in \mathcal{P}\}$ is a *compact* (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of Hurwitz matrices. Similarly to the case of a linear system with no switching, the following is true.

Theorem 3.4 *The switched linear system* (4) *is GUES if and only if it is locally attractive for every switching signal.*

The equivalence between local attractivity and global exponential stability is not very surprising. A more interesting finding is that uniformity with respect to σ is automatically guaranteed: it cannot happen that all switching signals produce solutions decaying to zero but the rate of decay can be made arbitrarily small by varying the switching signal. (This is in fact true for switched nonlinear systems that are uniformly Lyapunov stable.) Moreover, we saw earlier that stability properties of the switched linear system do not change if we replace the set $\{A_p : p \in \mathcal{P}\}$ by its convex hull (see Remark 3.3).

For switched linear systems, it is natural to consider *quadratic common Lyapunov functions*, i.e., functions of the form (33) such that for some positive definite symmetric matrix Q we have

$$A_p^T P + P A_p \leq -Q \qquad \forall\, p \in \mathcal{P}. \tag{34}$$

(The inequality $M \leq N$ or $M < N$ for two symmetric matrices M and N means that the matrix $M-N$ is nonpositive definite or negative definite, respectively.) In view of the compactness assumption made earlier, the inequality (34) is equivalent to the simpler one

$$A_p^T P + P A_p < 0 \qquad \forall\, p \in \mathcal{P} \tag{35}$$

(although in general they are different; cf. Example 3.1 in Section 4.2). One reason why quadratic common Lyapunov functions are attractive is that (35) is a system of *linear matrix inequalities* (LMIs) in P, and there are efficient methods for solving finite systems of such inequalities numerically. It is also known how to determine the infeasibility of (35): for $\mathcal{P} = \{1, 2, \dots, m\}$, a quadratic common Lyapunov function does not exist if and only if the equation

$$R_0 = \sum_{i=1}^{m} (A_i R_i + R_i A_i^T) \tag{36}$$

is satisfied by some nonnegative definite symmetric matrices R_0, R_1, $\dots$, R_m which are not all zero.

A natural question to ask is whether it is sufficient to work with quadratic common Lyapunov functions. In other words, is it true that if the switched linear system (4) is GUES and thus all systems in the family

$$\dot{x} = A_p x, \qquad p \in \mathcal{P} \tag{37}$$

share a common Lyapunov function (by virtue of Theorem 3.2), then one can always find a common Lyapunov function that is quadratic? The example given in the next section shows that the answer to this question is negative. However, it is always possible to find a common Lyapunov function that is homogeneous of degree 2, and in particular, one that takes the piecewise quadratic form

$$V(x) = \max_{1\leq i\leq k} (l_i^T x)^2,$$

where l_i, $i = 1, \dots, k$ are constant vectors. Level sets of such a function are given by surfaces of polyhedra, orthogonal to these vectors.

4.5 A counterexample

The following counterexample, taken from [3], demonstrates that even for switched linear systems GUES does not imply the existence of a *quadratic* common Lyapunov function. Take $\mathcal{P} = \{1, 2\}$, and let the two matrices be

$$A_1 := \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}.$$

These matrices are both Hurwitz.

Fact 1. The systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a quadratic common Lyapunov function of the form (33).

Without loss of generality, we can look for a positive definite symmetric matrix P in the form

$$P = \begin{pmatrix} 1 & q \\ q & r \end{pmatrix}$$

which satisfies the inequality (35). We have

$$-A_1^T P - P A_1 = \begin{pmatrix} 2 - 2q & 2q + 1 - r \\ 2q + 1 - r & 2q + 2r \end{pmatrix}$$

and this is positive definite only if

$$q^2 + \frac{(r-3)^2}{8} < 1. \tag{38}$$

(Recall that a symmetric matrix is positive definite if and only if all its leading principal minors are positive.) Similarly,

$$-A_2^T P - P A_2 = \begin{pmatrix} 2 - \dfrac{q}{5} & 2q + 10 - \dfrac{r}{10} \\ 2q + 10 - \dfrac{r}{10} & 20q + 2r \end{pmatrix}$$

is positive definite only if

$$q^2 + \frac{(r-300)^2}{800} < 100. \tag{39}$$

It is straightforward to check that the ellipses whose interiors are given by the formulas (38) and (39) do not intersect (see Figure 11 on the next page). Therefore, a quadratic common Lyapunov function does not exist.

Fact 2. The switched linear system $\dot{x} = A_\sigma x$ is GUES.

This claim can be verified by analyzing the behavior of the system under the “worst-case switching,” which is defined as follows. The vectors $A_1 x$ and $A_2 x$ are collinear on two lines going through the origin (the dashed lines in Figure 12 on the next page). At all other points in $\mathbb{R}^2$, one of the two vectors points outwards relative to the other, i.e., it forms a smaller angle with the exiting radial direction. The worst-case switching strategy consists of following the vector field that points outwards, with switches occurring on the two lines. It turns out that this produces a trajectory converging to the origin, because the distance from the origin after one rotation decreases (see Figure 12 on the next page). The trajectories produced by all other switching signals also converge to the origin, and the worst-case trajectory described above provides a uniform lower bound on the rate of convergence. Thus the system is GUES.

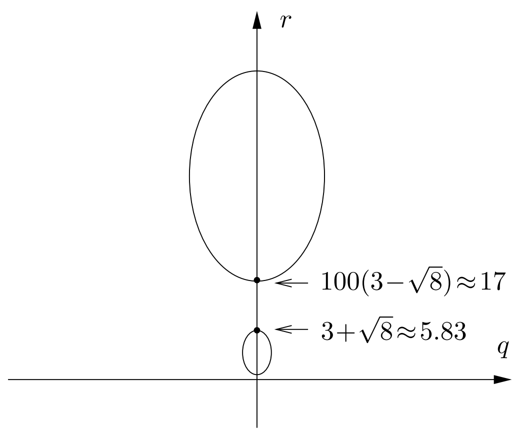

Fig. 11: Ellipses in the counterexample

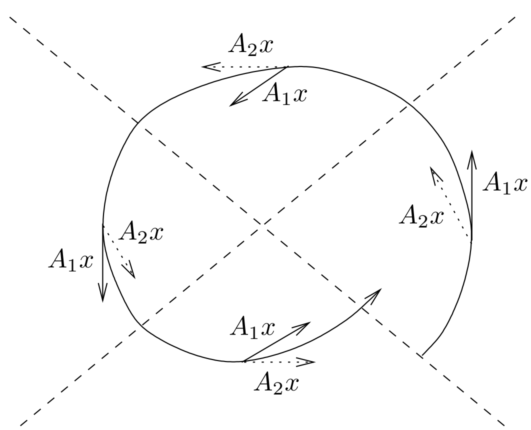

Fig. 12: Worst-case switching in the counterexample

5 Commutation relations and stability

This lecture is based on powerpoint slides, file name: `LA`. The following notes provide some further details.

The stability problem for switched systems can be studied from several different angles. In this section we explore a particular direction, namely, the role of commutation relations among the systems being switched.

5.1 Commuting systems

Linear systems

Consider the switched linear system (4), and assume for the moment that $\mathcal{P} = \{1, 2\}$ and that the matrices A_1 and A_2 commute: $A_1A_2 = A_2A_1$. We will often write the latter condition as $[A_1, A_2] = 0$, where the *commutator*, or *Lie bracket* $[\cdot, \cdot]$, is defined as

$$[A_1, A_2] := A_1A_2 - A_2A_1. \tag{40}$$

It is well known that in this case we have $e^{A_1}e^{A_2} = e^{A_2}e^{A_1}$, as can be seen from the definition of a matrix exponential via the series $e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots$, and more generally,

$$e^{A_1 t}e^{A_2 \tau} = e^{A_2 \tau}e^{A_1 t} \qquad \forall\, t, \tau > 0. \tag{41}$$

This means that the flows of the two individual subsystems $\dot{x} = A_1x$ and $\dot{x} = A_2x$ commute.

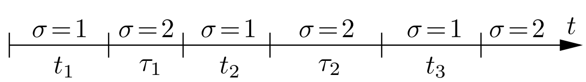

Fig. 13: Switching between two systems

Now consider an arbitrary switching signal σ, and denote by ρ_i and τ_i the lengths of the time intervals on which σ equals 1 and 2, respectively (see Figure 13). The solution of the system produced by this switching signal is

$$x(t) = \cdots e^{A_2\tau_2}e^{A_1\rho_2}e^{A_2\tau_1}e^{A_1\rho_1}x(0)$$

which in view of (41) equals

$$x(t) = \cdots e^{A_2\tau_2}e^{A_2\tau_1}\cdots e^{A_1\rho_2}e^{A_1\rho_1}x(0). \tag{42}$$

Another fact that we need is

$$[A, B] = 0 \quad \Rightarrow \quad e^Ae^B = e^{A+B}.$$

This is a consequence of the *Baker-Campbell-Hausdorff formula*

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[A,B]])+\dots}.$$

Scalar multiples of the same matrix clearly commute with each other, hence we can rewrite (42) as

$$x(t) = e^{A_2(\tau_1+\tau_2+\dots)}e^{A_1(\rho_1+\rho_2+\dots)}x(0). \tag{43}$$

Since at least one of the series $\rho_1 + \rho_2 + \dots$ and $\tau_1 + \tau_2 + \dots$ converges to ∞ as $t \to \infty$, the corresponding matrix exponential converges to zero in view of stability of the matrices A_1 and A_2 (recall that asymptotic stability of individual subsystems is assumed throughout this lecture). We have thus proved that $x(t) \to 0$ for an arbitrary switching signal. Generalization to the case when $\mathcal{P}$ has more than two elements is straightforward. In fact, the following result holds.

Theorem 3.5 *If $\{A_p : p \in \mathcal{P}\}$ is a finite set of commuting Hurwitz matrices, then the corresponding switched linear system (4) is GUES.*

The above argument only shows global attractivity for every switching signal. To prove Theorem 3.5, one can invoke Theorem 3.4. There is also a more direct way to arrive at the result, which is based on constructing a common Lyapunov function. The following iterative procedure, taken from [4], can be used to obtain a quadratic common Lyapunov function for a finite family of commuting asymptotically stable linear systems.

Let $\{A_1, A_2, \dots, A_m\}$ be the given set of commuting Hurwitz matrices. Let P_1 be the unique positive definite symmetric solution of the Lyapunov equation

$$A_1^T P_1 + P_1 A_1 = -I$$

(any other negative definite symmetric matrix could be used instead of $-I$ on the right-hand side). For $i = 1, \dots, m$, let P_i be the unique positive definite symmetric solution of the Lyapunov equation

$$A_i^T P_i + P_i A_i = -P_{i-1}.$$

Then the function

$$V(x) = x^T P_m x \tag{44}$$

is a desired quadratic common Lyapunov function for the given family of linear systems.

To see why this is true, observe that the matrix P_m is given by the formula

$$P_m = \int_0^\infty e^{A_m^T t_m} \dots \Big(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1 \Big) \dots e^{A_m t_m} dt_m$$

(see Example 2.1 in Section 3.3). Fix an arbitrary $i \in \{1, \dots, m\}$. Since the matrix exponentials in the above expression commute, we can regroup them to obtain

$$P_m = \int_0^\infty e^{A_i^T t_i}\, Q_i\, e^{A_i t_i}\, dt_i \tag{45}$$

where Q_i is given by an expression involving $m-1$ integrals. This matrix Q_i can thus be obtained by applying $m-1$ steps of the above algorithm (all except the ith step), hence it is positive definite. Since (45) implies that $A_i^T P_m + P_m A_i = -Q_i$, we conclude V given by (44) is a Lyapunov function for the ith subsystem. It is also not hard to prove this directly by manipulating Lyapunov equations, as done in [4]. Incidentally, from the above formulas we also see that changing the order of the matrices $\{A_1, A_2, \dots, A_m\}$ does not affect the resulting matrix P_m.

Nonlinear systems

To extend the above result to switched nonlinear systems, we first need the notion of a Lie bracket, or commutator, of two $\mathcal{C}^1$ vector fields. This is the vector field defined as follows:

$$[f_1, f_2](x) := \frac{\partial f_2(x)}{\partial x} f_1(x) - \frac{\partial f_1(x)}{\partial x} f_2(x).$$

For linear vector fields $f_1(x) = A_1 x$, $f_2(x) = A_2 x$ the right-hand side becomes $(A_2 A_1 - A_1 A_2)x$, which is consistent with the definition of the Lie bracket of two matrices (40) except for the difference in sign.

If the Lie bracket of two vector fields is identically zero, we will say that the two vector fields commute. The following result is a direct generalization of Theorem 3.5.

Theorem 3.6 *If $\{f_p : p \in \mathcal{P}\}$ is a finite set of commuting $\mathcal{C}^1$ vector fields and the origin is a globally asymptotically stable equilibrium for all systems in the family* (1)*, then the corresponding switched system* (3) *is GUAS.*

The proof of this result given in [5] establishes the GUAS property directly (in fact, commutativity of the flows is all that is needed, and the continuous differentiability assumption can be relaxed). It does not provide an explicit construction of a common Lyapunov function. Two alternative methods, discussed next, enable one to construct such a function. Unfortunately, they rely on the stronger assumption that the systems in the family (1) are *exponentially* stable, and provide a function that serves as a common Lyapunov function for this family only locally (in some neighborhood of the origin).

The first option is to employ Lyapunov's indirect method (described in Section 3.5). To this end, consider the linearization matrices

$$A_p := \frac{\partial f_p}{\partial x}(0), \qquad p \in \mathcal{P}. \tag{46}$$

If the nonlinear vector fields commute, then the linearization matrices also commute.

Exercise 3.2 Prove this (assuming that $f_p \in \mathcal{C}^1$ and $f_p(0) = 0$ for all $p \in \mathcal{P}$, and nothing else).

The converse does not necessarily hold, so commutativity of the linearization matrices is a weaker condition (which of course can be verified directly). The matrices A_p are Hurwitz if (and only if) the vector fields f_p are exponentially stable. Thus a quadratic common Lyapunov function for the linearized systems, constructed as explained earlier, serves as a local common Lyapunov function for the original finite family of nonlinear systems (1).

The second option is to use the iterative procedure described in [6]. This procedure, although not as constructive and practically useful as the previous one, parallels the procedure given earlier for commuting linear systems while working with the nonlinear vector fields directly. Let $\mathcal{P} = \{1, 2, \dots, m\}$ and suppose that the systems (1) are exponentially stable. For each $p \in \mathcal{P}$, denote by $\varphi_p(t, z)$ the solution of the system $\dot{x} = f_p(x)$ with initial condition $x(0) = z$. Define the functions

$$\begin{aligned} V_1(x) &:= \int_0^T |\varphi_1(\tau, x)|^2 d\tau \\ V_i(x) &:= \int_0^T V_{i-1}(\varphi_i(\tau, x)) d\tau, \qquad i = 2, \dots, m \end{aligned}$$

where T is a sufficiently large positive constant. Then V_m is a local common Lyapunov function for the family (1). Moreover, if the functions $f_p, p \in \mathcal{P}$ are globally Lipschitz, then we obtain a global common Lyapunov function. For the case of linear systems $f_p(x) = A_p x$, $p \in \mathcal{P}$ we recover the algorithm described earlier upon setting $T = \infty$.

5.2 Nilpotent and solvable Lie algebras

Linear systems

Consider again the switched linear system (4). In view of the previous discussion, it is reasonable to conjecture that if the matrices A_p , $p \in \mathcal{P}$ do not commute, then stability of the switched system may still depend on the commutation relations between them. A useful object which reveals the nature of these commutation relations is the Lie algebra $\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$ generated by the matrices A_p , $p \in \mathcal{P}$, with respect to the standard Lie bracket (40). This is a linear vector space of dimension at most n^2, spanned by the given matrices and all their iterated Lie brackets. Note that the Lie bracket of two Hurwitz matrices is no longer Hurwitz, as can be seen from the formula $\mathrm{tr}[A, B] = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$.

Beyond the commuting case, the simplest relevant classes of Lie algebras are *nilpotent* and *solvable* ones. A Lie algebra is called nilpotent if all Lie brackets of sufficiently high order are zero. Solvable Lie algebras form a larger class of Lie algebras, in which all Lie brackets of sufficiently high order having a certain structure are zero.

The first nontrivial case is when we have $\mathcal{P} = \{1, 2\}$, $[A_1, A_2] \neq 0$, and $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$. This means that $\mathfrak{g}$ is a nilpotent Lie algebra with order of nilpotency 2 and dimension 3 (as a basis we can choose $\{A_1, A_2, [A_1, A_2]\}$). Stability of the switched linear system corresponding to this situation—but in discrete time—was studied in [7]. The results obtained there for the discrete-time case can be easily adopted to continuous-time switched systems in which switching times are constrained to be integer multiples of a fixed positive number. In the spirit of the formula (43), in this case the solution of the switched system can be expressed as

$$x(t) = e^{A_2\tau_1} e^{A_1 t_1} e^{A_2\tau_2} e^{A_1 t_2} e^{A_2\tau_3} x(0)$$

where at least one of the quantities $\tau_1, t_1, \tau_2, t_2, \tau_3$ converges to ∞ as $t \to \infty$. This expression is a consequence of the Baker-Campbell-Hausdorff formula. Similarly to the commuting case, it follows that the switched linear system is GUES provided that the matrices A_1 and A_2 are Hurwitz.

The following general result, whose proof is sketched below, includes the above example and also Theorem 3.5 as special cases. (A further generalization will be obtained in Section 5.3.)

Theorem 3.7 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices and the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the switched linear system* (4) *is GUES.*

A standard example of a solvable Lie algebra is that generated by (nonstrictly) upper-triangular matrices, i.e., matrices with zero entries everywhere below the main diagonal. Such a Lie algebra is solvable because when one computes Lie brackets, additional zeros are generated on and then above the main diagonal. We will exploit the fact that, up to a coordinate transformation, all solvable Lie algebras can be characterized in this way. This is a consequence of the classical Lie's theorem from the theory of Lie algebras.

Proposition 3.8 (Lie) *If $\mathfrak{g}$ is a solvable Lie algebra, then there exists a (possibly complex) linear change of coordinates under which all matrices in $\mathfrak{g}$ are simultaneously transformed to the upper-triangular form.*

In view of this result we can assume, without loss of generality, that all matrices A_p, $p \in \mathcal{P}$ are upper-triangular. The following fact can now be used to finish the proof of Theorem 3.7.

Proposition 3.9 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of upper-triangular Hurwitz matrices, then the switched linear system* (4) *is GUES.*

To see why this proposition is true, suppose that $\mathcal{P} = \{1, 2\}$ and $x \in \mathbb{R}^2$. Let the two matrices be

$$A_1 := \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}. \tag{47}$$

Suppose for simplicity that their entries are real (the case of complex entries requires some care but the extension is not difficult). Since the eigenvalues of these matrices have negative real parts, we have

$a_i, c_i > 0$, $i = 1, 2$. Now, consider the switched system $\dot{x} = A_\sigma x$. The second component of x satisfies the equation

$$\dot{x}_2 = -c_\sigma x_2.$$

Therefore, x_2 decays to zero exponentially fast, at the rate corresponding to $\min\{c_1, c_2\}$. The first component of x satisfies the equation

$$\dot{x}_1 = -a_\sigma x_1 + b_\sigma x_2.$$

This can be viewed as the exponentially stable system $\dot{x}_1 = -a_\sigma x_1$ perturbed by the exponentially decaying input $b_\sigma x_2$. Thus x_1 also converges to zero exponentially fast. It is not hard to extend this argument to systems of arbitrary dimension (by induction, proceeding from the bottom component of x upwards) and to infinite index sets $\mathcal{P}$ (by using the compactness assumption).

An alternative method of proving Fact 2, and thus completing the proof of Theorem 3.7, consists of constructing a common Lyapunov function for the family of linear systems (37). This construction leads to a cleaner proof and is of independent interest. It turns out that in the present case it is possible to find a quadratic common Lyapunov function of the form (33), with P a diagonal matrix. We illustrate this on the example of the two matrices (47). Let us look for P taking the form

$$P = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

where $d_1, d_2 > 0$. A straightforward calculation gives

$$-A_i^T P - P A_i = \begin{pmatrix} 2d_1 a_i & -d_1 b_i \\ -d_1 b_i & 2d_2 c_i \end{pmatrix}, \qquad i = 1, 2.$$

To ensure that this matrix is positive definite, we can first pick an arbitrary $d_1 > 0$, and then choose $d_2 > 0$ large enough to have

$$4d_2 d_1 a_i c_i - d_1^2 b_i^2 > 0, \qquad i = 1, 2.$$

Again, this construction can be extended to the general situation by using the compactness assumption and induction on the dimension of the system.

Exercise 3.3 Verify whether or not the switched linear systems in the plane generated by the following pairs of matrices are GUES: (a) $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$; (b) $A_1 = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.

Nonlinear systems

Using Lyapunov's indirect method, we can obtain the following local version of Theorem 3.7 for switched nonlinear systems. Consider the family of nonlinear systems (1), assuming that each function f_p is $\mathcal{C}^1$ and satisfies $f_p(0) = 0$. Consider also the corresponding family of linearization matrices A_p, $p \in \mathcal{P}$ defined by the formula (46).

Corollary 3.10 *Suppose that the linearization matrices A_p, $p \in \mathcal{P}$ are Hurwitz, $\mathcal{P}$ is a compact set, and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each x in some neighborhood of the origin. If the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the switched system* (3) *is locally uniformly exponentially stable.*

This is a relatively straightforward application of Lyapunov's indirect method (see Section 3.5), although some additional technical assumptions are needed here because, unlike in Section 5.1, the set $\mathcal{P}$ is allowed to be infinite. The linearization matrices A_p, $p \in \mathcal{P}$ form a compact set because they are defined by the formula (46) and $\frac{\partial f_p}{\partial x}(x)$ is assumed to depend continuously on p. Moreover, since the matrices A_p, $p \in \mathcal{P}$ are Hurwitz and generate a solvable Lie algebra, the corresponding linear systems (37) share a quadratic common Lyapunov function (as we saw earlier). Then it is not hard to show that this function is also a common Lyapunov function for the original family (1) on a sufficiently small neighborhood of the origin.

5.3 More general Lie algebras

As before, we study the switched linear system (4), where $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices. Consider a decomposition of the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ into a semidirect sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is a solvable ideal and $\mathfrak{s}$ is a subalgebra. For our purposes, the best choice is to let $\mathfrak{r}$ be the radical, in which case $\mathfrak{s}$ is semisimple and we have a Levi decomposition. If $\mathfrak{g}$ is not solvable, then $\mathfrak{s}$ is not zero.

The following result is a direct extension of Theorem 3.7. It states that the system (4) is still GUES if the subalgebra $\mathfrak{s}$ is compact (which amounts to saying that all matrices in $\mathfrak{s}$ are diagonalizable and have purely imaginary eigenvalues).

Theorem 3.11 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices and the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is a semidirect sum of a solvable ideal and a compact subalgebra, then the switched linear system* (4) *is GUES.*

Example 3.2 Suppose that the matrices A_p, $p \in \mathcal{P}$ take the form $A_p = -\lambda_p I + S_p$ where $\lambda_p > 0$ and $S_p^T = -S_p$ for all $p \in \mathcal{P}$. These are automatically Hurwitz matrices. If $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ contains the identity matrix, then the condition of Theorem 3.11 is satisfied with $\mathfrak{r} := \mathbb{R}I$ (scalar multiples of the identity matrix) and $\mathfrak{s} := \{S_p : p \in \mathcal{P}\}_{LA}$, which is compact. If $\mathfrak{g}$ does not contain the identity matrix, then $\mathfrak{g}$ is a proper subalgebra of $\mathbb{R}I \oplus \{S_p : p \in \mathcal{P}\}_{LA}$; it is not difficult to see that the result is still valid in this case. □

If the condition of Theorem 3.11 is satisfied, then the linear systems (37) share a quadratic common Lyapunov function. (The proof of this fact exploits the Haar measure on the Lie group corresponding to $\mathfrak{s}$ and is not as constructive as in the case when $\mathfrak{g}$ is solvable.) Considering the family of nonlinear systems (1) with $f_p(0) = 0$ for all $p \in \mathcal{P}$, together with the corresponding linearization matrices (46), we immediately obtain the following generalization of Corollary 3.10.

Corollary 3.12 *Suppose that the linearization matrices A_p, $p \in \mathcal{P}$ are Hurwitz, $\mathcal{P}$ is a compact set, and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each x in some neighborhood of the origin. If the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is a semidirect sum of a solvable ideal and a compact subalgebra, then the switched system* (3) *is locally uniformly exponentially stable.*

The result expressed by Theorem 3.11 is in some sense the strongest one that can be given on the Lie algebra level. To explain this more precisely, we need to introduce a possibly larger Lie algebra $\widehat{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the scalar multiples of the identity matrix if necessary. In other words, define $\widehat{\mathfrak{g}} := \{I, A_p : p \in \mathcal{P}\}_{LA}$. The Levi decomposition of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathfrak{g}} = \widehat{\mathfrak{r}} \oplus \mathfrak{s}$ with $\widehat{\mathfrak{r}} \supset \mathfrak{r}$ (because the subspace $\mathbb{R}I$ belongs to the radical of $\widehat{\mathfrak{g}}$). Thus $\widehat{\mathfrak{g}}$ satisfies the hypothesis of Theorem 3.11 if and only if $\mathfrak{g}$ does.

It turns out that if $\widehat{\mathfrak{g}}$ cannot be decomposed as required by Theorem 3.11, then it can be generated by a family of Hurwitz matrices (which might in principle be different from $A_p\,,\ p \in \mathcal{P}$) with the property that the corresponding switched linear system is not GUES. On the other hand, there exists another set of Hurwitz generators for $\widehat{\mathfrak{g}}$ which does give rise to a GUES switched linear system. (In fact, both generator sets can always be chosen in such a way that they contain the same number of elements as the original set that was used to generate $\widehat{\mathfrak{g}}$.) Thus if the Lie algebra does not satisfy the hypothesis of Theorem 3.11, then this Lie algebra alone does not provide enough information to determine whether or not the original switched linear system is stable.

Theorem 3.13 *Suppose that a given matrix Lie algebra $\widehat{\mathfrak{g}}$ does not satisfy the hypothesis of Theorem 3.11. Then there exists a set of Hurwitz generators for $\widehat{\mathfrak{g}}$ such that the corresponding switched linear system is not GUES. There also exists another set of Hurwitz generators for $\widehat{\mathfrak{g}}$ such that the corresponding switched linear system is GUES.*

By virtue of this result, we have a complete characterization of all matrix Lie algebras $\widehat{\mathfrak{g}}$ that contain the identity matrix and have the property that every set of Hurwitz generators for $\widehat{\mathfrak{g}}$ gives rise to a GUES switched linear system. Namely, these are precisely the Lie algebras that admit a decomposition described in the statement of Theorem 3.11. The interesting—and rather surprising—discovery is that the above property depends only on the structure of $\widehat{\mathfrak{g}}$ as a Lie algebra, and not on the choice of a particular matrix representation of $\widehat{\mathfrak{g}}$.

5.4 Discussion of Lie-algebraic stability criteria

Lie-algebraic stability criteria for switched systems are appealing because nontrivial mathematical tools are brought to bear on the problem and lead to interesting results. Another attractive feature of these conditions is that they are formulated in terms of the original data. Take, for example, Theorem 3.7. The proof of this theorem relies on the facts that the matrices in a solvable Lie algebra can be simultaneously triangularized and that switching between triangular matrices preserves stability. It is important to recognize, however, that it is a nontrivial matter to find a basis in which all matrices take the triangular form or even decide whether such a basis exists. To apply Theorem 3.7, no such basis needs to be found. Instead, one can check directly whether the Lie algebra generated by the given matrices is solvable.

In fact, classical results from the theory of Lie algebras can be employed to check the various stability conditions for switched linear systems presented above. Assume for simplicity that the set $\mathcal{P}$ is finite or a maximal linearly independent subset has been extracted from $\{A_p : p \in \mathcal{P}\}$. Then one can verify directly, using the definitions, whether or not the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable (or nilpotent). To do this, one constructs a decreasing sequence of ideals by discarding lower-order Lie brackets at each step and checks whether the sequence of dimensions of these ideals strictly decreases to zero. In specific examples involving a small number of matrices, it is usually not difficult to derive the relevant commutation relations between them. To do this systematically in more complicated situations, it is helpful to use a canonical basis known as a P. Hall basis.

Alternatively, one can use the Killing form, which is a canonical symmetric bilinear form defined on every Lie algebra. Cartan's first criterion provides a necessary and sufficient condition for a Lie algebra to be solvable in terms of the Killing form. In view of these remarks, Lie-algebraic tools yield stability conditions for switched systems which are both mathematically appealing and computationally efficient.

The main disadvantage of the Lie-algebraic stability criteria is their limited applicability. Clearly, they provide only sufficient and not necessary conditions for stability. (This can be seen from the second statement of Theorem 3.13 and also from the fact that they imply the existence of quadratic common

Lyapunov functions—this property is interesting but, as we saw in Sections 4.4 and 4.5, does not hold for all GUES switched linear systems.)

Moreover, it turns out that even as sufficient conditions, the Lie-algebraic conditions are extremely nongeneric. To see why this is so, first note that the GUES property is robust with respect to sufficiently small perturbations of the matrices that define the individual subsystems. This follows via standard arguments from the converse Lyapunov theorem (Theorem 3.2). An especially transparent characterization of the indicated robustness property can be obtained in the case of linear systems sharing a quadratic common Lyapunov function, i.e., when there exist positive definite symmetric matrices P and Q satisfying the inequalities (34). Suppose that for every $p \in \mathcal{P}$, a perturbed matrix

$$\overline{A}_p := A_p + \Delta_p$$

is given. Let us derive an admissible bound on the perturbations $\Delta_p\,,\ p \in \mathcal{P}$ such that the matrices $\overline{A}_p\,,\ p \in \mathcal{P}$ still share the same quadratic common Lyapunov function $x^T P x$. This is guaranteed if we have

$$|2x^T \Delta_p^T P x| < x^T Q x \qquad \forall\, p \in \mathcal{P}, \quad \forall\, x \neq 0. \tag{48}$$

We denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a symmetric matrix, respectively. The right-hand side of (48) is lower-bounded by $\lambda_{\min}(Q)|x|^2$, while the left-hand side of (48) is upper-bounded by

$$2|\Delta_p x||Px| = 2\sqrt{x^T \Delta_p^T \Delta_p x} \cdot \sqrt{x^T P^2 x} \leq 2|x|^2 \sigma_{\max}(\Delta_p)\lambda_{\max}(P)$$

where $\sigma_{\max}(\Delta_p) := \sqrt{\lambda_{\max}(\Delta_p^T \Delta_p)}$ is the largest singular value of Δ_p. Therefore, a (conservative) admissible bound on $\Delta_p\,,\ p \in \mathcal{P}$ is given by the formula

$$\sigma_{\max}(\Delta_p) < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

(The right-hand side is maximized when $Q = I$.)

On the other hand, the Lie-algebraic conditions of the type considered here do not possess the above robustness property. This follows from the fact that in an arbitrarily small neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices that generate the entire Lie algebra $gl(n, \mathbb{R})$. In other words, the conditions given by Theorems 3.5, 3.7, and 3.11 are destroyed by arbitrarily small perturbations of the individual systems. To obtain more generic stability conditions, one needs to complement these results by a perturbation analysis.

Systems with special structure; common weak Lyapunov functions and observability

This lecture is based on powerpoint slides, file names: `structure, observability`. The following notes provide some further material on systems with special structure.

6 Systems with special structure

The results discussed so far apply to general switched systems. The questions related to stability of such systems are very difficult, and the findings discussed above certainly do not provide complete and satisfactory answers. On the other hand, specific structure of a given system can sometimes be utilized to obtain interesting results, even in the absence of a general theory. In this section we present a few results that are available for some special classes of switched systems.

6.1 Triangular systems

We already know (Proposition 3.9) that if $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices in the upper-triangular form, then the switched linear system (4) is GUES. In fact, under these hypotheses the linear systems in the family (37) share a quadratic common Lyapunov function. (The case of lower-triangular systems is completely analogous.) It is natural to ask to what extent this result is true for switched nonlinear systems.

Suppose that $\mathcal{P}$ is a compact set and that the family of systems (1) is such that for each $p \in \mathcal{P}$, the vector field f_p takes the upper-triangular form

$$f_p(x) = \begin{pmatrix} f_{p1}(x_1, x_2, \dots, x_n) \\ f_{p2}(x_2, \dots, x_n) \\ \vdots \\ f_{pn}(x_n) \end{pmatrix}. \tag{49}$$

If the linearization matrices (46) are Hurwitz and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p, then the linearized systems have a quadratic common Lyapunov function by virtue of the result mentioned above. It follows from Lyapunov's indirect method that in this case the original switched nonlinear system (3) is locally uniformly exponentially stable (cf. Corollary 3.10).

What about global stability results? One might be tempted to conjecture that under appropriate compactness assumptions, the switched system (3) is GUAS, provided that the individual subsystems (1) all share the origin as a globally asymptotically stable equilibrium. We now provide a counterexample showing that this is not true.

Example 4.3 Let $\mathcal{P} = \{1, 2\}$, and consider the vector fields

$$f_1(x) = \begin{pmatrix} -x_1 + 2\sin^2(x_1)x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

and

$$f_2(x) = \begin{pmatrix} -x_1 + 2\cos^2(x_1)x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

FACT 1. The systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ are globally asymptotically stable.

To see that the system $\dot{x} = f_1(x)$ is globally asymptotically stable, fix arbitrary initial values $x_1(0), x_2(0)$. We have $x_2(t) = x_2(0)e^{-t}$. As for x_1, note that $\sin(x_1)$ vanishes at the integer multiples of π. This implies that $|x_1(t)| \leq E$ for all $t \geq 0$, where E is the smallest integer multiple of π that is greater than or equal to $|x_1(0)|$. Since x_1 is bounded and x_2 converges to zero, it is easy to see that the linear term $-x_1$ eventually dominates and we have $x_1(t) \to 0$. We conclude that the system $\dot{x} = f_1(x)$ is globally attractive; its stability in the sense of Lyapunov can be shown by similar arguments. Global asymptotic stability of the system $\dot{x} = f_2(x)$ is established in the same way.

FACT 2. The switched system $\dot{x} = f_\sigma(x)$ is not GUAS.

If the switched system were GUAS, then Corollary 3.3 would guarantee global asymptotic stability of an arbitrary "convex combination"

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2(\alpha \sin^2(x_1) + (1-\alpha)\cos^2(x_1))x_1^2 x_2\\ \dot{x}_2 &= -x_2\end{aligned}$$

of the two subsystems, where $0 \leq \alpha \leq 1$. In particular, for $\alpha = 1/2$ we arrive at the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2\\ \dot{x}_2 &= -x_2.\end{aligned} \tag{50}$$

We will now show that this system is not globally asymptotically stable; in fact, it even has solutions that are not defined globally in time. Recall that solutions of the equation $\dot{x} = x^2$ escape to infinity in finite time (see Example 1.3). In view of this, it is not hard to see that for sufficiently large initial conditions, the x_1-component of the solution of the system (50) escapes to infinity before x_2 becomes small enough to slow it down. The system (50) was actually discussed in [8, p. 8] in the context of adaptive control. Its solutions are given by the formulas

$$\begin{aligned}x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t} + (2 - x_1(0)x_2(0))e^t}\\ x_2(t) &= x_2(0)e^{-t}.\end{aligned}$$

We see that solutions with $x_1(0)x_2(0) \geq 2$ are unbounded and, in particular, solutions with $x_1(0)x_2(0) > 2$ have a finite escape time. This proves that the switched system $\dot{x} = f_\sigma(x)$ is not GUAS. □

Thus in the case of switching among globally asymptotically stable nonlinear systems, the triangular structure alone is not sufficient for GUAS. One way to guarantee GUAS is to require that along solutions of the individual subsystems (1), each component of the state vector stay small if the subsequent components are small. The right notion in this regard turns out to be input-to-state stability (ISS), reviewed in Section 3.6.

Theorem 4.14 *Assume that $\mathcal{P}$ is a compact set, f_p is continuous in p for each x, and the systems* (1) *are globally asymptotically stable and take the triangular form* (49)*. If for each $i = 1, \dots, n-1$ and each $p \in \mathcal{P}$ the system*

$$\dot{x}_i = f_{pi}(x_i, x_{i+1}, \dots, x_n)$$

is ISS with respect to the input $u = (x_{i+1}, \dots, x_n)^T$, then the switched system (3) *is GUAS.*

The first two hypotheses are trivially satisfied if $\mathcal{P}$ is a finite set. The theorem can be proved by starting with the bottom component of the state vector x and proceeding upwards, using ISS-Lyapunov functions (this is in the same spirit as the argument we used earlier to prove Proposition 3.9). For asymptotically stable linear systems, the ISS assumption is automatically satisfied, which explains why we did not need it in Section 5.2. Under certain additional conditions, it is possible to extend Theorem 4.14 to block-triangular systems.

If the triangular subsystems are asymptotically stable, $\mathcal{P}$ is a compact set, and f_p depends continuously on p for each x, then the ISS hypotheses of Theorem 4.14 are automatically satisfied in a sufficiently small neighborhood of the origin. Indeed, asymptotic stability of $\dot{x} = f_p(x)$ guarantees that the system $\dot{x}_i = f_{pi}(x_i, 0)$ is asymptotically stable[5] for each i, and we know from Section 3.6 that this implies local ISS of $\dot{x}_i = f_{pi}(x_i, u)$. Thus the triangular switched system in question is always locally uniformly asymptotically stable, even if the linearization test mentioned earlier fails.

6.2 Feedback systems

Switched systems often arise from the feedback connection of different controllers with the same process. Such feedback switched systems therefore assume particular interest in control theory. The fact that the process is fixed imposes some structure on the closed-loop systems, which sometimes facilitates the stability analysis. Additional flexibility is gained if input-output properties of the process and the controllers are specified but one has some freedom in choosing state-space realizations. We now briefly discuss several stability results for switched systems of this kind.

Passivity, positive realness, and absolute stability

Consider the control system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

with $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}^m$. By (strict) *passivity* we mean the property of this system characterized by the existence of a $\mathcal{C}^1$ positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ (called a *storage function*) and a positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that we have

$$\frac{\partial V}{\partial x} f(x, u) \leq -W(x) + u^T h(x). \tag{51}$$

(It is usually assumed that the storage function is merely nonnegative definite, but in the case of strict passivity its positive definiteness is automatic.) Passive systems frequently arise in a variety of applications, for example, in models of electrical circuits and mechanical devices.

Suppose that the inequality (51) holds. It is easy to see that for every $K \geq 0$, the closed-loop system obtained by setting $u = -Ky$ is asymptotically stable, with Lyapunov function V whose derivative along solutions satisfies

$$\dot{V}(x) \leq -W(x) - y^T K y.$$

In other words, V is a common Lyapunov function for the family of closed-loop systems corresponding to all nonpositive definite feedback gain matrices. It follows that the switched system generated by this family is uniformly asymptotically stable (GUAS if V is radially unbounded). Clearly, the function V also serves as a common Lyapunov function for all nonlinear feedback systems obtained by setting $u = -\varphi(y)$, where φ satisfies $y^T \varphi(y) \geq 0$ for all y. In the single-input, single-output (SISO) case, this reduces to the sector condition

$$0 \leq y\varphi(y) \qquad \forall\, y. \tag{52}$$

[5] This is because all trajectories of the latter system are projections of trajectories of the former, for $x_{i+1}(0) = \cdots = x_n(0)$, onto the x_i-axis.

For linear systems, there is a very useful frequency-domain condition for passivity in terms of the concept of positive realness which we now define. We limit our discussion to SISO systems, although similar results hold for general systems. A proper rational function $g : \mathbb{C} \to \mathbb{C}$ is called *positive real* if $g(s) \in \mathbb{R}$ when $s \in \mathbb{R}$ and $\operatorname{Re} g(s) \geq 0$ when $\operatorname{Re} s \geq 0$, and *strictly positive real* if $g(s - \varepsilon)$ is positive real for some $\varepsilon > 0$. A positive real function has all its poles in the closed left half-plane; if all poles are in the open left half-plane, then it is enough to check the inequality $\operatorname{Re} g(s) \geq 0$ along the imaginary axis.

Every linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned}$$

with a Hurwitz matrix A and a strictly positive real transfer function

$$g(s) = c^T (sI - A)^{-1} b$$

is strictly passive. This follows from the famous Kalman-Yakubovich-Popov lemma, which guarantees the existence of a positive definite symmetric matrix P satisfying

$$\begin{aligned} A^T P + PA &\leq -Q < 0 \\ Pb &= c. \end{aligned}$$

Letting $V(x) := \frac{1}{2} x^T P x$, we obtain

$$\frac{\partial V}{\partial x}(Ax + Bu) = \frac{1}{2} x^T (A^T P + PA) x + x^T P b u \leq -\frac{1}{2} x^T Q x + u^T y.$$

We conclude that if the open-loop transfer function is strictly positive real, then the closed-loop systems for all nonpositive feedback gains ($u = -ky$, $k \geq 0$) share a quadratic common Lyapunov function. (For systems of dimension $n \leq 2$ the converse is also true: the existence of such a quadratic common Lyapunov function implies that the open-loop transfer function is strictly positive real.) We conclude that the corresponding switched linear system is GUES. Again, the above result immediately extends to nonlinear feedback systems

$$\dot{x} = Ax - b\varphi(c^T x) \tag{53}$$

with φ satisfying the inequality (52).

If the open-loop transfer function g is not strictly positive real but the function

$$\frac{1 + k_2 g}{1 + k_1 g} \tag{54}$$

is strictly positive real for some $k_2 > k_1 \geq 0$, where k_1 is a stabilizing gain, then a quadratic common Lyapunov function exists for the family of systems (53) under the following more restrictive sector condition on φ:

$$k_1 y^2 \leq y\varphi(y) \leq k_2 y^2 \qquad \forall\, y. \tag{55}$$

This result is usually referred to as the *circle criterion*, because the strict positive real property of the function (54) implies that the Nyquist locus of g lies outside the disk centered at the real axis which intersects the real axis at the points $(-1/k_1, 0)$ and $(-1/k_2, 0)$. For $k_1 = 0$ this disk becomes the half-plane $\{s : \operatorname{Re} s \leq -1/k_2\}$.

The problem of determining stability of the system (53) for all nonlinearities φ lying in some given sector such as (52) or (55) is the well-known *absolute stability* problem. In the investigation of this problem, conditions that lead to the existence of a *quadratic* Lyapunov function are in general too restrictive. Less conservative frequency-domain conditions for absolute stability are provided by *Popov's criterion*. One version of this criterion can be stated as follows: if g has one pole at zero and the rest in the open left half-plane and the function $(1+\alpha s)g(s)$ is positive real for some $\alpha \geq 0$, then the system (53) is globally asymptotically stable for every function φ that satisfies the sector condition $0 < y\varphi(y)$ for all $y \neq 0$. Alternatively, if $(1+\alpha s)g(s)$ is strictly positive real for some $\alpha \geq 0$, then the weaker sector condition (52) is sufficient. When Popov's criterion applies, there exists a Lyapunov function for the closed-loop system in the form of a quadratic term plus an integral of the nonlinearity. Since this Lyapunov function depends explicitly on φ, a common Lyapunov function in general no longer exists; in other words, switching between different negative feedback gains or sector nonlinearities may cause instability.

Small-gain theorem

Consider the output feedback switched linear system

$$\dot{x} = (A + BK_\sigma C)x. \tag{56}$$

Assume that A is a Hurwitz matrix and that $\|K_p\| \leq 1$ for all $p \in \mathcal{P}$, where $\|\cdot\|$ denotes the matrix norm induced by the Euclidean norm on $\mathbb{R}^n$. Then the classical small-gain theorem implies that (56) is GUES if

$$\|C(sI - A)^{-1}B\|_\infty < 1 \tag{57}$$

where $\|\cdot\|_\infty$ denotes the standard $\mathcal{H}_\infty$ norm of a transfer matrix, defined as $\|G\|_\infty := \max_{\mathrm{Re}\, s=0} \sigma_{\max}(G(s))$. This norm characterizes the $\mathcal{L}_2$ gain of the open-loop system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{58}$$

The condition (57) is satisfied if and only if there exists a solution $P > 0$ of the algebraic Riccati inequality

$$A^T P + PA + PBB^T P + C^T C < 0.$$

Under the present assumptions, this inequality actually provides a necessary and sufficient condition for the linear systems

$$\dot{x} = (A + BK_p C)x, \qquad p \in \mathcal{P} \tag{59}$$

to share a quadratic common Lyapunov function $V(x) = x^T Px$. A simple square completion argument demonstrates that the derivative of this Lyapunov function along solutions of the system (58) satisfies $\dot{V} \leq -|y|^2 + |u|^2 - \varepsilon|x|^2$ for some $\varepsilon > 0$. From this we see that V also serves as a common Lyapunov function for the family of nonlinear feedback systems that result from setting $u = \varphi(y)$ with

$$|\varphi(y)| \leq |y| \qquad \forall\, y. \tag{60}$$

Since V is quadratic and the bound on its decay rate is also quadratic, it follows from Remark 3.1 that the switched system is still GUES.

Note that the inequality (60) is equivalent to the sector condition (55) with $k_1 = -1$ and $k_2 = 1$. The circle criterion can be applied in such situations too, except that now the Nyquist locus must lie *inside* an appropriate disk. This observation points to a unified framework for small-gain and passivity conditions.

Exercise 4.4 Investigate stability of the system $\ddot{x} + \dot{x} = u$ under nonlinear feedbacks of the form $u = -\varphi(x)$ by checking which of the above results (passivity criterion, circle criterion, Popov's criterion, small-gain theorem) can be applied. Support your findings by Lyapunov analysis.

If a given switched linear system is not in the form (56), it may be possible to construct an auxiliary switched linear system whose stability can be checked with the help of the above result and implies stability of the original system. As an example, consider the switched linear system (4) with $\mathcal{P} = \{1, 2\}$. It can be recast, for instance, as

$$\dot{x} = \frac{1}{2}(A_1 + A_2)x + \sigma\frac{1}{2}(A_1 - A_2)x$$

where σ takes values in the set $\{-1, 1\}$. It follows that this system is GUES if the inequality (57) is satisfied for $A = \frac{1}{2}(A_1 + A_2)$, $B = \frac{1}{2}I$, and $C = A_1 - A_2$. A similar trick can be used if one wants to apply the passivity criterion. Rewriting the same switched linear system as, say,

$$\dot{x} = A_1 x - \sigma(A_1 - A_2)x$$

with σ now taking values in the set $\{0, 1\}$, we see that it is GUES if the open-loop system (58) with $A = A_1$, $B = I$, and $C = A_1 - A_2$ is strictly passive. Of course, the above choices of the auxiliary system parameters are quite arbitrary.

Stability under constrained switching; multiple Lyapunov functions

This lecture is based on powerpoint slides, file name: `constrained`. The following notes provide some further material.

7 Multiple Lyapunov functions

We begin by describing a useful tool for proving stability of switched systems, which relies on *multiple Lyapunov functions*, usually one or more for each of the individual subsystems being switched. To fix ideas, consider the switched system (3) with $\mathcal{P} = \{1, 2\}$. Suppose that both systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ are (globally) asymptotically stable, and let V_1 and V_2 be their respective (radially unbounded) Lyapunov functions. We are interested in the situation where a common Lyapunov function for the two systems is not known or does not exist. In this case, one can try to investigate stability of the switched system using V_1 and V_2.

In the absence of a common Lyapunov function, stability properties of the switched system in general depend on the switching signal σ. Let t_i, $i = 1, 2, \ldots$ be the switching times. If it so happens that the values of V_1 and V_2 coincide at each switching time, i.e., $V_{\sigma(t_{i-1})}(x(t_i)) = V_{\sigma(t_i)}(x(t_i))$ for all i, then V_σ is a continuous Lyapunov function for the switched system, and asymptotic stability follows. This situation is depicted in Figure 14(a).

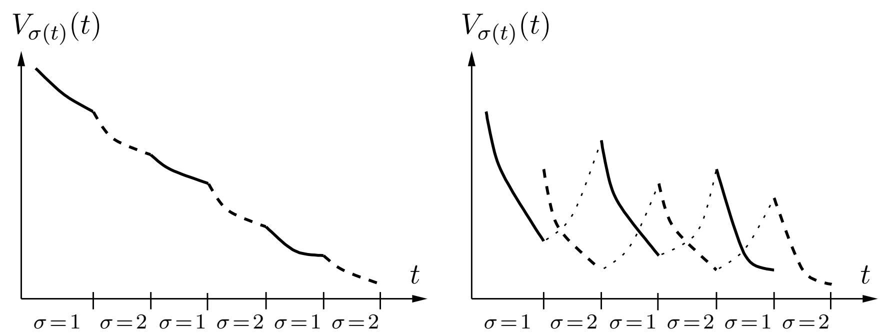

Fig. 14: Two Lyapunov functions (solid graphs correspond to V_1, dashed graphs correspond to V_2): (a) continuous V_σ, (b) discontinuous V_σ

In general, however, the function V_σ will be discontinuous. While each V_p decreases when the pth subsystem is active, it may increase when the pth subsystem is inactive. This behavior is illustrated in Figure 14(b). The basic idea that allows one to show asymptotic stability in this case is the following. Let us look at the values of V_p at the beginning of each interval on which $\sigma = p$. For the switched system to be asymptotically stable, these values[6] must form a decreasing sequence for each p.

Theorem 5.1 *Let* (1) *be a finite family of globally asymptotically stable systems, and let* V_p , $p \in \mathcal{P}$ *be a family of corresponding radially unbounded Lyapunov functions. Suppose that there exists a family of positive definite continuous functions* W_p, $p \in \mathcal{P}$ *with the property that for every pair of switching times* (t_i, t_j), $i < j$ *such that* $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ *and* $\sigma(t_k) \neq p$ *for* $t_i < t_k < t_j$, *we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)). \tag{61}$$

Then the switched system (3) *is globally asymptotically stable.*

[6] Alternatively, we could work with the values of V_p at the *end* of each interval on which $\sigma = p$.

PROOF. We first show stability of the origin in the sense of Lyapunov. Let m be the number of elements in $\mathcal{P}$. Without loss of generality, we assume that $\mathcal{P} = \{1, 2, \ldots, m\}$. Consider the ball around the origin of an arbitrary given radius $\varepsilon > 0$. Let $\mathcal{R}_m$ be a set of the form $\{x : V_m(x) \le c_m\}$, $c_m > 0$, which is contained in this ball. For $i = m-1, \ldots, 1$, let $\mathcal{R}_i$ be a set of the form $\{x : V_i(x) \le c_i\}$, $c_i > 0$, which is contained in the set $\mathcal{R}_{i+1}$. Denote by δ the radius of some ball around the origin which lies in the intersection of all nested sequences of sets constructed in this way for all possible permutations of $\{1, 2, \ldots, m\}$. Suppose that the initial condition satisfies $|x(0)| \le \delta$. If the first k values of σ are distinct, where $k \le m$, then by construction we have $|x(t_k)| \le \varepsilon$. After that, the values of σ will start repeating, and the condition (61) guarantees that the state trajectory will always belong to one of the above sets. Figure 15 illustrates this argument for the case $m = 2$.

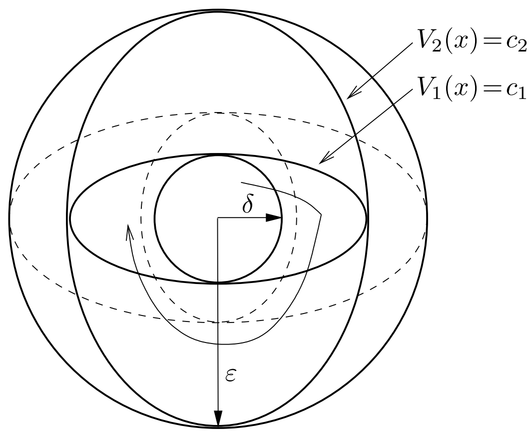

Fig. 15: Proving Lyapunov stability in Theorem 5.1

To show asymptotic stability, observe that due to the finiteness of $\mathcal{P}$ there exists an index $q \in \mathcal{P}$ that has associated with it an infinite sequence of switching times $t_{i_1}, t_{i_2}, \ldots$ such that $\sigma(t_{i_j}) = q$ (we are ruling out the trivial case when there are only finitely many switches). The sequence $V_q(x(t_{i_1}))$, $V_q(x(t_{i_2}))$, $\ldots$ is decreasing and positive and therefore has a limit $c \ge 0$. We have

$$\begin{aligned} 0 = c - c &= \lim_{j\to\infty} V_q(x(t_{i_{j+1}})) - \lim_{j\to\infty} V_q(x(t_{i_j})) \\ &= \lim_{j\to\infty} [V_q(x(t_{i_{j+1}})) - V_q(x(t_{i_j}))] \\ &\le \lim_{j\to\infty} [-W_q(x(t_{i_j}))] \le 0. \end{aligned}$$

Thus $W_q(x(t_{i_j})) \to 0$ as $j \to \infty$. We also know that W_q is positive definite. In view of radial unboundedness of V_p, $p \in \mathcal{P}$, an argument similar to the one used earlier to prove Lyapunov stability shows that $x(t)$ stays bounded. Therefore, $x(t_{i_j})$ must converge to zero as $j \to \infty$. It now follows from the Lyapunov stability property that $x(t) \to 0$ as $t \to \infty$. □

Remark 5.1 It is possible to obtain less conservative stability conditions involving multiple Lyapunov functions. In particular, one can relax the requirement that each V_p must decrease on the intervals on which the pth system is active, provided that the admissible growth of V_p on such intervals is bounded in a suitable way. Impulse effects can also be incorporated within the same framework. □

It is important to note that to apply Theorem 5.1, one must have some information about the solutions of the system. Namely, one needs to know the values of suitable Lyapunov functions at switching times, which in general requires the knowledge of the state at these times. This is to be contrasted with the classical Lyapunov stability results, which do not require the knowledge of solutions. (Of course, in both cases there remains the problem of finding candidate Lyapunov functions.) As we will see shortly, multiple Lyapunov function results such as Theorem 5.1 are useful when the class of admissible switching signals is constrained in a way that makes it possible to ensure the desired relationships between the values of Lyapunov functions at switching times.

8 Stability under slow switching

It is well known that a switched system is stable if all individual subsystems are stable and the switching is sufficiently slow, so as to allow the transient effects to dissipate after each switch. In this section we discuss how this property can be precisely formulated and justified using multiple Lyapunov function techniques.

8.1 Dwell time

The simplest way to specify slow switching is to introduce a number $\tau_d > 0$ and restrict the class of admissible switching signals to signals with the property that the switching times $t_1, t_2, \dots$ satisfy the inequality $t_{i+1} - t_i \geq \tau_d$ for all i. This number τ_d is usually called the *dwell time* (because σ "dwells" on each of its values for at least τ_d units of time).

It is a well-known fact that when all linear systems in the family (37) are asymptotically stable, the switched linear system (4) is asymptotically stable if the dwell time τ_d is sufficiently large. The required lower bound on τ_d can be explicitly calculated from the exponential decay bounds on the transition matrices of the individual subsystems.

Exercise 5.1 Consider a set of matrices $\{A_p : p \in \mathcal{P}\}$ with the property that for some positive constants c and λ_0 the inequality $\|e^{A_p t}\| \leq ce^{-\lambda_0 t}$ holds for all $t \geq 0$ and all $p \in \mathcal{P}$. Let the switched linear system (4) be defined by a switching signal σ with a dwell time τ_d. For an arbitrary number $\lambda \in (0, \lambda_0)$, derive a lower bound on τ_d that guarantees global exponential stability with stability margin λ (see Section 3.1).

Under suitable assumptions, a sufficiently large dwell time also guarantees asymptotic stability of the switched system in the nonlinear case. Probably the best way to prove most general results of this kind is by using multiple Lyapunov functions. We now sketch the relevant argument.

Assume for simplicity that all systems in the family (1) are globally exponentially stable. Then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p which for some positive constants a_p, b_p, and c_p satisfies

$$a_p|x|^2 \leq V_p(x) \leq b_p|x|^2 \tag{62}$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -c_p|x|^2. \tag{63}$$

Combining (62) and (63), we obtain

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -2\lambda_p V_p(x), \qquad p \in \mathcal{P}$$

where

$$\lambda_p := \frac{c_p}{2b_p}, \qquad p \in \mathcal{P}.$$

This implies that

$$V_p(x(t_0+\tau_d)) \le e^{-2\lambda_p\tau_d} V_p(x(t_0))$$

provided that $\sigma(t)=p$ for $t\in[t_0,t_0+\tau_d)$.

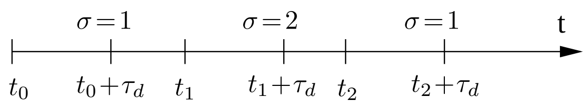

Fig. 16: A dwell-time switching signal

To simplify the next calculation, let us consider the case when $\mathcal{P}=\{1,2\}$ and σ takes the value 1 on $[t_0,t_1)$ and 2 on $[t_1,t_2)$, where $t_{i+1}-t_i\ge\tau_d$, $i=0,1$ (see Figure 16). From the above inequalities we have

$$V_2(t_1) \le \frac{b_2}{a_1}V_1(t_1) \le \frac{b_2}{a_1}e^{-2\lambda_1\tau_d}V_1(t_0)$$

and furthermore

$$V_1(t_2) \le \frac{b_1}{a_2}V_2(t_2) \le \frac{b_1}{a_2}e^{-2\lambda_2\tau_d}V_2(t_1) \le \frac{b_1b_2}{a_1a_2}e^{-2(\lambda_1+\lambda_2)\tau_d}V_1(t_0). \tag{64}$$

It is now straightforward to compute an explicit lower bound on τ_d which guarantees that the hypotheses of Theorem 5.1 are satisfied, implying that the switched system (3) is globally asymptotically stable. In fact, it is sufficient to ensure that

$$V_1(t_2)-V_1(t_0) \le -\gamma|x(t_0)|^2$$

for some $\gamma>0$. In view of (64), this will be true if we have

$$\left(\frac{b_1b_2}{a_1a_2}e^{-2(\lambda_1+\lambda_2)\tau_d}-1\right)V_1(t_0) \le -\gamma|x(t_0)|^2.$$

This will in turn hold, by virtue of (62), if

$$\left(\frac{b_1b_2}{a_1a_2}e^{-2(\lambda_1+\lambda_2)\tau_d}-1\right)a_1 \le -\gamma.$$

Since γ can be an arbitrary positive number, all we need to have is

$$\frac{b_1b_2}{a_2}e^{-2(\lambda_1+\lambda_2)\tau_d} < a_1$$

which can be equivalently rewritten as

$$-2(\lambda_1+\lambda_2)\tau_d < \log\frac{a_1a_2}{b_1b_2}$$

or finally as

$$\tau_d > \frac{1}{2(\lambda_1+\lambda_2)}\log\frac{b_1b_2}{a_1a_2}. \tag{65}$$

This is a desired lower bound on the dwell time.

We do not discuss possible extensions and refinements here because a more general result will be presented below. Note, however, that the above reasoning would still be valid if the quadratic estimates in (62) and (63) were replaced by, say, quartic ones. In essence, all we used was the fact that there exists a positive constant μ such that

$$V_p(x) \le \mu V_q(x) \qquad \forall\, x\in\mathbb{R}^n, \quad \forall\, p,q\in\mathcal{P}. \tag{66}$$

If this inequality does not hold globally in the state space for any $\mu>0$, then only local asymptotic stability can be established.

8.2 Average dwell time

In the context of controlled switching, specifying a dwell time may be too restrictive. If, after a switch occurs, there can be no more switches for the next τ_d units of time, then it is impossible to react to possible system failures during that time interval. When the purpose of switching is to choose the subsystem whose behavior is the best according to some performance criterion, as is often the case, there are no guarantees that the performance of the currently active subsystem will not deteriorate to an unacceptable level before the next switch is permitted. Thus it is of interest to relax the concept of dwell time, allowing the possibility of switching fast when necessary and then compensating for it by switching sufficiently slowly later.

The concept of average dwell time from [9] serves this purpose. Let us denote the number of discontinuities of a switching signal σ on an interval (t, T) by $N_\sigma(T, t)$. We say that σ has *average dwell time* τ_a if there exist two positive numbers N_0 and τ_a such that

$$N_\sigma(T,t) \le N_0 + \frac{T-t}{\tau_a} \qquad \forall\, T \ge t \ge 0. \tag{67}$$

For example, if $N_0 = 1$, then (67) implies that σ cannot switch twice on any interval of length smaller than τ_a. Switching signals with this property are exactly the switching signals with dwell time τ_a. Note also that $N_0 = 0$ corresponds to the case of no switching, since σ cannot switch at all on any interval of length smaller than τ_a. In general, if we discard the first N_0 switches (more precisely, the smallest integer greater than N_0), then the average time between consecutive switches is at least τ_a.

Besides being a natural extension of dwell time, the notion of average dwell time turns out to be very useful for analysis of the switching control algorithms to be studied later. Our present goal is to show that the property discussed earlier—namely, that asymptotic stability is preserved under switching with a sufficiently large dwell time—extends to switching signals with average dwell time. Although we cannot apply Theorem 5.1 directly to establish this result, the idea behind the proof is similar.

Exercise 5.2 Redo Exercise 5.1, this time working with the number of switches instead of assuming a fixed dwell time. Your answer should be of the form (67).

Theorem 5.2 *Consider the family of systems* (1)*. Suppose that there exist* $\mathcal{C}^1$ *functions* $V_p : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}$, *two class* $\mathcal{K}_\infty$ *functions* α_1 *and* α_2, *and a positive number* λ_0 *such that we have*

$$\alpha_1(|x|) \le V_p(x) \le \alpha_2(|x|) \qquad \forall\, x, \quad \forall\, p \in \mathcal{P} \tag{68}$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \le -2\lambda_0 V_p(x) \qquad \forall\, x, \quad \forall\, p \in \mathcal{P}. \tag{69}$$

Suppose also that (66) *holds. Then the switched system* (3) *is globally asymptotically stable for every switching signal* σ *with average dwell time*[7]

$$\tau_a > \frac{\log \mu}{2\lambda_0} \tag{70}$$

(and N_0 *arbitrary).*

[7] Note that $\log \mu > 0$ because $\mu > 1$ in view of the interchangeability of p and q in (66).

Let us examine the hypotheses of this theorem. If all systems in the family (1) are globally asymptotically stable, then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p which for all x satisfies

$$\alpha_{1,p}(|x|) \leq V_p(x) \leq \alpha_{2,p}(|x|)$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -W_p(x) \tag{71}$$

where W_p is positive definite. It is known (although nontrivial to prove) that there is no loss of generality in taking $W_p(x) = 2\lambda_p V_p(x)$ for some $\lambda_p > 0$, modifying V_p if necessary. Moreover, if $\mathcal{P}$ is a finite set or if it is compact and appropriate continuity assumptions are made, then we may choose functions α_1, α_2 and a constant λ_0, independent of p, such that the inequalities (68) and (69) hold. Thus the only really restrictive assumption is (66). It does not hold, for example, if V_p is quadratic for one value of p and quartic for another. If the systems (1) are globally exponentially stable, then the functions V_p, $p \in \mathcal{P}$ can be taken to be quadratic with quadratic decay rates, so that all hypotheses are verified.

Proof of Theorem 5.2. Pick an arbitrary $T > 0$, let $t_0 := 0$, and denote the switching times on the interval $(0, T)$ by $t_1, \ldots, t_{N_\sigma(T,0)}$. Consider the function

$$W(t) := e^{2\lambda_0 t} V_{\sigma(t)}(x(t)).$$

This function is piecewise differentiable along solutions of (3). On each interval $[t_i, t_{i+1})$ we have

$$\dot{W} = 2\lambda_0 W + e^{2\lambda_0 t} \frac{\partial V_{\sigma(t_i)}}{\partial x} f_{\sigma(t_i)}(x)$$

and this is nonpositive by virtue of (69), i.e., W is nonincreasing between the switching times. This together with (66) implies that

$$\begin{aligned} W(t_{i+1}) &= e^{2\lambda_0 t_{i+1}} V_{\sigma(t_{i+1})}(x(t_{i+1})) \leq \mu e^{2\lambda_0 t_{i+1}} V_{\sigma(t_i)}(x(t_{i+1})) \\ &= \mu W(t_{i+1}^-) \leq \mu W(t_i). \end{aligned}$$

Iterating this inequality from $i = 0$ to $i = N_\sigma(T, 0) - 1$, we have

$$W(T^-) \leq W(t_{N_\sigma(T,0)}) \leq \mu^{N_\sigma(T,0)} W(0).$$

It then follows from the definition of W that

$$e^{2\lambda_0 T} V_{\sigma(T^-)}(x(T)) \leq \mu^{N_\sigma(T,0)} V_{\sigma(0)}(x(0)). \tag{72}$$

Now suppose that σ has the average dwell time property expressed by the inequality (67). Then we can rewrite (72) as

$$\begin{aligned} V_{\sigma(T^-)}(x(T)) &\leq e^{-2\lambda_0 T + (N_0 + \frac{T}{\tau_a}) \log \mu} V_{\sigma(0)}(x(0)) \\ &= e^{N_0 \log \mu} e^{(\frac{\log \mu}{\tau_a} - 2\lambda_0) T} V_{\sigma(0)}(x(0)). \end{aligned}$$

We conclude that if τ_a satisfies the bound (70), then $V_{\sigma(T^-)}(x(T))$ converges to zero exponentially as $T \to \infty$; namely, it is upper-bounded by $\mu^{N_0} e^{-2\lambda T} V_{\sigma(0)}(x(0))$ for some $\lambda \in (0, \lambda_0)$. Using (68), we have $|x(T)| \leq \alpha_1^{-1}(\mu^{N_0} e^{-2\lambda T} \alpha_2(|x(0)|))$, which proves global asymptotic stability. □

Remark 5.2 Similarly to the way uniform stability over all switching signals was defined in Section 4.1, we can define uniform stability properties over switching signals from a certain class. Since the above argument gives the same conclusion for every switching signal with average dwell time satisfying the inequality (70), we see that under the assumptions of Theorem 5.2 the switched system (3) is GUAS in this sense over all such switching signals. (The switched system is GUES if the functions α_1 and α_2 are monomials of the same degree, e.g., quadratic; cf. Remark 3.1.) □

Remark 5.3 It is clear from the proof of Theorem 5.2 that exponential convergence of V_σ at the rate 2λ for an arbitrary $\lambda \in (0, \lambda_0)$ can be achieved by requiring that

$$\tau_a \geq \frac{\log \mu}{2(\lambda_0 - \lambda)}.$$

When the subsystems are linear, we can take the Lyapunov functions V_p, $p \in \mathcal{P}$ to be quadratic, and λ_0 corresponds to a common lower bound on stability margins of the individual subsystems. Thus the exponential decay rate λ for the switched linear system can be made arbitrarily close to the smallest one among the linear subsystems if the average dwell time is restricted to be sufficiently large. It is instructive to compare this with Exercise 5.2. □

The constant N_0 affects the overshoot bound for Lyapunov stability but otherwise does not change stability properties of the switched system. Also note that in the above stability proof we only used the bound on the number of switches on an interval starting at the initial time. The formula (67) provides a bound on the number of switches—and consequently a uniform decay bound for the state—on every interval, not necessarily of this form. For linear systems, this property guarantees that various induced norms of the switched system in the presence of inputs are finite.

9 Stability under state-dependent switching

In the previous section we studied stability of switched systems under time-dependent switching satisfying suitable constraints. Another example of constrained switching is state-dependent switching, where a switching event can occur only when the trajectory crosses a switching surface (see Section 1.1). In this case, stability analysis is often facilitated by the fact that properties of each individual subsystem are of concern only in the regions where this system is active, and the behavior of this system in other parts of the state space has no influence on the switched system.

Example 5.1 Consider the 2×2 matrices

$$A_1 := \begin{pmatrix} \gamma & -1 \\ 2 & \gamma \end{pmatrix}, \qquad A_2 := \begin{pmatrix} \gamma & -2 \\ 1 & \gamma \end{pmatrix} \tag{73}$$

where γ is a negative number sufficiently close to zero, so that the trajectories of the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ look, at least qualitatively, as depicted on the first two plots in Figure 8 on page 13. Now, define a state-dependent switched linear system in the plane by

$$\dot{x} = \begin{cases} A_1 x & \text{if } \; x_1 x_2 \leq 0 \\ A_2 x & \text{if } \; x_1 x_2 > 0\,. \end{cases} \tag{74}$$

It is easy to check that the function $V(x) := x^T x$ satisfies $\dot{V} < 0$ along all nonzero solutions of this switched system, hence we have global asymptotic stability. The trajectories of (74) look approximately as shown on the third plot in Figure 8.

For the above argument to apply, the individual subsystems do not even need to be asymptotically stable. Again, this is because the Lyapunov function only needs to decrease along solutions of each subsystem in an appropriate region, and not necessarily everywhere. If we set $\gamma = 0$, then V still decreases along all nonzero solutions of the switched system (74). From a perturbation argument it is clear that if γ is a sufficiently small positive number, then (74) is still globally asymptotically stable, even though the individual subsystems are unstable. (For one idea about how to prove asymptotic stability directly in the latter case, see Remark 5.1 in Section 7).

It is important to note that V serves as a Lyapunov function only in suitable regions for each subsystem. In fact, there is no global common Lyapunov function for the two subsystems. Indeed, if one changes the switching rule to

$$\dot{x} = \begin{cases} A_1 x & \text{if } \ x_1 x_2 > 0 \\ A_2 x & \text{if } \ x_1 x_2 \le 0 \end{cases}$$

then the resulting switched system is unstable (cf. the last plot in Figure 8). □

Observe that the state-dependent switching strategies considered in the above example can be converted to time-dependent ones, because the time needed for a linear time-invariant system to cross a quadrant can be explicitly calculated and is independent of the trajectory. This remark applies to most of the switched systems considered below.

If a stability analysis based on a single Lyapunov function breaks down, one can use multiple Lyapunov functions. The following modification of the previous example illustrates this method.

Example 5.2 Let us use the same matrices (73), with γ negative but close to zero, to define a different state-dependent switched linear system, namely,

$$\dot{x} = \begin{cases} A_1 x & \text{if } \ x_1 \ge 0 \\ A_2 x & \text{if } \ x_1 < 0 \,. \end{cases}$$

We know that the linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a quadratic common Lyapunov function. Moreover, it is also impossible to find a single quadratic function that, as in Example 5.1, decreases along solutions of each subsystem in the corresponding region. Indeed, since each region is a half-plane, by symmetry this would give a quadratic common Lyapunov function.

However, consider the two positive definite symmetric matrices

$$P_1 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The functions $V_1(x) := x^T P_1 x$ and $V_2(x) := x^T P_2 x$ are Lyapunov functions for the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$, respectively. Moreover, on the switching surface $\{x : x_1 = 0\}$ their values match. Thus the function V_σ, where σ is the switching signal taking the value 1 for $x_1 \ge 0$ and 2 for $x_1 < 0$, is continuous along solutions of the switched system and behaves as in Figure 14(a). This proves global asymptotic stability. The level sets of the Lyapunov functions in the appropriate regions and a typical trajectory of the switched system are plotted in Figure 17. □

Recall that Theorem 5.1 provides a less conservative condition for asymptotic stability, in the sense that multiple Lyapunov functions are allowed to behave like in Figure 14(b). If in Example 5.2 we multiplied the matrices P_1 and P_2 by two arbitrary positive numbers, the resulting Lyapunov functions

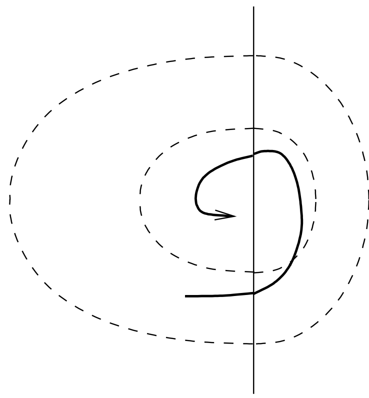

Fig. 17: Illustrating stability in Example 5.2

would still satisfy the hypotheses of Theorem 5.1. In general, however, it is more difficult to use that theorem when the values of Lyapunov functions do not coincide on the switching surfaces.

As before, there is in general no need to associate with each subsystem a global Lyapunov function. It is enough to require that each function V_p decrease along solutions of the pth subsystem in the region Ω_p where this system is active (or may be active, if the index of the active subsystem is not uniquely determined by the value of x). This leads to relaxed stability conditions. For the case of a switched linear system and quadratic Lyapunov functions $V_p(x) = x^T P_p x\,,\ p \in \mathcal{P}$ these conditions can be brought to a computationally tractable form. This is achieved by means of the following well-known result, which will also be useful later.

Lemma 5.3 ("S-procedure") *Let T_0 and T_1 be two symmetric matrices. Consider the following two conditions:*

$$x^T T_0 x > 0 \text{ whenever } x^T T_1 x \geq 0 \text{ and } x \neq 0 \tag{75}$$

and

$$\exists \beta \geq 0 \text{ such that } T_0 - \beta T_1 > 0. \tag{76}$$

Condition (76) *always implies condition* (75). *If there is some x_0 such that $x_0^T T_1 x_0 > 0$, then* (75) *implies* (76).

Suppose that there exist symmetric matrices $S_p\,,\ p \in \mathcal{P}$ such that $\Omega_p \subset \{x : x^T S_p x \geq 0\}$ for all $p \in \mathcal{P}$. This means that each operating region Ω_p is embedded in a conic region. Then the S-procedure allows us to replace the condition

$$x^T (A_p^T P_p + P_p A_p) x < 0 \qquad \forall\, x \in \Omega_p \setminus \{0\}$$

by the linear matrix inequality

$$A_p^T P_p + P_p A_p + \beta_p S_p < 0, \qquad \beta_p \geq 0.$$

We also need to restrict the search for Lyapunov functions to ensure their continuity across the switching surfaces. If the boundary between Ω_p and Ω_q is of the form $\{x : f_{pq}^T x = 0\}$, $f_{pq} \in \mathbb{R}^n$, then we must have $P_p - P_q = f_{pq} t_{pq}^T + t_{pq} f_{pq}^T$ for some $t_{pq} \in \mathbb{R}^n$.

One can further reduce conservatism by considering several Lyapunov functions for each subsystem. In other words, one can introduce further partitioning of the regions $\Omega_p\,,\ p \in \mathcal{P}$ and assign a function to each of the resulting regions. Stability conditions will be of the same form as before; we will simply have more regions, with groups of regions corresponding to the same dynamics. This provides greater flexibility in treating multiple subsystems and switching surfaces—especially in the presence of unstable subsystems—although the complexity of the required computations also grows.

10 Stabilization by state-dependent switching

In the previous section we discussed the problem of verifying stability of a given state-dependent switched linear system. In this section we study a related problem: given a family of linear systems, specify a state-dependent switching rule that makes the resulting switched linear system asymptotically stable. Of course, if at least one of the individual subsystems is asymptotically stable, this problem is trivial (just keep $\sigma(t) \equiv p$, where p is the index of this asymptotically stable subsystem). Therefore, in the present context it will be understood that none of the individual subsystems is asymptotically stable.

10.1 Stable convex combinations

Suppose that $\mathcal{P} = \{1, 2\}$ and that the individual subsystems are linear: $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$. As demonstrated in [10], one assumption that leads to an elegant construction of a stabilizing switching signal in this case is the following one:

ASSUMPTION 1. There exists an $\alpha \in (0, 1)$ such that the convex combination

$$A := \alpha A_1 + (1 - \alpha) A_2 \tag{77}$$

is Hurwitz. (The endpoints $\alpha = 0$ and $\alpha = 1$ are excluded because A_1 and A_2 are not Hurwitz.)

We know that in this case the switched system can be stabilized by fast switching designed so as to approximate the behavior of $\dot{x} = Ax$ (see Section 4.3). The procedure presented below allows one to avoid fast switching and is somewhat more systematic.

Under Assumption 1, there exist positive definite symmetric matrices P and Q which satisfy

$$A^T P + PA = -Q. \tag{78}$$

Using (77), we can rewrite (78) as

$$\alpha(A_1^T P + PA_1) + (1 - \alpha)(A_2^T P + PA_2) = -Q$$

which is equivalent to

$$\alpha x^T (A_1^T P + PA_1)x + (1 - \alpha)x^T (A_2^T P + PA_2)x = -x^T Qx < 0 \qquad \forall\, x \neq 0.$$

This implies that for every nonzero x we have either $x^T(A_1^T P + PA_1)x < 0$ or $x^T(A_2^T P + PA_2)x < 0$.

Let us define two regions

$$\Omega_i := \{x : x^T (A_i^T P + PA_i)x < 0\}, \qquad i = 1, 2.$$

These are open conic regions ($x \in \Omega_i \;\Rightarrow\; \lambda x \in \Omega_i \;\forall\, \lambda \in \mathbb{R}$) which overlap and together cover $\mathbb{R}^n \setminus \{0\}$. It is now clear that we want to orchestrate the switching in such a way that the system $\dot{x} = A_i x$ is active in the region Ω_i, because this will make the function $V(x) := x^T Px$ decrease along solutions.

In implementing the above idea, we will pay special attention to two issues. First, we would like to have a positive lower bound on the rate of decrease of V. This will be achieved by means of modifying the original regions Ω_1, Ω_2. Second, we want to avoid chattering on the boundaries of the regions. This will be achieved with the help of hysteresis (cf. Section 2.4). We now describe the details of this construction.

Pick two new open conic regions Ω_i', $i = 1, 2$ such that each Ω_i' is a strict subset of Ω_i and we still have $\Omega_1' \cup \Omega_2' = \mathbb{R}^n \setminus \{0\}$. The understanding here is that each Ω_i' is obtained from Ω_i by a small amount of shrinking. Then the number

$$\varepsilon_i := -\max_{x \in \operatorname{cl} \Omega_i',\ |x|=1} x^T(A_i^T P + PA_i)x$$

is well defined and positive for each $i \in \{1, 2\}$, where "cl" denotes the closure of a set. Choosing a positive number $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, we obtain

$$x^T(A_i^T P + PA_i)x < -\varepsilon|x|^2 \qquad \forall\, x \in \Omega_i', \quad i = 1, 2.$$

A hysteresis-based stabilizing switching strategy can be described as follows. Let $\sigma(0) = 1$ if $x(0) \in \Omega_1'$ and $\sigma(0) = 2$ otherwise. For each $t > 0$, if $\sigma(t^-) = i \in \{1, 2\}$ and $x(t) \in \Omega_i'$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = 1$ but $x(t) \notin \Omega_1'$, let $\sigma(t) = 2$. Similarly, if $\sigma(t^-) = 2$ but $x(t) \notin \Omega_2'$, let $\sigma(t) = 1$. Thus σ changes its value when the trajectory leaves one of the regions, and the next switch can occur only when the trajectory leaves the other region after having traversed the intersection $\Omega_1' \cap \Omega_2'$. This situation is illustrated in Figure 18.

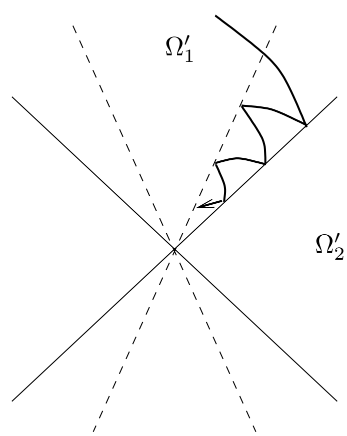

Fig. 18: Conic regions and a possible trajectory (the boundary of Ω_1' is shown by solid lines and the boundary of Ω_2' is shown by dashed lines)

The above discussion implies that the derivative of V along the solutions of the resulting state-dependent switched linear system satisfies

$$\frac{d}{dt} x^T Px < -\varepsilon|x|^2 \qquad \forall\, x \neq 0. \tag{79}$$

This property is known as *quadratic stability* and is in general stronger than just global asymptotic stability, even for switched linear systems (see Example 5.3 below). We arrive at the following result.

Theorem 5.4 *If the matrices A_1 and A_2 have a Hurwitz convex combination, then there exists a state-dependent switching strategy that makes the switched linear system* (4) *with $\mathcal{P} = \{1, 2\}$ quadratically stable.*

Exercise 5.3 Construct an example of two unstable 2×2 matrices with a Hurwitz convex combination and implement the above procedure via computer simulation.

When the number of individual subsystems is greater than 2, one can try to single out from the corresponding set of matrices a pair that has a Hurwitz convex combination. If that fails, it might be possible to find a Hurwitz convex combination of three or more matrices from the given set, and then the above method for constructing a stabilizing switching signal can be applied with minor modifications.

A converse result

An interesting observation made in [11] is that Assumption 1 is not only sufficient but also necessary for quadratic stabilizability via state-dependent switching. This means that we cannot hope to achieve quadratic stability unless a given pair of matrices has a Hurwitz convex combination.

Theorem 5.5 *If there exists a state-dependent switching strategy that makes the switched linear system* (4) *with* $\mathcal{P}=\{1,2\}$ *quadratically stable, then the matrices* A_1 *and* A_2 *have a Hurwitz convex combination.*

Proof. Suppose that the switched linear system is quadratically stable, i.e., there exists a Lyapunov function $V(x)=x^TPx$ whose derivative along solutions of the switched system satisfies $\dot V<-\varepsilon|x|^2$ for some $\varepsilon>0$. Since the switching is state-dependent, this implies that for every nonzero x we must have either $x^T(A_1^TP+PA_1)x<-\varepsilon|x|^2$ or $x^T(A_2^TP+PA_2)x<-\varepsilon|x|^2$. We can restate this as follows:

$$x^T(-A_1^TP-PA_1-\varepsilon I)x>0 \quad\text{whenever}\quad x^T(A_2^TP+PA_2+\varepsilon I)x\ge 0 \tag{80}$$

and

$$x^T(-A_2^TP-PA_2-\varepsilon I)x>0 \quad\text{whenever}\quad x^T(A_1^TP+PA_1+\varepsilon I)x\ge 0. \tag{81}$$

If $x^T(A_1^TP+PA_1+\varepsilon I)x\le 0$ for all $x\ne 0$, then the matrix A_1 is Hurwitz and there is nothing to prove. Similarly, if $x^T(A_2^TP+PA_2+\varepsilon I)x\le 0$ for all $x\ne 0$, then A_2 is Hurwitz. Discarding these trivial cases, we can apply the S-procedure (Lemma 5.3) to one of the last two conditions, say, to (80), and conclude that for some $\beta\ge 0$ we have

$$A_1^TP+PA_1+\beta(A_2^TP+PA_2)<-(1+\beta)\varepsilon I$$

or, equivalently,

$$\frac{(A_1+\beta A_2)^T}{1+\beta}P+P\frac{(A_1+\beta A_2)}{1+\beta}<-\varepsilon I.$$

Therefore, the matrix $(A_1+\beta A_2)/(1+\beta)$, which is a convex combination of A_1 and A_2, is Hurwitz, and so Assumption 1 is satisfied. □

We emphasize that the above result is limited to two linear subsystems, state-dependent switching signals, and quadratic stability.

10.2 Unstable convex combinations

A given pair of matrices may not possess a Hurwitz convex combination, in which case we know from Theorem 5.5 that quadratic stabilization is impossible. Also note that even if Assumption 1 is satisfied, in order to apply the procedure of Section 10.1 we need to identify a Hurwitz convex combination explicitly, which is a nontrivial task (in fact, this problem is known to be NP-hard). However, global asymptotic stabilization via state-dependent switching may still be possible even if no Hurwitz convex combination can be found.

Example 5.3 Consider the matrices

$$A_1 := \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Define a two-dimensional state-dependent switched linear system by the rule

$$\dot{x} = \begin{cases} A_1 x & \text{if } \ x_1 x_2 \leq 0 \\ A_2 x & \text{if } \ x_1 x_2 > 0\,. \end{cases}$$

This is the system of Example 5.1 with $\gamma = 0$. The trajectories of the individual subsystems and the switched system are shown in Figure 19.

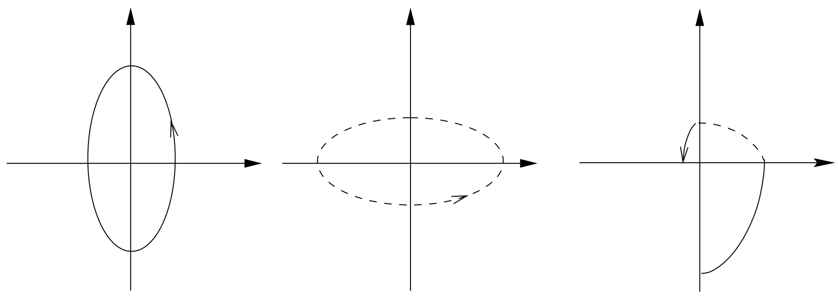

Fig. 19: Switching between critically stable systems

It is not hard to see that this switched system is globally asymptotically stable. For example, the derivative of the function $V(x) := x^T x$ along solutions is negative away from the coordinate axes, i.e., we have $\dot{V} < 0$ when $x_1 x_2 \neq 0$. Moreover, the smallest invariant set contained in the union of the coordinate axes is the origin, thus global asymptotic stability follows from LaSalle's invariance principle (Section 3.4).

Since the derivative of V vanishes for some nonzero x, we do not have quadratic stability. More precisely, the inequality (79) cannot be satisfied with any $P = P^T > 0$ and $\varepsilon > 0$, and actually with any other choice of a state-dependent switching signal either. This follows from Theorem 5.5 because all convex combinations of the matrices A_1 and A_2 have purely imaginary eigenvalues.

The switching law used to asymptotically stabilize the switched system in this example is a special case of what is called a *conic switching law.* The switching occurs on the lines where the two vector fields are collinear, and the active subsystem is always the one whose vector field points inwards relative to the other. The system is globally asymptotically stable because the distance to the origin decreases after each rotation. No other switching signal would lead to better convergence.

It is interesting to draw a comparison with the system considered in Section 4.5. There, the opposite switching strategy was considered, whereby the active subsystem is always the one whose vector field points *outwards* relative to the other. If this does not destabilize the system, then no other switching signal will.

In the above example, both subsystems rotate in the same direction. It is possible to apply similar switching laws to planar subsystems rotating in opposite directions. This may produce different types of trajectories, such as the one shown in Figure 18. □

Multiple Lyapunov functions

Both in Section 10.1 and in Example 5.3, the stability analysis was carried out with the help of a single Lyapunov function. When this does not seem possible, in view of the results presented in Sections 7 and 9 one can try to find a stabilizing switching signal and prove stability by using multiple Lyapunov functions.

Suppose again that we are switching between two linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$. Associate with the first system a function $V_1(x) = x^T P_1 x$, with $P_1 = P_1^T > 0$, which decreases along its solutions in a nonempty conic region Ω_1. It can be shown that this is always possible unless A_1 is a nonnegative multiple of the identity matrix. Similarly, associate with the second system a function $V_2(x) = x^T P_2 x$, with $P_2 = P_2^T > 0$, which decreases along its solutions in a nonempty conic region Ω_2. If the union of the regions Ω_1 and Ω_2 covers $\mathbb{R}^n \setminus \{0\}$, then one can try to orchestrate the switching in such a way that the conditions of Theorem 5.1 are satisfied.

Using the ideas discussed in Section 9, one can derive algebraic conditions (in the form of bilinear matrix inequalities) under which such a stabilizing switching signal exists and can be constructed explicitly. Suppose that the following condition holds:

Condition 1. We have

$$x^T(A_1^T P_1 + P_1 A_1)x < 0 \quad \text{whenever} \quad x^T P_1 x \le x^T P_2 x \quad \text{and} \quad x \neq 0$$

and

$$x^T(A_2^T P_2 + P_2 A_2)x < 0 \quad \text{whenever} \quad x^T P_1 x \ge x^T P_2 x \quad \text{and} \quad x \neq 0.$$

If this condition is satisfied, then a stabilizing switching signal can be defined by

$$\sigma(t) := \arg\min\{V_i(x(t)) : i = 1, 2\}.$$

Indeed, let us first suppose that no sliding motion occurs on the switching surface $\mathcal{S} := \{x : x^T P_1 x = x^T P_2 x\}$. Then the function V_σ is continuous and decreases along solutions of the switched system, which guarantees global asymptotic stability. The existence of a sliding mode, on the other hand, is easily seen to be characterized by the inequalities

$$x^T\big(A_1^T(P_1 - P_2) + (P_1 - P_2)A_1\big)x \ge 0$$

and

$$x^T\big(A_2^T(P_1 - P_2) + (P_1 - P_2)A_2\big)x \le 0 \tag{82}$$

for $x \in \mathcal{S}$. If a sliding motion occurs on $\mathcal{S}$, then σ is not uniquely defined, so we let $\sigma = 1$ on $\mathcal{S}$ without loss of generality. Let us show that V_1 decreases along the corresponding Filippov solution. For every $\alpha \in (0, 1)$, we have

$$\begin{aligned}
&x^T\Big(\big(\alpha A_1 + (1-\alpha)A_2\big)^T P_1 + P_1\big(\alpha A_1 + (1-\alpha)A_2\big)\Big)x \\
&\quad = \alpha x^T(A_1^T P_1 + P_1 A_1)x + (1-\alpha)x^T(A_2^T P_1 + P_1 A_2)x \\
&\quad \le \alpha x^T(A_1^T P_1 + P_1 A_1)x + (1-\alpha)x^T(A_2^T P_2 + P_2 A_2)x < 0
\end{aligned}$$

where the first inequality follows from (82) while the last one follows from Condition 1. Therefore, the switched system is still globally asymptotically stable.

Condition 1 holds if the following condition is satisfied (by virtue of the S-procedure, the two conditions are equivalent, provided that there exist $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T(P_2 - P_1)x_1 > 0$ and $x_2^T(P_1 - P_2)x_2 > 0$):

Condition 2. There exist $\beta_1, \beta_2 \ge 0$ such that we have

$$-A_1^T P_1 - P_1 A_1 - \beta_1(P_2 - P_1) > 0 \tag{83}$$

and

$$-A_2^T P_2 - P_2 A_2 - \beta_2(P_1 - P_2) > 0. \tag{84}$$

The problem of finding a stabilizing switching signal can thus be reduced to finding two positive definite matrices P_1 and P_2 such that the above inequalities are satisfied. Similarly, if $\beta_1, \beta_2 \leq 0$, then a stabilizing switching signal can be defined by

$$\sigma(t) := \arg\max\{V_i(x(t)) : i = 1, 2\}.$$

A somewhat surprising difference, however, is that this alternative approach does not guarantee stability of a sliding mode, so sliding modes need to be ruled out.

Switched systems with inputs and outputs

This lecture is based on powerpoint slides, file name: `ISS-invertibility`.

More information can be found in these references: [12, 13, 14].

Lyapunov functions for switched systems: computational aspects; Brockett's condition and nonholonomic systems

The first part of this lecture is based on powerpoint slides, file name: `gradient`. The reference for this part is [15].

The second part of this lecture is based on the material given below. See also powerpoint slides, file name: `parking`.

The remaining lectures are devoted to *switching control*. This material is motivated primarily by problems of the following kind: given a process, typically described by a continuous-time control system, find a controller such that the closed-loop system displays a desired behavior. In some cases, this can be achieved by applying a continuous static or dynamic feedback control law. In other cases, a continuous feedback law that solves the problem may not exist. A possible alternative in such situations is to incorporate logic-based decisions into the control law and implement switching among a family of controllers. This yields a switched (or hybrid) closed-loop system, shown schematically in Figure 20.

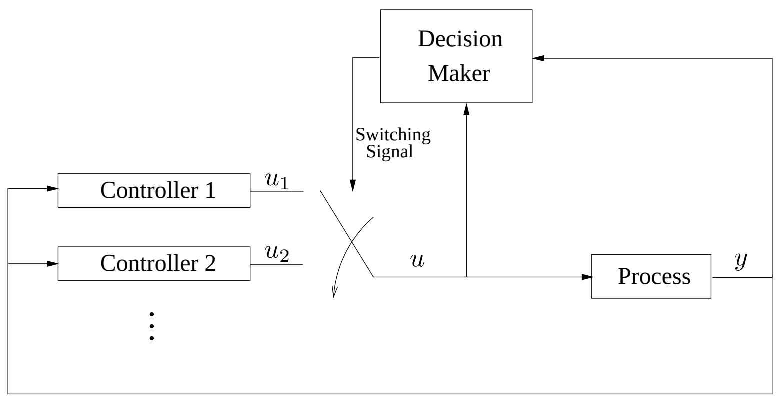

Fig. 20: Switching control

We single out the following categories of control problems for which one might want—or need—to consider switching control (of course, combinations of two or more of these are also possible):

1. Due to the nature of the problem itself, continuous control is not suitable.
2. Due to sensor and/or actuator limitations, continuous control cannot be implemented.
3. The model of the system is highly uncertain, and a single continuous control law cannot be found.

There are actually several different scenarios that fit into the first category. If the given process is prone to unpredictable environmental influences or component failures, then it may be necessary to consider logic-based mechanisms for detecting such events and providing counteractions. If the desired system trajectory is composed of several pieces of significantly different types (e.g., aircraft maneuvers), then one might need to employ different controllers at different stages. The need for logic-based decisions also arises when the state space of the process contains obstacles. Perhaps more interestingly, there exist systems that are smooth and defined on spaces with no obstacles (e.g., $\mathbb{R}^n$) yet do not admit continuous feedback laws for tasks as basic as asymptotic stabilization. In other words, an obstruction to continuous stabilization may come from the mathematics of the system itself. A well-known class of such systems is given by nonholonomic control systems.

The second of the categories mentioned above also encompasses several different classes of problems. The simplest example of an actuator limitation is when the control input is bounded, e.g., due to saturation. It is well known that optimal control of such systems involves switching (bang-bang) control. Control using output feedback, when the number of outputs is smaller than the number of states, can be viewed as control under sensor limitations. Typically, stabilization by a static output feedback is not possible, while implementing a dynamic output feedback may be undesirable. On the other hand, a simple switching control strategy can sometimes provide an effective solution to the output feedback stabilization problem. Perhaps more interestingly, a switched system naturally arises if the process dynamics are continuous-time but information is communicated only at discrete instants of time or over a finite-bandwidth channel, or if event-driven actuators are used. Thus we view switching control as a natural set of tools that can be applied to systems with sensor and actuator constraints.

The third category includes problems of controlling systems with large modeling uncertainty. As an alternative to traditional adaptive control, where controller selection is achieved by means of continuous tuning, it is possible to carry out the controller selection procedure with the help of logic-based switching among a family of control laws. This latter approach turns out to have some advantages over more conventional adaptive control algorithms, having to do with modularity of the design, simplicity of the analysis, and wider applicability.

11 Obstructions to continuous stabilization

Some systems cannot be globally asymptotically stabilized by smooth (or even continuous) feedback. This is not simply a lack of controllability. It might happen that, while every state can be steered to the origin by some control law, these control laws cannot be patched together in a continuous fashion to yield a globally defined stabilizing feedback. In this section we discuss how this situation can occur.

11.1 State-space obstacles

Consider a continuous-time system $\dot{x} = f(x)$ defined on some state space $\mathcal{X} \subset \mathbb{R}^n$. Assume that it has an asymptotically stable equilibrium, which we take to be the origin with no loss of generality. The region of attraction, which we denote by $\mathcal{D}$, must be a contractible set. This means that there exists a continuous mapping $H : [0,1] \times \mathcal{D} \to \mathcal{D}$ such that $H(0,x) = x$ and $H(1,x) \equiv 0$. The mapping H can be constructed in a natural way using the flow of the system.

One implication of the above result is that if the system is globally asymptotically stable, then its state space $\mathcal{X}$ must be contractible. Therefore, while there exist globally asymptotically stable systems on $\mathbb{R}^n$, no system on a circle can have a single globally asymptotically stable equilibrium.

Now, suppose that we are given a control system

$$\dot{x} = f(x,u), \qquad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m. \tag{85}$$

If a feedback law $u = k(x)$ is sufficiently regular (i.e., smooth or at least continuous) and renders the closed-loop system

$$\dot{x} = f(x, k(x)) \tag{86}$$

asymptotically stable, then the previous considerations apply to (86). In particular, global asymptotic stabilization is impossible if $\mathcal{X}$ is not a contractible space (for example, a circle).

An intuitive explanation of this fact can be given with the help of Figure 21. Solutions with initial conditions on the right side need to move clockwise, whereas solutions on the left side need to move

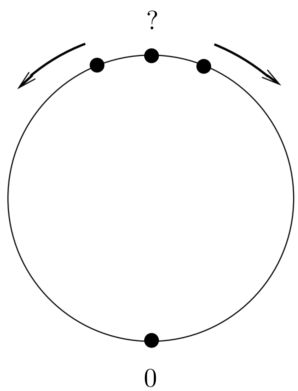

Fig. 21: A system on a circle: global continuous stabilization is impossible

counterclockwise. Since there can be no equilibria other than the origin, there must be a point on the circle at which the feedback law is discontinuous. We can say that at that point a logical decision (whether to move left or right) is necessary.

Instead of a system evolving on a circle, we may want to consider a system evolving on the plane with a circular (or similar) obstacle. The same argument shows that such a system cannot be continuously globally asymptotically stabilized. Indeed, it is intuitively clear that there is no globally defined continuous feedback strategy for approaching a point behind a table.

When a continuous feedback law cannot solve the stabilization problem, what are the alternative ways to stabilize the system? One option is to use static discontinuous feedback control. In the context of the above example, this means picking a feedback function k that is continuous everywhere except at one point. At this discontinuity point, we simply make an arbitrary decision, say, to move to the right.

One of the shortcomings of this solution is that it requires precise information about the state and so the resulting closed-loop system is highly sensitive to measurement errors. Namely, in the presence of small random measurement noise it may happen that, near the discontinuity point, we misjudge which side of this point we are currently on and start moving toward it instead of away from it. If this happens often enough, the solution will oscillate around the discontinuity point and may never reach the origin.

Using a different logic to define the control law, it is possible to achieve robustness with respect to measurement errors. For example, if we sample and hold each value of the control for a long enough period of time, then we are guaranteed to move sufficiently far away from the discontinuity point, where small errors will no longer cause a motion in the wrong direction. After that we can go back to the usual feedback implementation in order to ensure convergence to the equilibrium. This initial discussion illustrates potential advantages of using logic-based switching control algorithms.

11.2 Brockett's condition

The examples discussed above illustrate possible obstructions to global asymptotic stabilization which arise due to certain topological properties of the state space. We now present an important result which shows that even local asymptotic stabilization by continuous feedback is impossible for some systems. Since a sufficiently small neighborhood of an equilibrium point has the same topological properties as $\mathbb{R}^n$, this means that such an obstruction to continuous stabilization has nothing to do with the properties of the state space and is instead embedded into the system equations.

Theorem 9.1 (Brockett) *Consider the control system* (85) *with* $\mathcal{X} = \mathbb{R}^n$ *and* $\mathcal{U} = \mathbb{R}^m$*, and suppose that there exists a continuous*[8] *feedback law* $u = k(x)$ *satisfying* $k(0) = 0$ *which makes the origin a (locally) asymptotically stable equilibrium of the closed-loop system* (86)*. Then the image of every neighborhood of* $(0, 0)$ *in* $\mathbb{R}^n \times \mathbb{R}^m$ *under the map*

$$(x, u) \mapsto f(x, u) \tag{87}$$

contains some neighborhood of zero in $\mathbb{R}^n$*.*

Sketch of proof. If k is asymptotically stabilizing, then by a converse Lyapunov theorem there exists a Lyapunov function V which decreases along nonzero solutions of (86), locally in some neighborhood of zero. Consider the set $\mathcal{R} := \{x : V(x) \leq c\}$, where c is a sufficiently small positive number. Then the vector field $f(x, k(x))$ points inside $\mathcal{R}$ everywhere on its boundary. By compactness of $\mathcal{R}$, it is easy to see that the vector field $f(x, k(x)) - \xi$ also points inside on the boundary of $\mathcal{R}$, where $\xi \in \mathbb{R}^n$ is chosen so that $|\xi|$ is sufficiently small.

We can now apply a standard fixed point argument to show that we must have $f(x, k(x)) = \xi$ for some x in $\mathcal{R}$. Namely, suppose that the vector field $f(x, k(x)) - \xi$ is nonzero on $\mathcal{R}$. Then for every $x \in \mathcal{R}$ we can draw a ray in the direction opposite to the one provided by this vector field until it hits the boundary of $\mathcal{R}$ (see Figure 22). This yields a continuous map from $\mathcal{R}$ onto its boundary, and it is well known that such a map cannot exist. **Please note that this figure from the book is wrong: the arrows should run in the reverse directions.**

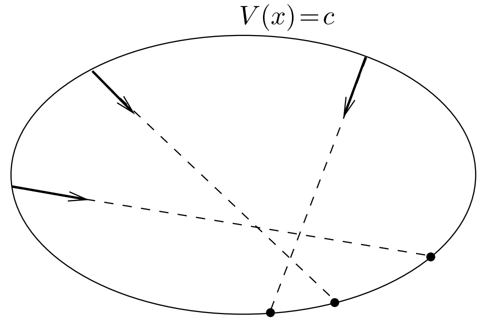

Fig. 22: Illustrating the proof of Theorem 9.1

We have thus shown that the equation $f(x, k(x)) = \xi$ can be solved for x in a given sufficiently small neighborhood of zero in $\mathbb{R}^n$, provided that $|\xi|$ is sufficiently small. In other words, the image of every neighborhood of zero under the map

$$x \mapsto f(x, k(x)) \tag{88}$$

contains a neighborhood of zero. Moreover, if x is small, then $k(x)$ is small in view of continuity of k and the fact that $k(0) = 0$. It follows that the image under the map (87) of every neighborhood of $(0, 0)$ contains a neighborhood of zero. □

Theorem 9.1 provides a very useful necessary condition for asymptotic stabilizability by continuous feedback. *Intuitively, it means that, starting near zero and applying small controls, we must be able to move in all directions.* (The statement that a set of admissible velocity vectors does not contain a neighborhood of zero means that there are some directions in which we cannot move, even by a small amount.) Note that this condition is formulated in terms of the original open-loop system. If the map (87) satisfies the hypothesis of the theorem, then it is said to be *open* at zero.

[8] To ensure uniqueness of solutions, we in principle need to impose stronger regularity assumptions, but we ignore this issue here because the result is valid without such additional assumptions.

Clearly, a system cannot be feedback stabilizable unless it is asymptotically open-loop controllable to the origin. It is important to keep in mind the difference between these two notions. The latter one says that, given an initial condition, we can find a control law that drives the state to the origin; the former property is stronger and means that there exists a feedback law that drives the state to the origin, regardless of the initial condition. In the next section we study an important class, as well as some specific examples, of controllable nonlinear systems which fail to satisfy Brockett's condition. Since Theorem 9.1 does not apply to switched systems, we will see that switching control provides an effective approach to the feedback stabilization problem for systems that are not stabilizable by continuous feedback.

Note that when the system (85) has a controllable (or at least stabilizable) linearization $\dot{x} = Ax+Bu$, it can be locally asymptotically stabilized (by linear feedback). However, controllability of a nonlinear system in general does not imply controllability of its linearization.

Exercise 9.1 Show directly that every controllable linear system satisfies Brockett's condition.

12 Nonholonomic systems

Consider the system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i = G(x)u, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad G \in \mathbb{R}^{n\times m}. \tag{89}$$

Systems of this form are known as (driftless, kinematic) *nonholonomic control systems*. Nonholonomy means that the system is subject to constraints involving both the state x (position) and its derivative $\dot{x}$ (velocity). Namely, since there are fewer control variables than state variables, the velocity vector $\dot{x}$ at each x is constrained to lie in the proper subspace of $\mathbb{R}^n$ spanned by the vectors $g_i(x)$, $i = 1, \dots, m$. Under the assumptions that $\operatorname{rank} G(0) = m$ and $m < n$, the system (89) violates Brockett's condition, and we have the following corollary of Theorem 9.1.

Corollary 9.2 *The system* (89) *with* $\operatorname{rank} G(0) = m < n$ *cannot be asymptotically stabilized by a continuous feedback law.*

PROOF. Rearrange coordinates so that G takes the form

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix}$$

where $G_1(x)$ is an $m \times m$ matrix which is nonsingular in some neighborhood N of zero. Then the image of $N \times \mathbb{R}^m$ under the map

$$(x, u) \mapsto G(x)u = \begin{pmatrix} G_1(x)u \\ G_2(x)u \end{pmatrix}$$

does not contain vectors of the form $\binom{0}{a}$, where $a \in \mathbb{R}^{n-m} \neq 0$. Indeed, if $G_1(x)u = 0$, then we have $u = 0$ since G_1 is nonsingular, and this implies $G_2(x)u = 0$. □

In the singular case, i.e., when $G(x)$ drops rank at 0, the above result does not hold. Thus the full-rank assumption imposed on G is essential for making the class of systems (89) interesting in the present context. Nonholonomic systems satisfying this assumption are called *nonsingular*.

Even though infinitesimally the state of the system (89) can only move along linear combinations of the m available control directions, it is possible to generate motions in other directions by a suitable choice of controls. For example, consider the following (switching) control strategy. First, starting at some x_0, move along the vector field g_1 for ε units of time (by setting $u_1 = 1$ and $u_i = 0$ for all $i \neq 1$). Then move along the vector field g_2 for ε units of time. Next, move along $-g_1$ for ε units of time ($u_1 = -1$, $u_i = 0$ for $i \neq 1$), and finally along $-g_2$ for ε units of time. It is straightforward (although quite tedious) to check that for small ε the resulting motion is approximated, up to the second order in ε, by $\varepsilon^2[g_1, g_2](x_0)$ (see Section 5.1 for the definition of the Lie bracket of two nonlinear vector fields). This situation is depicted in Figure 23.

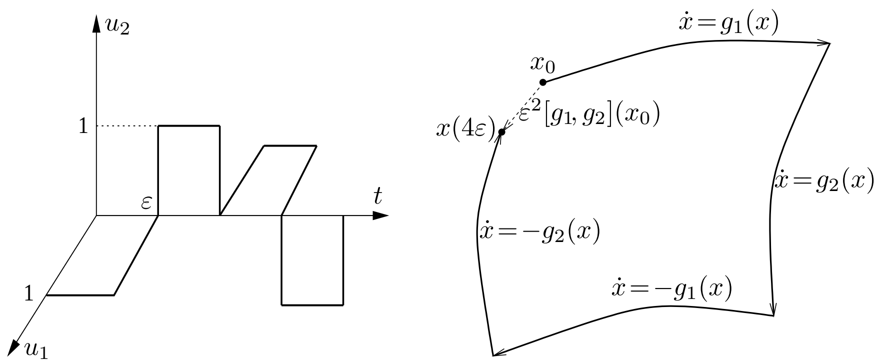

Fig. 23: (a) Switching between two control directions, (b) the resulting approximate motion along the Lie bracket

The above example illustrates the general principle that by switching among the principal control directions, one can generate slower "secondary" motions in the directions of the corresponding Lie brackets. More complicated switching patterns give rise to motions in the directions of higher-order iterated Lie brackets. This explains the importance of the *controllability Lie algebra* $\{g_i : i = 1, \dots, m\}_{LA}$ spanned by the control vector fields g_i. If this Lie algebra has rank n for all x, the system is said to satisfy the *Lie algebra rank condition* (LARC). In this case, it is well known that the system is completely controllable, in the sense that every state can be steered to every other state (Chow's theorem). The LARC guarantees that the motion of the system is not confined to any proper submanifold of $\mathbb{R}^n$ (Frobenius's theorem). In other words, the nonholonomic constraints are not integrable, i.e., cannot be expressed as constraints involving x only. Control systems of the form (89) satisfying the LARC are referred to as *completely nonholonomic*.

Nonsingular, completely nonholonomic control systems are of special interest to us. Indeed, we have shown that such systems cannot be stabilized by continuous feedback, even though they are controllable (and, in particular, asymptotically open-loop controllable to the origin). One way to overcome this difficulty is to employ switching control techniques.

As we already mentioned in Section 1.3, the system (89) becomes equivalent to the switched system (3) with $\mathcal{P} = \{1, 2, \dots, m\}$ and $f_i = g_i$ for all i if we restrict the admissible controls to be of the form $u_k = 1$, $u_i = 0$ for $i \neq k$ (this gives $\sigma = k$). In particular, the bilinear system

$$\dot{x} = \sum_{i=1}^{m} A_i x u_i$$

corresponds to the switched linear system (4). It is intuitively clear that asymptotic stability of the switched system (3) for arbitrary switching corresponds to a lack of controllability for (89). Indeed, it implies that for every admissible control function, the resulting solution trajectory of (89) must approach the origin. As we have just seen, Lie algebras naturally arise in characterizing controllability of (89); this perhaps makes their relevance for stability analysis of (3), unveiled by the results discussed in Section 5, less surprising.

12.1 The unicycle and the nonholonomic integrator

As a simple example of a nonholonomic system, we consider the wheeled mobile robot of unicycle type shown in Figure 24. We henceforth refer to it informally as the unicycle.

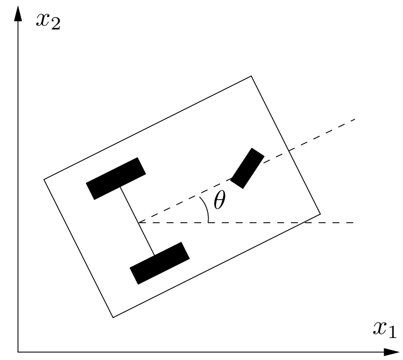

Fig. 24: The unicycle

The state variables are x_1, x_2, and θ, where x_1, x_2 are the coordinates of the point in the middle of the rear axle and θ denotes the angle that the vehicle makes with the x_1-axis (for convenience, we can assume that θ takes values in $\mathbb{R}$). The front wheel turns freely and balances the front end of the robot above the ground. When the same angular velocity is applied to both rear wheels, the robot moves straight forward. When the angular velocities applied to the rear wheels are different, the robot turns. The kinematics of the robot can be modeled by the equations

$$\begin{aligned} \dot{x}_1 &= u_1 \cos\theta \\ \dot{x}_2 &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{aligned} \tag{90}$$

where u_1 and u_2 are the control inputs (the forward and the angular velocity, respectively). We assume that the no-slip condition is imposed on the wheels, so the robot cannot move sideways (this is precisely the nonholonomic constraint). Asymptotic stabilization of this system amounts to parking the unicycle at the origin and aligning it with the x_1-axis.

The system (90) is a nonsingular completely nonholonomic system. Indeed, it takes the form (89) with $n = 3$, $m = 2$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix}, \qquad g_1(x) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The vectors g_1 and g_2 are linearly independent for all x_1, x_2, θ. Moreover, we have

$$[g_1, g_2](x) = -\begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}.$$

Thus the vector field $[g_1, g_2]$ is orthogonal to both g_1 and g_2 everywhere, hence the LARC is satisfied.

Incidentally, controllability of the system (90) justifies the common strategy for parallel parking. After normalization, this strategy can be modeled by switching between the control laws $u_1 = -1, u_2 = -1$ (moving backward, the steering wheel turned all the way to the right), $u_1 = -1, u_2 = 1$ (moving backward, the wheel turned all the way to the left), $u_1 = 1, u_2 = -1$ (moving forward, the wheel turned to the right), $u_1 = 1, u_2 = 1$ (moving forward, the wheel turned to the left). As we explained earlier (see Figure 23), the resulting motion is approximately along the Lie bracket of the corresponding vector fields, which is easily seen to be the direction perpendicular to the straight motion (i.e., sideways). The factor ε^2 explains the frustration often associated with parallel parking.

We know from Corollary 9.2 that the parking problem for the unicycle cannot be solved by means of continuous feedback. It is actually not hard to show directly that the system (90) fails to satisfy Brockett's necessary condition for continuous stabilizability (expressed by Theorem 9.1). The map (87) in this case is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \theta \\ u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \cos\theta \\ u_1 \sin\theta \\ u_2 \end{pmatrix}.$$

Pick a neighborhood in the x, u space where $|\theta| < \pi/2$. The image of such a neighborhood under the above map does not contain vectors of the form

$$\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}, \qquad a \neq 0.$$

Indeed, $u_1 \cos\theta = 0$ implies $u_1 = 0$ because $\cos\theta \neq 0$, hence we must also have $u_1 \sin\theta = 0$. Thus for small values of the angle θ we cannot move in all directions, which is an immediate consequence of the fact that the wheels are not allowed to slip.

We can intuitively understand why there does not exist a continuous stabilizing feedback. The argument is similar to the one we gave earlier for systems on a circle (see Section 11.1). Since the unicycle cannot move sideways, we need to decide which way to turn. If we start rotating clockwise from some initial configurations and counterclockwise from others, then the need for a logical decision will arise for a certain set of initial configurations (see Figure 25). Thus the nonholonomic constraint plays a role similar to that of a state-space obstacle.

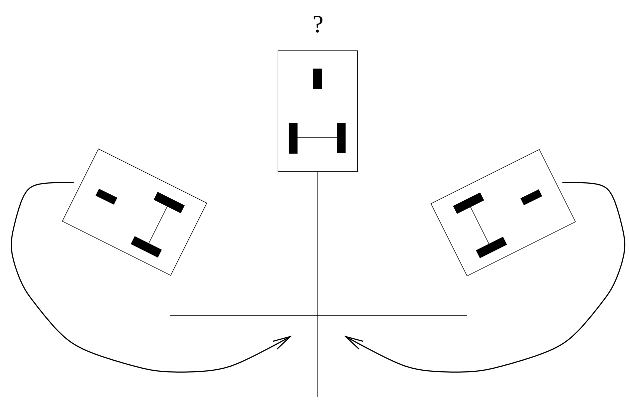

Fig. 25: Parking the unicycle

Our next objective is to demonstrate how the system (90) can be asymptotically stabilized by a switching feedback control law. To this end, it is convenient to consider the following state and control coordinate transformation:

$$
\begin{aligned}
x &= x_1 \cos\theta + x_2 \sin\theta \\
y &= \theta \\
z &= 2(x_1 \sin\theta - x_2 \cos\theta) - \theta(x_1 \cos\theta + x_2 \sin\theta) \\
u &= u_1 - u_2(x_1 \sin\theta - x_2 \cos\theta) \\
v &= u_2.
\end{aligned}
\tag{91}
$$

This transformation is well defined and preserves the origin, and in the new coordinates the system takes the form

$$
\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= xv - yu.
\end{aligned}
\tag{92}
$$

The system (92) is known as Brockett's *nonholonomic integrator*. Given a feedback law that stabilizes this system, by reversing the above change of coordinates one obtains a stabilizing feedback law for the unicycle. It is therefore clear that a continuous feedback law that stabilizes the nonholonomic integrator does not exist. This can also be seen directly: the image of the map

$$
\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ xv - yu \end{pmatrix}
$$

does not contain vectors of the form

$$
\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \qquad a \neq 0.
$$

Note that, unlike in the case of the unicycle, we did not even need to restrict the above map to a sufficiently small neighborhood of the origin.

The nonholonomic integrator is also controllable, and in fact its controllability has an interesting geometric interpretation. Suppose that we steer the system from the origin to some point $(x(t), y(t), z(t))^T$. Then from (92) and Green's theorem we have

$$
z(t) = \int_0^t (x\dot{y} - y\dot{x})dt = \int_{\partial\mathcal{D}} xdy - ydx = 2\int_{\mathcal{D}} dxdy
$$

where $\mathcal{D}$ is the area defined by the projection of the solution trajectory onto the xy-plane, completed by the straight line from $(x(t), y(t))$ to $(0, 0)$, and $\partial\mathcal{D}$ is its boundary (see Figure 26). Note that the integral along the line is zero. Thus the net change in z equals twice the signed area of $\mathcal{D}$. Since the subsystem of (92) that corresponds to the "base" coordinates x, y is obviously controllable, it is not difficult to see how to drive the system from the origin to a desired final state $(x(t), y(t), z(t))^T$. For example, we can first find a control law that generates a closed path in the xy-plane of area $z(t)/2$, and then apply a control law that induces the motion from the origin to the point $(x(t), y(t))^T$ along a straight line.

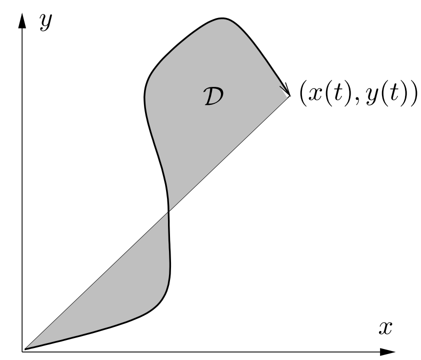

Fig. 26: Illustrating controllability of the nonholonomic integrator

We now describe a simple example of a switching feedback control law that asymptotically stabilizes the nonholonomic integrator (92). Let us consider another state and control coordinate transformation, given by

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi \\ \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \end{aligned}$$

Of course, the above transformation is only defined when $x^2 + y^2 = r^2 \neq 0$. We obtain the following equations in the new cylindrical coordinates:

$$\begin{aligned} \dot{r} &= \bar{u} \\ \dot{\psi} &= \bar{v}/r \\ \dot{z} &= r\bar{v}. \end{aligned}$$

The feedback law

$$\bar{u} = -r^2, \qquad \bar{v} = -z \tag{93}$$

yields the closed-loop system

$$\begin{aligned} \dot{r} &= -r^2 \\ \dot{z} &= -rz \\ \dot{\psi} &= -\frac{z}{r}. \end{aligned} \tag{94}$$

Its solutions satisfy the formulas

$$r(t) = \frac{1}{t + \frac{1}{r(0)}}$$

and

$$z(t) = e^{-\int_0^t r(\tau)d\tau} z(0)$$

from which it is not difficult to conclude that if $r(0) \neq 0$, then we have $r(t)$, $z(t) \to 0$ as $t \to \infty$, and moreover $r(t) \neq 0$ for all t. This implies that x, y, and z converge to zero (the exact behavior of the angle variable ψ is not important).

We now need to explain what to do if $r(0) = 0$. The simplest solution is to apply some control law that moves the state of (92) away from the z-axis (for example, $u = v = 1$) for a certain amount of time T, and then switch to the control law defined by (93). We formally describe this procedure by introducing a logical variable s, which is initially set to 0 if $r(0) = 0$ and to 1 otherwise. If $s(0) = 0$, then at the switching time T it is reset to 1. In fact, it is possible to achieve asymptotic stability in the Lyapunov sense, e.g., if we move away from the singularity line $r = 0$ with the speed proportional to $z(0)$, as in

$$u = z(0), \qquad v = z(0). \tag{95}$$

Figure 27 shows a computer-like diagram illustrating this switching logic, as well as a typical trajectory of the resulting switched system. A reset integrator is used to determine the switching time.

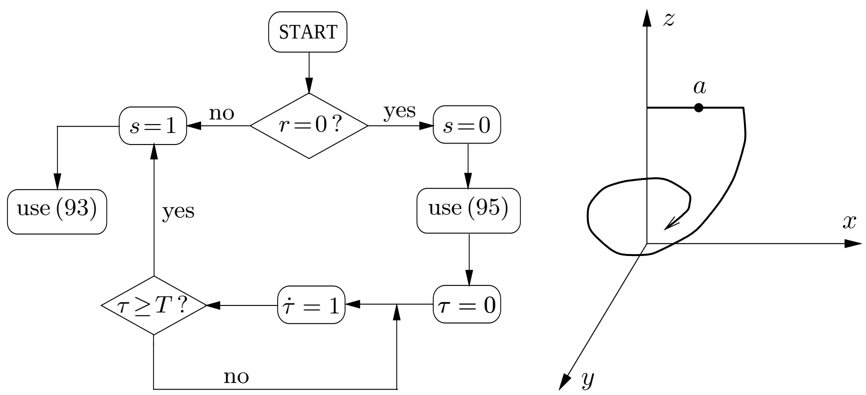

Fig. 27: Stabilizing the nonholonomic integrator: (a) the switching logic, (b) a typical trajectory of the closed-loop system

Exercise 9.2 Implement the above switching stabilizing control law for the unicycle via computer simulation.

Note that the closed-loop system is a truly hybrid system, in the sense that its description necessarily involves discrete dynamics. Indeed, the value of the control is not completely determined by the value of the state, but also depends on the value of the discrete variable s. For example, at the point a shown in Figure 27(b), the control takes different values for $s = 0$ and $s = 1$. Thus the control law that we have just described is a *hybrid control law*, with two discrete states. To increase robustness with respect to measurement errors, we could replace the condition $r(0) = 0$ with $|r(0)| < \varepsilon$ for some $\varepsilon > 0$ (this condition would be checked only once, at $t = 0$).

Control with limited information

This lecture is based on powerpoint slides, file name: `liminfo`.

Supervisory control of uncertain systems

This lecture is based on powerpoint slides, file name: `supervisory`.

Conclusion

Answers to questions, brief discussions of topics not covered, open problems and future directions. There are no notes or slides for this lecture, this will be an informal discussion.

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