Invariance-Like Theorems and "lim inf" Convergence Properties

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Abstract—**Several theorems, inspired by the Krasovskii-LaSalle invariance principle, to establish "lim inf" convergence results are presented in a unified framework. These properties are useful to "describe" the oscillatory behavior of the solutions of dynamical systems. The theorems resemble "lim inf" Matrosov and Small-gain theorems and are based on a "lim inf" Barbalat's Lemma. Additional technical assumptions to have "lim" convergence are given: the "lim inf"/"lim" relation is discussed in-depth and the role of some of the assumptions is illustrated by means of examples.**

Index Terms—**Krasovskii-LaSalle invariance, Lyapunov theorems, Matrosov theorem.**

I. INTRODUCTION

THE qualitative study of asymptotic properties of trajectories of nonlinear systems is a key problem in systems and control theory, see, *e.g.*, [1]–[5]. Among these asymptotic properties, the most important is attractiveness, which is often established by means of Lyapunov functions. Although this formulation is convenient from a practical point of view, it is in general hard to find a function that fulfills the sufficient (and in some cases necessary) conditions of the Lyapunov theorems. It is somewhat easier to find a weak Lyapunov function, *i.e.*, a positive definite function with negative semi-definite time derivative along the trajectories of the systems. In this last case, for time-invariant systems, the Krasovskii-LaSalle invariance principle allows to establish attractiveness, under additional assumptions on the Ω-limit sets of the solutions (see, *e.g.*, [6]–[9]).

Another tool that is used to replace the negative definiteness condition of the Lyapunov theorem and, in addition, can be used for time-varying systems is Matrosov Theorem (see *e.g.*, [10]–[12]). This theorem allows proving attractiveness of equilibrium points, provided that a linear combination of positive semi-definite functions is positive definite and their time derivatives along the trajectories of the system have a triangular

structure. However, to apply Matrosov Theorem it is necessary to assume stability of the equilibrium point.

In [13], the authors posed the question of what can be established if this stability assumption and the positive definiteness assumption are removed from Matrosov Theorem. The answer to this question is that it is still possible to establish some convergence result, although, with reference to the positive definiteness assumption, not as strong as one could think borrowing from the Krasovskii-LaSalle invariance principle. We call this convergence "lim inf" convergence, in the sense that we cannot establish asymptotic stability of the equilibrium point, but we can show that there is an oscillating behavior with some "nice" asymptotic properties.

Since (the classical) Matrosov Theorem relies upon the study of a triangular system of two differential inequalities, it can be extended in different directions. A straightforward extension consists in the so-called nested Matrosov Theorem [12], in which several inequalities are considered. Another extension, which changes radically the reach of the theorem, consists in removing the triangular structure of the system of differential inequalities. In this case the Matrosov inequalities, which can also be interpreted within the framework of vector Lyapunov functions (see [14] for instance), lead (assuming additional hypotheses) to the Lyapunov formulation of the Small-gain Theorem [15].

The Small-gain Theorem is an important tool to assess the asymptotic properties of the trajectories of a system resulting from the interconnection of two or more subsystems. The Small-gain Theorem has been developed in different formulations depending on which property is used to describe the input-output behavior of each of the subsystems. For linear systems the L_p Small-gain Theorem has been successfully used in input-output formulations of the problem (see *e.g.*, [16, Chapter 6] and [17]). For nonlinear systems versions based on L_p-gains (see [18]), but using Lyapunov functions, have been presented in [19]–[21]. In this paper, the Lyapunov formulation given in [15] and derived from the property of input-to-state stability (ISS) (see [22] and [23]) is used. Note that within this framework other formulations in which interconnections between possibly non-ISS subsystems are considered (see, *e.g.*, [24]–[26]), have been proposed. While the Small-gain Theorem is usually formulated for two interconnected subsystems it is often interesting, for practical applications, to study its large-scale version. A large-scale version of the theorem for linear systems can be found in [16], whereas a nonlinear formulation has recently been presented in [27] and [28] (see also [14]).

Preliminary versions of our work have been published in [13], [29], and [30]. One of the contributions of this paper

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is to generalize the results given in those papers. In addition, the proofs of the lemmas and theorems are presented, together with new technical results that are of independent interest. Finally, this paper sheds light on some of the problems left open in the preliminary versions. In fact, the role of some of the assumptions (*e.g.*, the boundedness assumption), the meaning of the results (*e.g.*, the "lim inf"/"lim" relation) and some connections with stability theory (*e.g.*, Barbalat's Lemma) are analyzed in detail in this paper.

The outcome of this series of papers is a class of theorems inspired by the Krasovskii-LaSalle invariance principle that can establish "lim inf" convergence results, thus can "describe" the oscillatory behavior of the solutions of dynamical systems. These theorems lead to "lim inf" Matrosov and Small-gain Theorems which are based on a "lim inf" Barbalat's Lemma. In addition, technical assumptions to have "lim" convergence[1] are given, and the "lim inf"/"lim" relation and the role of some of the assumptions are discussed. Applications of the results of this paper are currently under investigation.

The rest of the paper is organized as follows. This section continues with the formulation of the problem and with a discussion on the connections with Matrosov Theorem and the Small-gain Theorem. In Section II some properties of the so-called M-matrices are recalled. We also give a small-gain-like condition and we extend the concept of irreducibility to non-constant matrices. Section III presents a series of technical lemmas which forms the core theoretical part of the paper. The irreducible case is analyzed and connections with Barbalat's Lemma are drawn before studying the general reducible case. In Section IV the use of "linear gains" as opposed to "nonlinear gains" is justified and supported by a series of counterexamples. In Section V the theorems are applied to the study of the asymptotic behavior of solutions of dynamical systems. Sections VI and VII contain examples illustrating the theoretical results and Section VIII gives some concluding remarks.

Notation: We use standard notation. $\mathbb{R}_+$ denotes the set of non-negative real numbers. A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class K_∞, if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(s) \to +\infty$ as $s \to +\infty$. Id denotes the identity map, *i.e.*, $\mathrm{Id}(s) = s$. $[v]_i$ denotes the i-th component of the vector v and the notation $u \leq v$ has to be understood component-wise.

A. Problem Formulation

Motivated by the attempt to add a new tool to "comparison theory" for studying the behavior of the solutions of dynamical systems, we consider the following problem.

Problem 1: Let $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\bar{a}, \bar{a}]$ be absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \bar{b}]$ be continuous functions. Consider continuous, positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous func-

tions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, with $i \neq j$, satisfying $\beta_{ij}(0) = 0$, such that the differential inequalities

$$
\begin{aligned}
\dot{a}_1 &\leq -\alpha_1(b_1) + \beta_{12}(b_2) + \cdots + \beta_{1p}(b_p), \\
\dot{a}_2 &\leq -\alpha_2(b_2) + \beta_{21}(b_1) + \cdots + \beta_{2p}(b_p), \\
&\vdots \\
\dot{a}_p &\leq -\alpha_p(b_p) + \beta_{p1}(b_1) + \cdots + \beta_{p(p-1)}(b_{p-1}), \quad (1)
\end{aligned}
$$

hold for almost all t in $\mathbb{R}_+$.

The *"lim inf"/"lim" convergence problem* consists in determining the asymptotic properties of the functions b_i, more precisely, in determining conditions on the functions α_i and β_{ij} such that

$$
\liminf_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0, \quad (2)
$$

or

$$
\lim_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0. \quad (3)
$$

The key feature of the inequalities (1) is, in line with the approach followed in [13], [29], and [30], that the arguments b_i of the functions α_i and β_{ij} are not related *a-priori* with the functions a_k in the left-hand side. To illustrate this statement we recall the (simplest) formulation of the Matrosov Theorem and of the Small-gain Theorem.

Consider a nonlinear system described by the equation

$$
\dot{x} = f(x), \quad (4)
$$

where $x \in \mathbb{R}^n$, is the state of the system and the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. Assume there exists an equilibrium point which, without loss of generality, we choose as the origin of the coordinate system, *i.e.*, $f(0) = 0$.

Theorem 1 (Matrosov Theorem [10]–[13]): Consider system (4). Let $i \in \{1, 2\}$. Assume there exist

1) a differentiable, positive definite and radially unbounded function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ such that $\dot{V}_0 \leq 0$ along all the solutions of system (4);
2) two differentiable functions $V_i : \mathbb{R}^n \to \mathbb{R}$ and two continuous, positive semi-definite functions $h_i : \mathbb{R}^n \to \mathbb{R}_+$ such that the function $h_1 + h_2$ is positive definite;
3) a continuous function $\beta_{21} : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\beta_{21}(0) = 0$;

satisfying, along all the solutions of system (4), the inequalities

$$
\begin{aligned}
\dot{V}_1 &\leq -h_1, \\
\dot{V}_2 &\leq -h_2 + \beta_{21}(h_1). \quad (5)
\end{aligned}
$$

Then the equilibrium $x = 0$ of system (4) is globally asymptotically stable.

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[1] "lim" convergence is an abuse of language that we use to indicate the usual concept of convergence and that allows to clarify the distinction between the two "types of convergence."

Theorem 2 (Small-gain Theorem[2] [15], [29]): Consider system (4). Let $i \in \{1, 2\}$. Assume there exist

1) two $\mathcal{C}^1$ functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$ such that the function $V_1 + V_2$ is positive definite and radially unbounded;
2) two class K_∞ functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and two continuous functions $\beta_{12}, \beta_{21} : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\beta_{12}(0) = \beta_{21}(0) = 0$;

satisfying, along all the solutions of system (4), the inequalities

$$\dot{V}_1 \leq -\alpha_1(V_1) + \beta_{12}(V_2),$$
$$\dot{V}_2 \leq -\alpha_2(V_2) + \beta_{21}(V_1). \tag{6}$$

If the small-gain condition

$$\beta_{21} \circ \alpha_1^{-1} \circ \beta_{12} \circ \alpha_2^{-1} < \text{Id}, \tag{7}$$

holds, then the equilibrium $x = 0$ of system (4) is globally asymptotically stable.

Note that in Problem 1 and Theorem 1 and 2 differential inequalities with similar structure are studied; in Theorem 1 and 2 the inequalities must hold when the functions are evaluated along any solution. Instead in Problem 1, we restrict our attention to those particular solutions which are bounded.

We are also interested in generalizing Theorem 1, removing the stability assumption and not requiring that a linear combination of positive-semidefinite functions be positive definite (in the spirit of LaSalle invariance principle), and Theorem 2, allowing the arguments h_i of the functions α_i and β_{ij} to be not related *a-priori* with the functions V_k in the left-hand side (in the spirit of Matrosov Theorem). Note that as anticipated in [29] and illustrated in detail here, the result that we prove may not hold when the nonlinear functions α_i and β_{ij} satisfy the nonlinear small-gain condition (7): a more restrictive linear small-gain-like condition may be required.

II. PRELIMINARY RESULTS ON THE TEST MATRIX

In this section we define the notion of "test matrix" associated to the inequalities (1) and we recall or prove properties which are instrumental to establish the results of the following sections.

Definition 1: A principal minor of order j of an $n \times n$ matrix A is the determinant of the $j \times j$ sub-matrix obtained from A by deleting $n - j$ columns and $n - j$ rows with the same indices.

A leading principal minor of order j of a matrix A is the determinant of its upper-left j by j sub-matrix and is indicated by the notation $\mathcal{M}_j(A)$.

Definition 2 [31]: A Z-matrix is a matrix with non-positive off-diagonal elements.

Definition 3 [31, Condition E_{17}, Theorem 6.2.3]: A Z-matrix having all its leading principal minors strictly positive is called a non-singular M-matrix.

Definition 4 [31]: A matrix is reducible if, after some permutation of the rows and the columns, it can be written in a lower block triangular form. Otherwise it is said to be irreducible.

Lemma 1 [31, Theorem 6.2.7]: The inverse of a non-singular M-matrix A has non-negative entries. Moreover, if A is irreducible, the inverse has strictly positive entries.

In the following we call *test matrix* the matrix Q with the (i, j) element equal to $-\beta_{ij}(b_j)$, if $i \neq j$, or to $\alpha_j(b_j)$, if $i = j$, namely[3]

$$Q = \begin{bmatrix} \alpha_1(b_1) & -\beta_{12}(b_2) & \cdots & -\beta_{1p}(b_p) \\ -\beta_{21}(b_1) & \alpha_2(b_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -\beta_{(p-1)1}(b_1) & \cdots & \alpha_{p-1}(b_{p-1}) & -\beta_{(p-1)p}(b_p) \\ -\beta_{p1}(b_1) & \cdots & -\beta_{p(p-1)}(b_{p-1}) & \alpha_p(b_p) \end{bmatrix}.$$

Note that Q is a Z-matrix.

When, for all $k = 1, \dots, p$, $l = 1, \dots, p$, with $k \neq l$, there exist non-negative real numbers γ_{kl} satisfying

$$\sup_{s \in (0, \bar{b}]} \frac{\beta_{kl}(s)}{\alpha_l(s)} \leq \gamma_{kl}, \tag{8}$$

we associate to the test matrix Q a matrix Γ defined as the matrix with off-diagonal elements equal to $-\sup_{s \in (0, \bar{b}]}(\beta_{kl}(s)/\alpha_j(s))$ and diagonal elements equal to one. Again Γ is a Z-matrix.

Lemma 2: Assume the following.

1) The test matrix Q satisfies the following *linear small-gain-like condition*: there exists a strictly positive real number ε such that, for all $j = 1, \dots, p$ and all $(b_1, \dots, b_p)$ in $[0, \bar{b}]^p$, we have

$$\mathcal{M}_j(Q(b_1, \dots, b_p)) \geq \varepsilon \prod_{k=1}^{j} \alpha_k(b_k). \tag{9}$$

2) Each function $s \mapsto (\beta_{kl}(s)/\alpha_l(s))$ is bounded.
Then the matrix Γ satisfies

$$\mathcal{M}_j(\Gamma) \geq \varepsilon, \qquad \forall j = 1, \dots, p. \tag{10}$$

Proof: Condition (9) is equivalent to

$$\mathcal{M}_j\left(Q \operatorname{diag}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_p}\right)\right) \geq \varepsilon, \quad \forall j \text{ and } \forall b_i \in (0, \bar{b}].$$

By definition of supremum, there exist p sequences $\{b_{ln}\}$ such that

$$\frac{\beta_{kl}(b_{ln})}{\alpha_l(b_{ln})} \leq \sup_{s \in (0, \bar{b}]} \frac{\beta_{kl}(s)}{\alpha_l(s)} \leq \frac{\beta_{kl}(b_{ln})}{\alpha_l(b_{ln})} + \frac{1}{n}.$$

Since a minor is a polynomial in the entries of the matrix and the b_{ln} are bounded, this yields

$$\mathcal{M}_j(\Gamma) \geq \varepsilon + p\left(\frac{1}{n}\right), \qquad \forall j = 1, \dots, p,$$

---

[2]The Small-gain Theorem is usually applied in the study of the stability properties of the equilibrium point of an interconnected system. In this paper, following the Lyapunov formulation given in [15], we abuse the terminology saying that the Small-gain Theorem holds for the inequalities (6), ignoring if these arise as the result of a composition of systems.

[3]Omitting the arguments of Q.

where $p(1/n)$ is a polynomial in $1/n$ that goes to zero as $n \to \infty$, *i.e.*, $p(0) = 0$, hence the claim. ∎

Another way to make sure that (10) holds when the second assumption of Lemma 2 is satisfied is by defining a matrix $\overline{\Gamma}$ with off-diagonal elements equal to some $-\gamma_{kl}$ satisfying (8) and diagonal elements equal to one and check if we have

$$\mathcal{M}_j(\overline{\Gamma}) \geq \overline{\varepsilon} > 0, \qquad \forall j = 1, \dots, p.$$

Indeed in this case, we have

$$\mathcal{M}_j(\Gamma) \geq \mathcal{M}_j(\overline{\Gamma}).$$

This follows from the fact that Lemma 1 implies that $\mathcal{M}_j(\overline{\Gamma})$ is a non-increasing function of γ_{kl}.

We show now that, when the small-gain-like condition (9) is satisfied, the irreducibility of Q implies the boundedness of the functions $s \mapsto (\beta_{ij}(s)/\alpha_j(s))$ on $(0, \overline{b}]$.

Lemma 3: Assume the test matrix Q satisfies the linear small-gain-like condition (9). If, for some index j, there exists a vector $(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^*)$ in $(0, \overline{b}]^{p-1}$ such that, for all $b_j \in (0, \overline{b}]$, the matrix $Q(b_1^*, \dots, b_{j-1}^*, b_j, b_{j+1}^* \dots, b_p^*)$ is irreducible then, for all $i \neq j$, the functions $s \mapsto (\beta_{ij}(s)/\alpha_j(s))$ are bounded on $(0, \overline{b}]$.

Proof: By definition, the small-gain-like condition (8) implies that $Q(b_1, \dots, b_p)$ is an M-matrix for all $(b_1, \dots, b_p)$ in $[0, \overline{b}]^p$. It implies also that $\det(Q(b_1, \dots, b_p))$ is strictly positive for all $(b_1, \dots, b_p)$ in $(0, \overline{b}]^p$.

Let j and $(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^*)$ in $(0, \overline{b}]^{p-1}$ be such that the matrix $Q(b_1^*, \dots, b_{j-1}^*, s, b_{j+1}^* \dots, b_p^*)$ is irreducible for all s in $(0, \overline{b}]$. Omitting the argument $(b_1^*, \dots, b_{j-1}^*, s, b_{j+1}^* \dots, b_p^*)$ when it is not necessary, let q_k, for all k, be the k-th entry of the j-th row of $\text{adj}(Q) = \det(Q)Q^{-1}$. From Lemma 1 q_k is strictly positive. Also q_k does not depend on s. Indeed the row j of Q^T depends only on s, and q_k is the determinant of the sub-matrix formed by deleting the j-th row and the k-th column of Q^T. Finally, the j-th diagonal element of the matrix identity

$$\det(Q)I = \text{adj}(Q)Q,$$

yields

$$0 < \det(Q) = q_j \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^*, \dots, b_p^* \right) \alpha_j(s)$$

$$- \sum_{\substack{k=1 \\ k \neq j}}^{p} q_k \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^*, \dots, b_p^* \right) \beta_{kj}(s) \quad \forall s \in (0, \overline{b}].$$

Since for any $i \neq j$, q_i is strictly positive, this implies

$$\beta_{ij}(s) + \sum_{\substack{k=1 \\ k \neq j, i}}^{p} \frac{q_k \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^*, \dots, b_p^* \right)}{q_i \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^* \right)} \beta_{kj}(s)$$

$$< \frac{q_j \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^* \right)}{q_i \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^* \right)} \alpha_j(s), \quad \forall s \in (0, \overline{b}], \forall i \neq j$$

and therefore

$$\frac{\beta_{ij}(s)}{\alpha_j(s)} < \frac{q_j \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^* \right)}{q_i \left(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^* \right)}, \quad \forall s \in (0, \overline{b}], \forall i \neq j.$$

∎

In view of this result we define what we mean by the fact that Q as a function of $(b_1, \dots, b_p)$ is irreducible.

Definition 5: A test matrix is said to be irreducible as a function if, for each index j, there exists a vector $(b_1^*, \dots, b_{j-1}^*, b_{j+1}^* \dots, b_p^*)$ in $(0, \overline{b}]^{p-1}$ such that, for all $b_j \in (0, \overline{b}]$, the matrix $Q(b_1^*, \dots, b_{j-1}^*, b_j, b_{j+1}^* \dots, b_p^*)$ is irreducible.

The outcome of Lemma 3 is that if the inequalities in (1) cannot be re-written in triangular form by means of a permutation of rows and columns or more precisely if the associated test matrix is irreducible as a function, then the linear small-gain-like condition implies the existence of the matrix Γ with no additional hypotheses. In other words, when Q is irreducible as a function and (9) holds there is no need to assume that the functions $s \mapsto (\beta_{ij}(s)/\alpha_j(s))$ are bounded.

III. MAIN TECHNICAL RESULTS

In this section we present lemmas which constitute the core theoretical part of the paper. They will be used to establish the results of the following sections dealing with the study of the behavior of solutions of ordinary differential equations which are known to exist on $[0, +\infty)$, and taking values in a compact set, as detailed in Problem 1. For this reason we assume, without loss of generality, that all functions are bounded.

We begin with the irreducible case in the first subsection, we study the triangular reducible case in the second and we conclude with the triangular block reducible case in the last.

A. Irreducible Case

Lemma 4: Let $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\overline{a}, \overline{a}]$ be absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \overline{b}]$ be continuous functions. Consider continuous, positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, with $i \neq j$, satisfying $\beta_{ij}(0) = 0$, such that the following hold.

1) The differential inequalities

$$\dot{a}_1 \leq -\alpha_1(b_1) + \beta_{12}(b_2) + \cdots + \beta_{1p}(b_p),$$

$$\dot{a}_2 \leq -\alpha_2(b_2) + \beta_{21}(b_1) + \cdots + \beta_{2p}(b_p),$$

$$\vdots$$

$$\dot{a}_p \leq -\alpha_p(b_p) + \beta_{p1}(b_1) + \cdots + \beta_{p(p-1)}(b_{p-1}), \quad (11)$$

hold for almost all t in $\mathbb{R}_+$.

2) The test matrix Q associated to (11) is irreducible as a function and satisfies the linear small-gain-like condition (9).

Then we have

$$\liminf_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0. \quad (12)$$

If the functions b_i are uniformly continuous then we have

$$\lim_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0. \quad (13)$$

To prove Lemma 4, we observe in the next statement, given without proof to save space, that Cesàro's summability implies the "lim inf" convergence.

Lemma 5: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function. If

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma(s)ds = 0,$$

then

$$\liminf_{t \to \infty} |\sigma(t)| = 0.$$

Remark 1: If σ has constant sign then it is sufficient that σ be piecewise continuous.

Proof of Lemma 4: By Lemmas 2 and 3 we know the γ_{ij} defined in (8) exist and the inequality (10) holds. Hence the i-th line in (11) gives

$$\dot{a}_i \leq -\alpha_i(b_i) + \sum_{\substack{j=1 \\ j \neq i}}^p \gamma_{ij}\alpha_j(b_j). \qquad (14)$$

To rewrite this inequality in more compact notation let

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}, \quad \boldsymbol{\alpha}(\boldsymbol{b}) = \begin{bmatrix} \alpha_1(b_1) \\ \vdots \\ \alpha_p(b_p) \end{bmatrix}.$$

Then (14) reads

$$\dot{a}_i \leq \begin{bmatrix} \gamma_{i1} & \cdots & \gamma_{i(i-1)} & -1 & \gamma_{i(i+1)} & \cdots & \gamma_{ip} \end{bmatrix} \boldsymbol{\alpha}(\boldsymbol{b}). \quad (15)$$

With the definition of the matrix Γ, this reduces further to

$$[\dot{\boldsymbol{a}}]_i \leq [-\Gamma\boldsymbol{\alpha}]_i. \qquad (16)$$

Since, by (10), Γ has all leading principal minors with strictly positive determinant, by Lemma 1, Γ^{-1} has all positive entries, hence the relation

$$[\Gamma^{-1}\dot{\boldsymbol{a}}]_i \leq [-\boldsymbol{\alpha}]_i, \qquad (17)$$

holds. In fact each of the inequalities in (17) is obtained as a weighted sum, with positive weights, of the inequalities in (16). Integrating from 0 to t each of these relations yields

$$\int_0^t [\boldsymbol{\alpha}(\boldsymbol{b}(s))]_i \, ds \leq -\int_0^t [\Gamma^{-1}\dot{\boldsymbol{a}}(s)]_i \, ds$$

$$\leq [\Gamma^{-1}(\boldsymbol{a}(t) - \boldsymbol{a}(0))]_i.$$

Since the functions a_i are bounded, there exists a positive real number $\overline{\alpha}$ such that, for all i

$$\int_0^t [\boldsymbol{\alpha}(\boldsymbol{b}(s))]_i \, ds \leq \overline{\alpha}.$$

By adding all the above inequalities we have that

$$\int_0^t \sum_{i=1}^p \alpha_i(b_i(s)) \, ds < +\infty, \qquad (18)$$

hence, by Lemma 5, we conclude

$$\liminf_{t \to \infty} \sum_{i=1}^p \alpha_i(b_i(t)) = 0.$$

Since the functions α_i are positive definite, this implies (12). When the functions b_i are also uniformly continuous, the functions $t \mapsto \alpha_i(b_i(t))$ are uniformly continuous. So in this case, by Barbalat's Lemma, (18) gives

$$\lim_{t \to \infty} \sum_{i=1}^p \alpha_i(b_i(t)) = 0$$

and therefore (13) follows. ∎

B. Triangular Reducible Case

Lemma 6: Let $p \geq 3$, $i \in \{1, \ldots, p\}$ and $j \in \{2, \ldots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\overline{a}, \overline{a}]$ be absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \overline{b}]$ be continuous functions. Consider continuous, positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, with $j < i$, satisfying $\beta_{ij}(0) = 0$, such that the differential inequalities

$$\dot{a}_1 \leq -\alpha_1(b_1),$$

$$\dot{a}_2 \leq -\alpha_2(b_2) + \beta_{21}(b_1),$$

$$\vdots$$

$$\dot{a}_p \leq -\alpha_p(b_p) + \beta_{p1}(b_1) + \cdots \beta_{p(p-1)}(b_{p-1}),$$

hold for almost all t in $\mathbb{R}_+$. Then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{i=1}^p b_i(s)ds = 0 \qquad (19)$$

and therefore

$$\liminf_{t \to \infty} \sum_{i=1}^p b_i(t) = 0. \qquad (20)$$

Remark 2: As opposed to the irreducible case given in Lemma 4, in the triangular reducible case boundedness of the functions $s \mapsto (\beta_{ij}(s)/\alpha_j(s))$ does not play any role.

To prove Lemma 6 we use the following sufficient condition to have Cesàro's summability of an integral [32], *i.e.*, convergence of the mean, stated without proof to save space.

Lemma 7: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. If, for all $\varepsilon > 0$, there exits a positive number μ such that

$$\left| \int_0^t \sigma(s)ds \right| \leq \varepsilon t + \mu, \qquad (21)$$

for all $t \geq 0$, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma(s) ds = 0.$$

Remark 3: Lemma 5 and 7 provide a weaker version of Barbalat's Lemma (see *e.g.*, [5]). In fact, the classical Barbalat's Lemma can be recovered when the function σ is uniformly continuous and (21) holds for $\epsilon = 0$.

Another notion that we need to introduce concerns a function φ associated with a pair of functions (α, β).

Let $\bar{b}$ be a non-negative real number. To a continuous positive definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $\beta(0) = 0$, we associate the function $\varphi : [1, +\infty) \to \mathbb{R}$ defined as

$$\varphi(\rho) = \max_{b \in [0, \bar{b}]} (\beta(b) - (\rho - 1)\alpha(b)). \tag{22}$$

Lemma 8: The function φ takes non-negative values and is non-increasing, Lipschitz and such that $\lim_{\rho \to +\infty} \varphi(\rho) = 0$.

Proof: Since α and β are continuous and $[0, \bar{b}]$ is compact, for each $\rho \in [1, +\infty)$ there exists (at least one) $b(\rho)$ in $[0, \bar{b}]$ such that

$$\varphi(\rho) = \beta(b(\rho)) - (\rho - 1)\alpha(b(\rho)).$$

As a result, for any $\rho' \geq \rho'' \geq 1$

$$\begin{aligned}
\varphi(\rho'') &= \beta(b(\rho'')) - (\rho'' - 1)\alpha(b(\rho'')) \\
&\geq \beta(b(\rho')) - (\rho'' - 1)\alpha(b(\rho')) \\
&\geq \beta(b(\rho')) - (\rho' - 1)\alpha(b(\rho')) \\
&= \varphi(\rho') \\
&\geq \beta(b(\rho'')) - (\rho' - 1)\alpha(b(\rho'')).
\end{aligned}$$

This yields

$$0 \leq \varphi(\rho'') - \varphi(\rho') \leq (\rho' - \rho'')\bar{\alpha},$$

where

$$\bar{\alpha} = \max_{b \in [0, \bar{b}]} \alpha(b),$$

i.e., the function φ is Lipschitz and non-increasing.

Note now that, since α is continuous and positive definite, for any sequence $\{\rho_n\}$, such that $\lim_{n \to +\infty} \rho_n = +\infty$, there exists $N > 0$ and a sequence $\{b_n\} \subset [0, \bar{b}]$, satisfying $\lim_{n \to +\infty} b_n = 0$, and $\alpha(b_n) = 1/n\rho_n$, for all $n \geq N$. In addition, since $b(\rho_n) \in [0, \bar{b}]$,

$$\beta(b_n) + \alpha(b_n) - \frac{1}{n} \leq \varphi(\rho_n) = \beta(b(\rho_n)) - (\rho_n - 1)\alpha(b(\rho_n))$$

and therefore

$$0 \leq \rho_n \alpha(b(\rho_n)) + \beta(b_n) + \alpha(b_n) \leq \beta(b(\rho_n)) + \alpha(b(\rho_n)) + \frac{1}{n}.$$

This implies that $\lim_{n \to \infty} \alpha(b(\rho_n)) = 0$ and, since α is continuous and positive definite, that $\lim_{n \to \infty} b(\rho_n) = 0$. Finally, since β is zero at zero and continuous

$$\lim_{\rho \to +\infty} \varphi(\rho) = 0,$$

which also proves that φ takes non-negative values. ∎

Note that Lemma 8 holds also for a linear combination of functions $\beta(b)$. In this case we use the notation

$$\varphi_j(\rho) = \max_{b_j \in [0, \bar{b}]} (\beta_j(b_j) - (\rho - 1)\alpha_j(b_j)), \tag{23}$$

with $\beta_j(b_j) = \sum_{\substack{i=1 \\ i \neq j}}^p k_i \beta_{ij}(b_j)$, where the weights k_i are non-negative.

We are now ready to prove Lemma 6.

Proof of Lemma 6: The claim is proved by contradiction. To simplify the discussion consider the case $p = 3$, which contains all ingredients necessary for the general proof.

Let $\varphi_2 : [1, +\infty) \to \mathbb{R}$ be defined as

$$\varphi_2(\rho) = \max_{b_2 \in [0, \bar{b}]} (\beta_{32}(b_2) - (\rho - 1)\alpha_2(b_2)).$$

Let also ε be an arbitrarily chosen strictly positive real number. Since by Lemma 8, φ_2 is non-increasing and $\lim_{\rho \to +\infty} \varphi_2(\rho) = 0$, we can select $\psi_2(\varepsilon)$ in $[1, +\infty)$ such that

$$\varphi_2(\psi_2(\varepsilon)) \leq \frac{\varepsilon}{2}.$$

Let $\varphi_1 : [1, +\infty) \to \mathbb{R}$ be defined as

$$\varphi_1(\rho) = \max_{b_2 \in [0, \bar{b}]} (\psi_2(\varepsilon)\beta_{21}(b_1) + \beta_{31}(b_1) - (\rho - 1)\alpha_1(b_1)).$$

Similarly, for all $\varepsilon > 0$ we can select $\psi_1(\varepsilon)$ in $[1, +\infty)$ such that

$$\varphi_1(\psi_1(\varepsilon)) \leq \frac{\varepsilon}{2}.$$

Note now that

$$\begin{aligned}
\overbrace{\psi_1(\varepsilon)a_1 + \psi_2(\varepsilon)a_2 + a_3}^{} \\
\leq -\psi_1(\varepsilon)\alpha_1(b_1) + \psi_2(\varepsilon)\beta_{21}(b_1) + \beta_{31}(b_1) \\
- \psi_2(\varepsilon)\alpha_2(b_2) + \beta_{32}(b_2) - \alpha_3(b_3) \\
\leq (\varphi_1(\psi_1(\varepsilon)) + \varphi_2(\psi_2(\varepsilon))) - (\alpha_1(b_1) + \alpha_2(b_2) + \alpha_3(b_3)).
\end{aligned}$$

As a result

$$\begin{aligned}
\int_0^t \sum_{i=1}^3 \alpha_i(b_i(s)) \, ds \\
\leq (\varphi_1(\psi_1(\varepsilon)) + \varphi_2(\psi_2(\varepsilon))) t \\
+ (\psi_1(\varepsilon)a_1(0) + \psi_2(\varepsilon)a_2(0) + a_3(0)) \\
- (\psi_1(\varepsilon)a_1(t) + \psi_2(\varepsilon)a_2(t) + a_3(t)) \\
\leq (\varphi_1(\psi_1(\varepsilon)) + \varphi_2(\psi_2(\varepsilon))) t + 2(\psi_1(\varepsilon) + \psi_2(\varepsilon) + 1)\bar{a} \\
\leq \varepsilon t + 2(\psi_1(\varepsilon) + \psi_2(\varepsilon) + 1)\bar{a}.
\end{aligned}$$

Since ε is arbitrary, the claim follows by Lemmas 5 and 7.

In the case $p > 3$ the claim can be proved along the same lines defining $p - 1$ functions φ_j.   ∎

Remark 4: If in Lemma 6 we assume that the functions β_{ij}/α_j are bounded and the functions b_i are uniformly continuous, then we have the "lim" convergence result

$$\lim_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0.$$

In fact in the previous proof we can pick

$$\varepsilon = 0, \quad \psi_2 = 1 + \gamma_{21}, \quad \psi_1 = 1 + \gamma_{31} + \gamma_{32}\psi_2$$

and follow the same arguments as in the proof of Lemma 4.

C. Triangular Block Reducible Case

We are now ready to study the triangular block reducible case that can be regarded as a generalization of the previous results. To this end, let $s_l = r_l - r_{l-1}$, with $s_0 = r_0 = 0$, be the dimension of the column vectors

$$\boldsymbol{a}_l = \begin{bmatrix} a_{(r_{l-1}+1)} & a_{(r_{l-1}+2)} & \cdots & a_{r_l} \end{bmatrix}^T,$$

$$\boldsymbol{b}_l = \begin{bmatrix} b_{(r_{l-1}+1)} & b_{(r_{l-1}+2)} & \cdots & b_{r_l} \end{bmatrix}^T$$

and define

$$\boldsymbol{\alpha}_l(\boldsymbol{b}_l)$$

$$= \begin{bmatrix} \alpha_{(r_{l-1}+1)}(b_{(r_{l-1}+1)}) & \alpha_{(r_{l-1}+2)}(b_{(r_{l-1}+2)}) & \cdots & \alpha_{r_l}(b_{r_l}) \end{bmatrix}^T,$$

$$\boldsymbol{\delta}_l(\boldsymbol{b}_l) = \sum_{j=(r_{l-1}+1)}^{r_l} \begin{bmatrix} \delta_{(r_{l-1}+1)j}(b_j) \\ \delta_{(r_{l-1}+2)j}(b_j) \\ \vdots \\ \delta_{(r_l)j}(b_j) \end{bmatrix}, \quad \text{with}$$

$$\delta_{kj} = \begin{cases} -\alpha_j & \text{if } k = j, \\ \beta_{kj} & \text{if } k \neq j, \end{cases}$$

$$\boldsymbol{\mu}_{lm}(\boldsymbol{b}_m) = \sum_{j=(r_{m-1}+1)}^{r_m} \boldsymbol{\mu}_{lm}(b_j)$$

$$= \sum_{j=(r_{m-1}+1)}^{r_m} \begin{bmatrix} \beta_{(r_{l-1}+1)j}(b_j) \\ \beta_{(r_{l-1}+2)j}(b_j) \\ \vdots \\ \beta_{(r_l)j}(b_j) \end{bmatrix}.$$

Proposition 1: Let $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\bar{a}, \bar{a}]$ be absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \bar{b}]$ be continuous functions. Consider continuous, positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, with $i \neq j$, satisfying $\beta_{ij}(0) = 0$. Let $\boldsymbol{a}_l$, $\boldsymbol{b}_l$, $\boldsymbol{\delta}_l$, and $\boldsymbol{\mu}_{lm}$ be vectors of dimension s_l, with components

obtained from the a_i's, b_i's, α_i's, and β_{ij}'s, such that the following hold.

1) The differential inequalities

$$\dot{\boldsymbol{a}}_1 \leq \boldsymbol{\delta}_1(\boldsymbol{b}_1),$$

$$\dot{\boldsymbol{a}}_2 \leq \boldsymbol{\mu}_{21}(\boldsymbol{b}_1) + \boldsymbol{\delta}_2(\boldsymbol{b}_2),$$

$$\vdots$$

$$\dot{\boldsymbol{a}}_q \leq \boldsymbol{\mu}_{q1}(\boldsymbol{b}_1) + \boldsymbol{\mu}_{q2}(\boldsymbol{b}_2) + \cdots + \boldsymbol{\delta}_q(\boldsymbol{b}_q), \quad (24)$$

with $r_q = p$, hold for almost all t in $\mathbb{R}_+$.

2) The matrix Q_l for each diagonal element $\boldsymbol{\delta}_l$ is irreducible as a function and satisfies the linear small-gain-like condition (9).

Then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{i=1}^{p} b_i(s) ds = 0 \quad (25)$$

and therefore

$$\liminf_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0. \quad (26)$$

Remark 5: As expected, (26) holds with no additional restrictions on the off-diagonal elements $\boldsymbol{\mu}_{lm}$. However, as discussed in Remark 4, if all functions β_{ij}/α_j in the off-diagonal element $\boldsymbol{\mu}_{lm}$ are bounded and the b_i are uniformly continuous then (26) can be replaced by

$$\lim_{t \to \infty} \sum_{i=1}^{p} b_i(t) = 0. \quad (27)$$

Proof: We consider the case with $q = 3$ blocks, namely

$$\dot{\boldsymbol{a}}_1 \leq \boldsymbol{\delta}_1(\boldsymbol{b}_1),$$

$$\dot{\boldsymbol{a}}_2 \leq \boldsymbol{\mu}_{21}(\boldsymbol{b}_1) + \boldsymbol{\delta}_2(\boldsymbol{b}_2),$$

$$\dot{\boldsymbol{a}}_3 \leq \boldsymbol{\mu}_{31}(\boldsymbol{b}_1) + \boldsymbol{\mu}_{32}(\boldsymbol{b}_2) + \boldsymbol{\delta}_3(\boldsymbol{b}_3),$$

which contains all the ingredients necessary for the general proof. Define Γ_l as the matrix corresponding to the test matrix Q_l attached to the vector $\boldsymbol{\delta}_l$ and $\mathbf{1}_l$ as the row vector with s_l elements equal to 1. Let also ε be an arbitrarily chosen strictly positive real number. In a way similar to the one followed to get (16), we obtain

$$\boldsymbol{\delta}_l(\boldsymbol{b}_l) \leq -\Gamma_l \boldsymbol{\alpha}(\boldsymbol{b}_l)$$

and therefore

$$\Gamma_l^{-1} \boldsymbol{\delta}_l(\boldsymbol{b}_l) \leq -\boldsymbol{\alpha}(\boldsymbol{b}_l).$$

This leads to

$$\Gamma_1^{-1} \dot{\boldsymbol{a}}_1 \leq -\boldsymbol{\alpha}_1(\boldsymbol{b}_1),$$

$$\Gamma_2^{-1} \dot{\boldsymbol{a}}_2 \leq \Gamma_2^{-1} \boldsymbol{\mu}_{21}(\boldsymbol{b}_1) - \boldsymbol{\alpha}_2(\boldsymbol{b}_2),$$

$$\Gamma_3^{-1} \dot{\boldsymbol{a}}_3 \leq \Gamma_3^{-1} \boldsymbol{\mu}_{31}(\boldsymbol{b}_1) + \Gamma_3^{-1} \boldsymbol{\mu}_{32}(\boldsymbol{b}_2) - \boldsymbol{\alpha}_3(\boldsymbol{b}_3).$$

To deal with the terms in $\boldsymbol{b}_2$, we define s_2 functions $\varphi_i : [1, +\infty) \to \mathbb{R}$ as

$$\varphi_{r_1+1}(\rho) = \max_{b_{r_1+1} \in [0, \bar{b}]} \big(\mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{32}(b_{r_1+1})$$
$$- (\rho - 1)\alpha_{r_1+1}(b_{r_1+1}) \big),$$

$$\vdots$$

$$\varphi_{r_2}(\rho) = \max_{b_{r_2} \in [0, \bar{b}]} \big(\mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{32}(b_{r_2}) - (\rho - 1)\alpha_{r_2}(b_{r_2}) \big).$$

$$(28)$$

By Lemma 8, the functions φ_i are non-increasing and $\lim_{\rho \to +\infty} \varphi_i(\rho) = 0$. As a result, we can select a vector $\boldsymbol{\psi}_2(\varepsilon)$ of size s_2 with components $\psi_i(\varepsilon)$ in $[1, +\infty)$ such that

$$\sum_{i=r_1+1}^{r_2} \varphi_i(\psi_i(\varepsilon)) \le \frac{\varepsilon}{2}.$$

This gives

$$\mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{32}(\boldsymbol{b}_2) - \boldsymbol{\psi}_2(\varepsilon)\boldsymbol{\alpha}_2(\boldsymbol{b}_2) \le \frac{\varepsilon}{2} - \mathbf{1}_2 \boldsymbol{\alpha}_2(\boldsymbol{b}_2).$$

Similarly, to deal with the terms in $\boldsymbol{b}_1$, we define s_1 functions $\varphi_i : [1, +\infty) \to \mathbb{R}$ as

$$\varphi_1(\rho) = \max_{b_1 \in [0, \bar{b}]} \big(\boldsymbol{\psi}_2(\varepsilon) \Gamma_2^{-1} \boldsymbol{\mu}_{21}(b_1)$$
$$+ \mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{31}(b_1) - (\rho - 1)\alpha_1(b_1) \big)$$

$$\vdots$$

$$\varphi_{r_1}(\rho) = \max_{b_{r_1} \in [0, \bar{b}]} \big(\boldsymbol{\psi}_2(\varepsilon) \Gamma_2^{-1} \boldsymbol{\mu}_{21}(b_{r_1}) + \mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{31}(b_{r_1})$$
$$- (\rho - 1)\alpha_{r_1}(b_{r_1}) \big). \quad (29)$$

Again, by Lemma 8, the functions φ_i are non-increasing and $\lim_{\rho \to +\infty} \varphi_i(\rho) = 0$. So we can select a vector $\boldsymbol{\psi}_1(\varepsilon)$ of size s_1 with components $\psi_i(\varepsilon)$ in $[1, +\infty)$ such that

$$\sum_{i=1}^{r_1} \varphi_i(\psi_i(\varepsilon)) \le \frac{\varepsilon}{2}.$$

This gives

$$\boldsymbol{\psi}_2(\varepsilon) \Gamma_2^{-1} \boldsymbol{\mu}_{21}(\boldsymbol{b}_1) + \mathbf{1}_3 \Gamma_3^{-1} \boldsymbol{\mu}_{31}(\boldsymbol{b}_1)$$
$$- \boldsymbol{\psi}_1(\varepsilon)\boldsymbol{\alpha}_1(\boldsymbol{b}_1) \le \frac{\varepsilon}{2} - \mathbf{1}_1 \boldsymbol{\alpha}_1(\boldsymbol{b}_1).$$

So we have obtained

$$\mathbf{1}_3 \Gamma_3^{-1} \dot{\boldsymbol{a}}_3 + \boldsymbol{\psi}_2(\varepsilon) \Gamma_2^{-1} \dot{\boldsymbol{a}}_2 + \boldsymbol{\psi}_1(\varepsilon) \Gamma_1^{-1} \dot{\boldsymbol{a}}_1$$
$$\le \varepsilon - \mathbf{1}_3 \boldsymbol{\alpha}_3(\boldsymbol{b}_3) - \mathbf{1}_2 \boldsymbol{\alpha}_2(\boldsymbol{b}_2) - \mathbf{1}_1 \boldsymbol{\alpha}_1(\boldsymbol{b}_1)$$
$$\le \varepsilon - \sum_{i=1}^{r_3} \alpha_i. \quad (30)$$

The claim follows by integrating both sides of (30) from 0 to t and applying Lemmas 5 and 7. ∎

IV. ON THE LINEAR SMALL-GAIN-LIKE CONDITION

In this section we discuss the linear small-gain condition and explain why it is necessary to use this in the assumptions of Proposition 1 instead of the nonlinear condition. The discussion, for simplicity, is limited to the case $p = 2$, in which (9) yields

$$\beta_{12}(b_2)\beta_{21}(b_1) \le (1 - \varepsilon)\alpha_2(b_2)\alpha_1(b_1), \quad \forall (b_1, b_2) \in [0, \bar{b}]^2. \quad (31)$$

To simplify the discussion we restrict ourselves to consider the case in which the functions α_i are invertible and the above inequality (31) holds for all non-negative real numbers b_1 and b_2. Then, from the theory of interconnected nonlinear systems, we would expect that stability properties be related to the nonlinear small-gain condition

$$\beta_{21} \circ \alpha_1^{-1} \circ \beta_{12} \circ \alpha_2^{-1}(s) < s, \quad \forall s > 0. \quad (32)$$

Lemma 9: If β_{12} (β_{21} respectively) is positive definite, condition (31) implies, but it is not implied by, condition (32).

Proof: We first show that the linear condition implies the nonlinear one. Pick any pair (b_1, b_2) in $]0, \bar{b}]^2$ and note that the linear condition (31) yields

$$\big[\beta_{12} \circ \alpha_2^{-1}(b_2) \big] \big[\beta_{21} \circ \alpha_1^{-1}(b_1) \big] \le (1 - \varepsilon)b_1 b_2.$$

In particular, the selection

$$s > 0, \quad b_2 = s, \quad b_1 = \beta_{12} \circ \alpha_2^{-1}(s),$$

yields

$$b_1 \beta_{21} \circ \alpha_1^{-1} \circ \beta_{12} \circ \alpha_2^{-1}(s) \le (1 - \varepsilon) b_1 s,$$

which implies condition (32).

To show that the converse is not true, let $\alpha_1(s) = s, \beta_{12}(s) = s^2, \alpha_2(s) = s$ and $\beta_{21}(s) = \gamma\sqrt{s}$. The nonlinear small-gain reduces to $\gamma s \le (1 - \varepsilon)s$ which holds for all $0 \le \gamma < 1$, whereas the linear condition reduces to $\gamma(b_2/\sqrt{b_1}) < (1 - \varepsilon)$ which does not hold whatever the positive value of γ is. ∎

As usual for small-gain conditions it is difficult to establish the true necessity of (31). We now show that violation of the non-strict inequality yields the existence of functions a_i and b_i such that the convergence result of Lemma 4 does not hold.

Lemma 10: Assume there exist strictly positive real numbers b_{1a}, b_{2b} and b_{2c} such that

$$\frac{\beta_{12}(b_{2b})\beta_{21}(b_{1a})}{\alpha_2(b_{2b})\alpha_1(b_{1a})} > 1, \quad \frac{\beta_{12}(b_{2c})\beta_{21}(b_{1a})}{\alpha_2(b_{2c})\alpha_1(b_{1a})} < 1. \quad (33)$$

Then there exist functions a_i and b_i such that the convergence result in Lemma 4 does not hold.

Remark 6: Condition (33) says that, with $b_1 = b_{1a}$, the inequality (31) holds for $b_2 = b_c$ but does not for $b_2 = b_{2b}$.

Proof: Assume for the time being that we can find strictly positive real numbers ε_1 and ε_2 such that there exist strictly

positive real numbers T_b and T_c satisfying the linear equations

$$\begin{bmatrix} \beta_{12}(b_{2b}) - \alpha_1(\varepsilon_1) & \beta_{12}(b_{2c}) - \alpha_1(\varepsilon_1) \\ \alpha_2(b_{2b}) - \beta_{21}(\varepsilon_1) & \alpha_2(b_{2c}) - \beta_{21}(\varepsilon_1) \end{bmatrix} \begin{bmatrix} T_b \\ T_c \end{bmatrix}$$
$$= -\begin{bmatrix} \beta_{12}(\varepsilon_2) - \alpha_1(b_{1a}) \\ \alpha_2(\varepsilon_2) - \beta_{21}(b_{1a}) \end{bmatrix}. \quad (34)$$

Then, let b_1 and b_2 be piecewise constant and $(1 + T_b + T_c)$-periodic functions defined as

$$b_1(t) = \begin{cases} b_{1a} & \text{if } t \in [0, 1], \\ \varepsilon_1 & \text{if } t \in]1, 1 + T_b + T_c[, \end{cases}$$

$$b_2(t) = \begin{cases} \varepsilon_2 & \text{if } t \in [0, 1], \\ b_{2b} & \text{if } t \in]1, 1 + T_b], \\ b_{2c} & \text{if } t \in]1 + T_b, 1 + T_b + T_c[. \end{cases}$$

As a result

$$a_1(1 + T_b + T_c) - a_1(0)$$
$$\leq -[\alpha_1(b_{1a}) - \beta_{12}(\varepsilon_2)] + [-\alpha_1(\varepsilon_1) + \beta_{12}(b_{2b})] T_b$$
$$+ [-\alpha_1(\varepsilon_1) + \beta_{12}(b_{2c})] T_c$$
$$= 0$$

and

$$a_2(1 + T_b + T_c) - a_2(0)$$
$$\leq -[\alpha_2(\varepsilon_2) - \beta_{21}(b_{1a})] + [-\alpha_2(b_{2b}) + \beta_{21}(\varepsilon_1)] T_b$$
$$+ [-\alpha_2(b_{2c}) + \beta_{21}(\varepsilon_1)] T_c$$
$$= 0.$$

Therefore the result holds with a_i any constant function.

Now to prove that T_b, T_c do exist we note that when ε_1 and ε_2 are both zero, the solution of the equations (34) is

$$T_b = \frac{\alpha_2(b_{2c})\alpha_1(b_{1a}) - \beta_{12}(b_{2c})\beta_{21}(b_{1a})}{\alpha_2(b_{2c})\beta_{12}(b_{2b}) - \alpha_2(b_{2b})\beta_{12}(b_{2c})},$$

$$T_c = \frac{\beta_{12}(b_{2b})\beta_{21}(b_{1a}) - \alpha_2(b_{2b})\alpha_1(b_{1a})}{\alpha_2(b_{2c})\beta_{12}(b_{2b}) - \alpha_2(b_{2b})\beta_{12}(b_{2c})}.$$

By condition (33) T_b and T_c are strictly positive if the denominator is strictly positive. This is indeed the case since, multiplying the inequalities in (33), yields

$$\beta_{12}(b_{2b})\beta_{21}(b_{1a})\alpha_2(b_{2c})\alpha_1(b_{1a})$$
$$> \alpha_2(b_{2b})\alpha_1(b_{1a})\beta_{12}(b_{2c})\beta_{21}(b_{1a}),$$

where $\alpha_1(b_{1a}) > 0$, since $b_{1a} > 0$ and $\beta_{12}(b_{2b}) > 0$ because of (33). Therefore, by continuity, T_b and T_c are strictly positive when ε_1 and ε_2 are strictly positive, and sufficiently small. ∎

Thus, (31) is necessary to guarantee that there do not exist functions b_1 and b_2 such that the convergence result of Lemma 4 does not hold.

V. "LIM INF" ASYMPTOTIC PROPERTIES IN DYNAMICAL SYSTEMS

Proposition 1 can be applied to study asymptotic properties of the solutions of dynamical systems. In particular the follow-

ing theorem solve Problem 1 and gives conditions to establish the "lim inf" or "lim" convergence of such solutions.

Theorem 3: Let $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, p\}$. Consider system (4) and let $V_i : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions and $h_i : \mathbb{R}^n \to \mathbb{R}_+$ be continuous functions. Consider continuous, positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$, continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $\beta_{ij}(0) = 0$. Let $\boldsymbol{V}_l$, $\boldsymbol{h}_l$, $\boldsymbol{\delta}_l$, and $\boldsymbol{\mu}_{lm}$ be vectors of dimension s_l, with components obtained from the V_i's, h_i's, α_i's, and β_{ij}'s,[4] such that the following hold.

1) Along the solutions of system (4), we have

$$\dot{\boldsymbol{V}}_1(x) \leq \boldsymbol{\delta}_1 (\boldsymbol{h}_1(x)),$$
$$\dot{\boldsymbol{V}}_2(x) \leq \boldsymbol{\mu}_{21} (\boldsymbol{h}_1(x)) + \boldsymbol{\delta}_2 (\boldsymbol{h}_2(x)),$$
$$\vdots$$
$$\dot{\boldsymbol{V}}_q(x) \leq \boldsymbol{\mu}_{q1} (\boldsymbol{h}_1(x)) + \boldsymbol{\mu}_{q2} (\boldsymbol{h}_2(x)) + \cdots + \boldsymbol{\delta}_q (\boldsymbol{h}_q(x)),$$
$$(35)$$

$\forall x \in \mathbb{R}^n$, with $r_q = p$.
2) The matrix Q_l for each diagonal element $\boldsymbol{\delta}_l$ is irreducible as a function and satisfies the linear small-gain-like condition (9).

Then, for any bounded solution $t \mapsto X(x, t)$ of (4)

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{i=1}^p h_i (X(x, s)) \, ds = 0$$

and therefore

$$\liminf_{t \to \infty} \sum_{i=1}^p h_i (X(x, t)) = 0. \quad (36)$$

Moreover,
3a) if all functions β_{ij}/α_j of all off-diagonal elements $\boldsymbol{\mu}_{lm}$ are bounded,

or

3b) if the largest invariant set $\mathcal{H}_p$ contained in the set

$$\Omega_{h_1, \ldots, h_p} = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots = h_p(x) = 0\}$$

is stable,

then

$$\lim_{t \to \infty} \sum_{i=1}^p h_i (X(x, t)) = 0. \quad (37)$$

Proof: Property (36) follows directly from Proposition 1 with $h_i(X(x, t))$ playing the role of $b_i(t)$.

If 3a) holds, (37) follows directly from Remark 5 with $h_i(X(x, t))$ playing the role of $b_i(t)$. Note that the uniform continuity of $t \mapsto h_i(X(x, t))$ follows from the continuity of

[4]In particular

$$\boldsymbol{V}_l = \begin{bmatrix} V_{(r_{l-1}+1)} & V_{(r_{l-1}+2)} & \cdots & V_{r_l} \end{bmatrix}^T$$

$$\boldsymbol{h}_l = \begin{bmatrix} h_{(r_{l-1}+1)} & h_{(r_{l-1}+2)} & \cdots & h_{r_l} \end{bmatrix}^T.$$

h_i, and the boundedness of the locally Lipschitz function $t \mapsto X(x, t)$ (since f is continuous).

If 3*b*) holds suppose that all the blocks have dimension one and $p = 3$. This contains all ingredients necessary for the general case.

Since V_1 is bounded and decreasing along all the trajectories of the system by assumption, the first inequality in (35), namely $\dot{V}_1 \leq -\alpha_1(h_1)$ implies that

$$\lim_{t \to \infty} h_1(X(x, t)) = 0.$$

Since the solution is bounded, $X(x, t)$ has an ω-limit set $\Omega(x)$ which is invariant and compact, the previous limit implies

$$\Omega(x) \subset \Omega_{h_1} = \{x \in \mathbb{R}^n : h_1(x) = 0\}.$$

For every $x_{\omega_1} \in \Omega(x)$, $h_1(x_{\omega_1}) = 0$ which, by the second inequality in (35), implies

$$\lim_{t \to \infty} h_2(X(x_{\omega_1}, t)) = 0$$

and similarly to the previous discussion, this implies

$$\Omega(x_{\omega_1}) \subset \Omega_{h_1, h_2} = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = 0\}.$$

For every $x_{\omega_2} \in \Omega(x_{\omega_1})$, $h_1(x_{\omega_2}) = h_2(x_{\omega_2}) = 0$ which, by the third inequality in (35), implies

$$\lim_{t \to \infty} h_3(X(x_{\omega_2}, t)) = 0$$

and again, this implies

$$\Omega(x_{\omega_2}) \subset \Omega_{h_1, h_2, h_3} = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = h_3(x) = 0\}.$$

This proves that, if the differential inequalities (35) are in triangular form and $\Omega_{h_1, \ldots, h_p}$ is stable, then (37) holds. Note that, if the first block of the differential inequalities (35) has dimension greater than one, then (37) follows directly from Lemma 4 applied to that block. The proof of the general triangular block case can be derived from this last fact and the discussion carried out for the triangular case.

Now we prove that it is sufficient that the largest invariant set $\mathcal{H}_p$ contained in $\Omega_{h_1, \ldots, h_p}$ is stable. Again, for simplicity, consider the case $p = 3$. Assume, by contradiction, that there exist x in $\mathbb{R}^n$, ε strictly positive and a sequence t_m going to infinity with m such that

$$d(X(x, t_m), \mathcal{H}_3) > \varepsilon.$$

Since $\mathcal{H}_3$ is stable there exists δ strictly positive such that, for any χ in $\mathbb{R}^n$ satisfying

$$d(\chi, \mathcal{H}_3) \leq \delta, \tag{38}$$

we have

$$d(X(\chi, s), \mathcal{H}_3) \leq \varepsilon, \qquad \forall s \geq 0. \tag{39}$$

Then, since $\Omega(x)$ is a closed invariant set we have $\Omega(x_{\omega_1}) \subset \Omega(x)$ and since $\mathcal{H}_3$ is the largest invariant set contained in the set Ω_{h_1, h_2, h_3} one has $\Omega(x_{\omega_2}) \subset \mathcal{H}_3$. Now because of the

convergence of $X(x_{\omega_2}, t)$ to its ω-limit set $\Omega(x_{\omega_2})$, hence there exists T_2 such that

$$d(X(x_{\omega_2}, T_2), \mathcal{H}_3) \leq d(X(x_{\omega_2}, T_2), \Omega(x_{\omega_2})) \leq \frac{\delta}{2}$$

and

$$X(x_{\omega_2}, T_2) \in \Omega(x_{\omega_1}) \subset \Omega(x).$$

This means that $X(x_{\omega_2}, T_2)$ is an ω-limit point of $X(x, t)$, there exists T such that

$$|X(x, T) - X(x_{\omega_2}, T_2)| \leq \frac{\delta}{2}.$$

As a result the triangular inequality yields

$$\begin{aligned} d(X(x, T), \mathcal{H}_3) \\ \leq |X(x, T) - X(x_{\omega_2}, T_2)| + d(X(x_{\omega_2}, T_2), \mathcal{H}_3) \\ \leq \delta. \end{aligned}$$

Therefore $\chi = X(x, T)$ satisfies (38), which by (39) yields a contradiction. ∎

Remark 7: The fact that (37) is implied by the stability of $\Omega_{h_1, \ldots, h_p}$ is a restatement of a well-known result, see for instance [23, Lemma I.4]. The fact that (37) is implied by the stability of the largest invariant set $\mathcal{H}_p$ contained in $\Omega_{h_1, \ldots, h_p}$ is a new result.

Remark 8: If 3*b*) holds then (37) implies that $\mathcal{H}_p$ is asymptotically stable.

VI. AN ELEMENTARY EXAMPLE

In this section we present an elementary example which gives a simple illustration of how the results of the paper can be used. Of course, the convergence properties we obtain could be established with classical tools. A more involved example is presented in the next section.

Example 1: Consider the 2-dimensional system describing the Duffing oscillator, namely

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_1 - \beta x_2 - \gamma x_1^3 \end{aligned} \tag{40}$$

with $(x_1, x_2) \in \mathbb{R}^2$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. The equilibrium points are $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (\pm\sqrt{\alpha/\gamma}, 0)$. Let

$$V_1(x_1, x_2) = \frac{1}{\beta} \left(\gamma \frac{x_1^4}{4} - \alpha \frac{x_1^2}{2} + \frac{x_2^2}{2} \right)$$

and

$$V_2(x_1, x_2, x_3) = -\left(\alpha x_1 - \gamma x_1^3\right) x_2.$$

Then

$$\dot{V}_1 = -x_2^2,$$
$$\dot{V}_2 = -\left(\alpha - 3\gamma x_1^2\right) x_2^2 - \left(\alpha x_1 - \gamma x_1^3\right)^2 + \beta \left(\alpha x_1 - \gamma x_1^3\right) x_2. \tag{41}$$

Since V_1 is radially unbounded, the first equality in (41) implies that all trajectories are bounded. Then, selecting

$$c \geq \sup_t |X_1(x_1, x_2, t)|,$$

(41) yields

$$\dot{V}_2 \leq -\frac{1}{2}\left(\alpha x_1 - \gamma x_1^3\right)^2 + \left(3\gamma c^2 - \alpha + \frac{\beta^2}{2}\right) x_2^2,$$

which motivates the choice

$$\alpha_1(s) = s, \qquad \alpha_2(s) = \frac{1}{2}s,$$

$$\beta_{12}(s) = 0, \qquad \beta_{21}(s) = \left(3\gamma c^2 - \alpha + \frac{\beta^2}{2}\right)s,$$

$$b_1 = x_2^2, \qquad b_2 = \left(\alpha x_1 - \gamma x_1^3\right)^2.$$

Note that we have a triangular structure and that

$$\sup_{s>0} \frac{\beta_{21}(s)}{\alpha_1(s)} = \frac{6\gamma c^2 - 2\alpha + \beta^2}{2}$$

is finite, hence Theorem 3 yields

$$\lim_{t\to+\infty} X_2(x_1, x_2, t)^2 + \left(\alpha X_1(x_1, x_2, t) - \gamma X_1(x_1, x_2, t)^3\right)^2 = 0,$$

which implies that the solutions of the system are converging to at least one equilibrium point.

VII. A More Elaborate Example

Consider the class of systems described by the differential equations

$$\begin{aligned}
\dot{x}_1 &= \eta\, x_1 \left(1 - x_1^2 - x_2^2\right) x_3^r + k_1 x_2 \left[\Psi(x_{1+}) + x_3^p\right], \\
\dot{x}_2 &= \eta\, x_2 \left(1 - x_1^2 - x_2^2\right) x_3^r - k_1 x_1 \left[\Psi(x_{1+}) + x_3^p\right], \\
\dot{x}_3 &= -k_2 x_3^q,
\end{aligned} \tag{42}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$, $x_{1+} = \max\{x_1, 0\}$, Ψ is a positive definite function, p and $r \geq p$ are positive even integers, q is a positive odd integer, k_1, k_2 are positive and $\eta \geq 0$. The set of equilibrium points is given by $\{(x_1, x_2, x_3) : x_{1+} = x_3 = 0\}$. Note that

$$\overbrace{x_1^2 + x_2^2 + x_3^2}^{\textstyle\cdot} = -\eta\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 1\right) x_3^r - k_2 x_3^{q+1}.$$

This shows that all solutions are bounded. Let

$$V_1(x_1, x_2, x_3) = \frac{x_3^2}{2k_2}$$

and

$$V_2(x_1, x_2, x_3) = \frac{x_2}{k_1}.$$

Then

$$\dot{V}_1 = -x_3^{q+1},$$

$$\dot{V}_2 = \frac{\eta}{k_1}\, x_2 \left(1 - x_1^2 - x_2^2\right) x_3^r - x_1 \left[\Psi(x_{1+}) + x_3^p\right]$$

$$= -x_1 \Psi(x_{1+}) + x_3^p \left[-x_1 + \frac{\eta}{k_1} x_2 \left(1 - x_1^2 - x_2^2\right) x_3^{r-p}\right]$$

$$\leq -x_{1+} \Psi(x_{1+}) + x_3^p \left[x_{1-} + \frac{\eta}{k_1}\left(x_2\left(1 - x_1^2 - x_2^2\right)\right)_+ x_3^{r-p}\right], \tag{43}$$

where $x_{1-} = -\min\{x_1, 0\}$. These inequalities motivate the choice

$$\boldsymbol{\delta}_1(s) = \alpha_1(s) = s^{\frac{q+1}{2}}, \qquad \boldsymbol{\delta}_2(s) = \alpha_2(s) = s,$$

$$\boldsymbol{\mu}_{21}(s) = \beta_{21}(s) = cs^{\frac{p}{2}},$$

$$\boldsymbol{h}_1(x) = b_1 = x_3^2, \qquad \boldsymbol{h}_2(x) = b_2 = x_{1+}\Psi(x_{1+}),$$

where

$$c \geq x_{1-} + \frac{\eta}{k_1}\left(x_2\left(1 - x_1^2 - x_2^2\right)\right)_+ x_3^{r-p}.$$

Note that we have a triangular structure and that, for any strictly positive $\bar{b}$, we have

$$\sup_{b\in(0,\bar{b}]} \frac{\beta_{21}(b)}{\alpha_1(b)} = \sup_{b\in(0,\bar{b}]} \frac{c}{b^{\frac{q-p+1}{2}}},$$

which is finite for $p \geq q + 1$ and infinite otherwise. By Theorem 3

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \left[X_3(x_1, x_2, x_3, t)^2\right.$$

$$\left. + X_{1+}(x_1, x_2, x_3, t)\Psi\left(X_{1+}(x_1, x_2, x_3, t)\right)\right] = 0.$$

Since

$$\lim_{t\to+\infty} \int_0^t X_3(x_1, x_2, x_3, t)^{q+1}(\tau)d\tau < +\infty,$$

$$\lim_{t\to+\infty} X_3(x_1, x_2, x_3, t) = 0,$$

then

$$\lim_{t\to+\infty} X_{1+}(x_1, x_2, x_3, t)\Psi\left(X_{1+}(x_1, x_2, x_3, t)\right)$$

$$= 0, \qquad \text{if } p \geq q + 1,$$

$$\liminf_{t\to+\infty} X_{1+}(x_1, x_2, x_3, t)\Psi\left(X_{1+}(x_1, x_2, x_3, t)\right)$$

$$= 0, \qquad \text{otherwise.} \tag{44}$$

In what follows we focus on the case $p = q - 1$ and we show that the asymptotic property expressed by the second of equations (44) cannot be improved. To this end re-write the system using polar coordinates (θ, ρ) in the (x_1, x_2)-plane, i.e.,

$$\dot{\rho} = \eta\rho(1 - \rho^2)x_3^r,$$

$$\dot{\theta} = -k_1\left(\Psi\left(\rho(\cos\theta)_+\right) + x_3^{q-1}\right),$$

$$\dot{x}_3 = -k_2 x_3^q.$$

From the first equation, we obtain

$$\rho(t) = \frac{\rho(0)\exp\left(\eta \int_0^t x_3(s)^r ds\right)}{\sqrt{1 - \rho(0)^2 + \rho(0)^2 \exp\left(2\eta \int_0^t x_3(s)^r ds\right)}}.$$

This implies

$$\eta = 0 \qquad\qquad \Rightarrow \rho(t) = \rho_0$$
$$\eta > 0, \; x_3(t)^r \text{ integrable} \qquad \Rightarrow \min(1, \rho_0) \leq \rho(t)$$
$$\leq \max(1, \rho_0)$$
$$\eta > 0, \; x_3(t)^r \text{ not integrable} \quad \Rightarrow \rho(t) \to 1.$$

From the second equation

$$\theta(t) \leq \theta(0) - k_1 \int_0^t x_3(s)^{q-1} ds$$

and from the third equation

$$\frac{dx_3}{x_3} = -k_2 x_3^{q-1} dt$$

and then

$$\frac{1}{k_2} \left(\log(x_3(t)) - \log(x_3(0)) \right) = -\int_0^t x_3(s)^{q-1} ds,$$

that can be directly substituted in the right hand side of $\theta(t)$. As a result

$$\theta(t) \leq \left[\theta(0) - \frac{k_1}{k_2} \log(x_3(0)) \right] + \frac{k_1}{k_2} \log(x_3(t))$$

and, since $\lim_{t\to\infty} x_3(t) = 0$, $\theta(t)$ tends to $-\infty$ modulo 2π, *i.e.*, $\theta(t)$ does not converge. Hence the vector $(x_1(t), x_2(t))$ does not stop turning around the origin. This implies that

$$\limsup_{t \to +\infty} x_{1+}(t) \neq 0,$$

for all $(x_1(0), x_2(0)) \in \mathbb{R}^2/\{0\}$. This last equation shows that the asymptotic property expressed by the second of equations (44) cannot be improved.

A. "lim inf" Convergence Case

Let $p = 2$, $q = 3$, $k_1 = k_2 = 1$, $\Psi(s) = |s|$ and consider the three cases $\eta = 0$; $\eta = 1$ with $r = 2$; and $\eta = 1$ with $r = 4$. Fig. 1 shows the trajectory of the system with initial condition $x(0) = [0.5 \;\; 0 \;\; 1]'$ for the three cases, whereas Fig. 2 shows the time histories of the states x_1, x_2, and x_3. Note that the time axis is in log-scale. Fig. 2 highlights that all trajectories with initial condition off the (x_1, x_2)-plane have an oscillatory behavior with a period that tends to infinity.

Note that trajectories with initial conditions such that $x_3(0) = 0$ converge to the set

$$\left\{ (x_1, x_2) \mid x_1^2 + x_2^2 = x_1(0)^2 + x_2(0)^2, \; x_1 \leq 0 \right\},$$

i.e., to a semi-circle centered at the origin, the size of which depends upon the initial conditions. This set is not stable, hence condition 3*b*) does not hold.
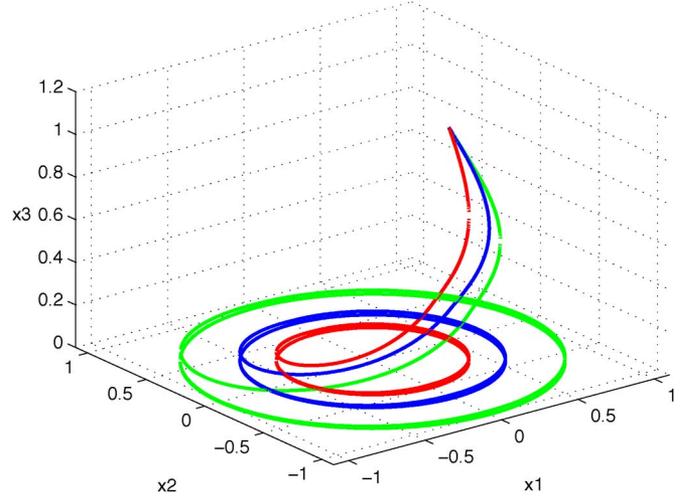

Fig. 1. The trajectory of system (42), with $p = 2$, $q = 3$, $k_1 = k_2 = 1$, $\Psi(s) = |s|$, $x(0) = [0.5 \;\; 0 \;\; 1]'$, for the three considered cases. The trajectory converges to the circle of radius $\rho_0 = 0.5$ for $\eta = 0$ (red/dotted); of radius $\min(1, \rho_0) \leq \rho \leq \max(1, \rho_0)$ for $\eta = 1$ and $r = 4$ (blue/solid); of radius 1 for $\eta = 1$ and $r = 2$ (green/dashed).

Remark 9: The ω-limit set of the trajectories of the system starting off the (x_1, x_2)-plane is, as detailed in [33], a chain recurrent set, which strictly contains the ω-limit set of the trajectories of the system starting in the (x_1, x_2)-plane, consistently with the results in [33] and [34] on asymptotically autonomous semiflows.

Remark 10: As a consequence of the discussion in this section, the (x_1, x_2)-subsystem of system (42), with $p = 2$ and $q = 3$, and x_3 regarded as an input, does not possess the converging-input converging-state property, see [35]–[37]. This does not contradict the result in [35], which highlights (among other things, and similarly to what is done in this paper) the importance of asymptotic stability (of an equilibrium, or of a set) to establish asymptotic properties of solutions.

Remark 11: In [13], further connections with the results in [38] are drawn for one particular case of systems (42).

B. "lim" Convergence Case

Let $p = 2$, $q = 3$, $\Psi(s) = |s|$, $\eta = 0$ and consider the two cases $k_1 = k_2 = 1$; and $k_1 = 1000$, $k_2 = 1$. Fig. 3 shows a trajectory with initial state $x(0) = [1 \;\; 0 \;\; 1]'$. Unlike the previous case $x_1(t)$ and $x_2(t)$ converge to a point such that $x_{1+} = 0$. Note that this is the case also if x_1 and x_2 undergo fast transient (solid line). Finally, Fig. 4 shows the projection of the phase portrait on the (x_1, x_2)-plane for $p = 2$, $q = 3$, $k_1 = k_2 = 1$, $\Psi(s) = |s|$ and $\eta = 0$ and the set of initial conditions $\{(x_1(0), x_2(0), x_3(0)) : x_1(0)^2 + x_2(0)^2 \leq 1, \; x_3(0) = 1\}$.

VIII. CONCLUSION

A class of theorems inspired by the Krasovskii-LaSalle invariance principle has been presented in a unified framework. The contribution of the paper is a tool to study "lim inf" convergence properties of solutions of dynamical systems. In particular the theorems give sufficient conditions to determine the
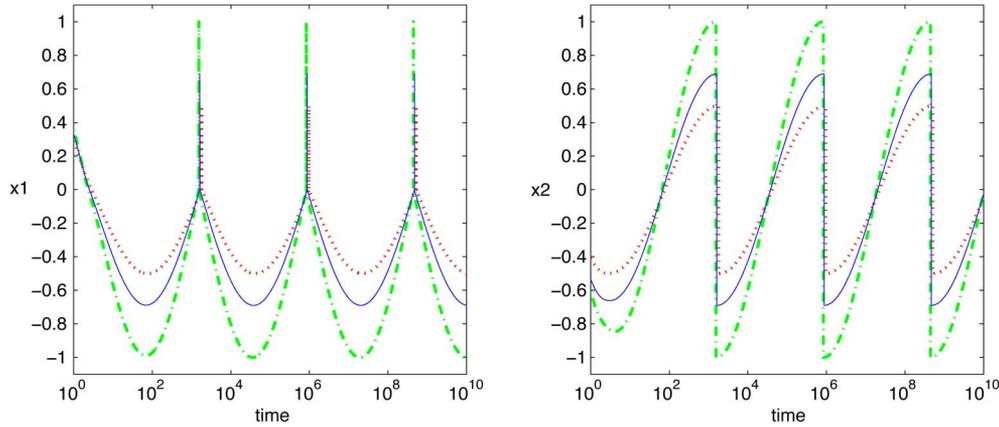
Fig. 2.   Time histories of the states of the system (42) with $p = 2$, $q = 3$, $k_1 = k_2 = 1$, $\Psi(s) = |s|$ and $x(0) = [0.5 \; 0 \; 1]'$: $\eta = 0$ (red/dotted); $\eta = 1$ and $r = 4$ (blue/solid); $\eta = 1$ and $r = 2$ (green/dashed).
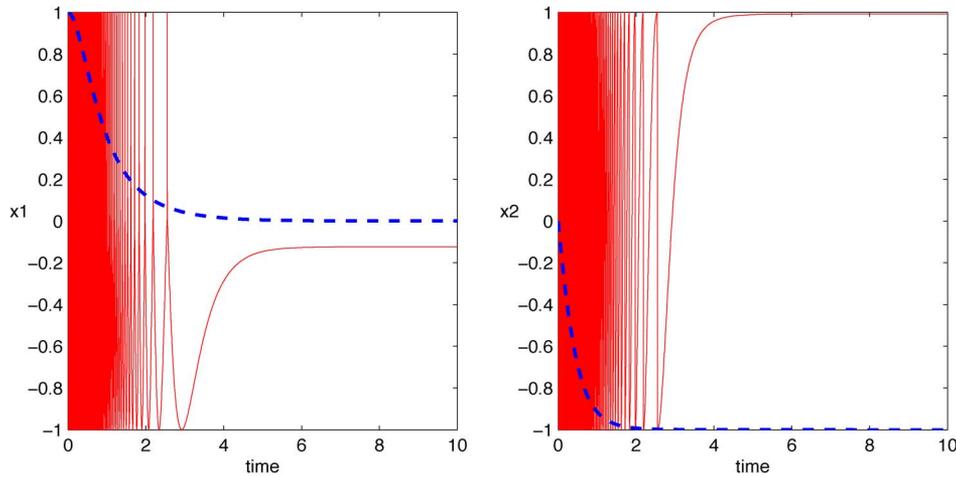

Fig. 3.   Time histories of the states of the system (42) with $p = 2$, $q = 1$, $\Psi(s) = |s|$, $\eta = 0$ and $x(0) = [1 \; 0 \; 1]'$: $k_1 = k_2 = 1$ (blue/dashed); $k_1 = 1000$, $k_2 = 1$ (red/solid).
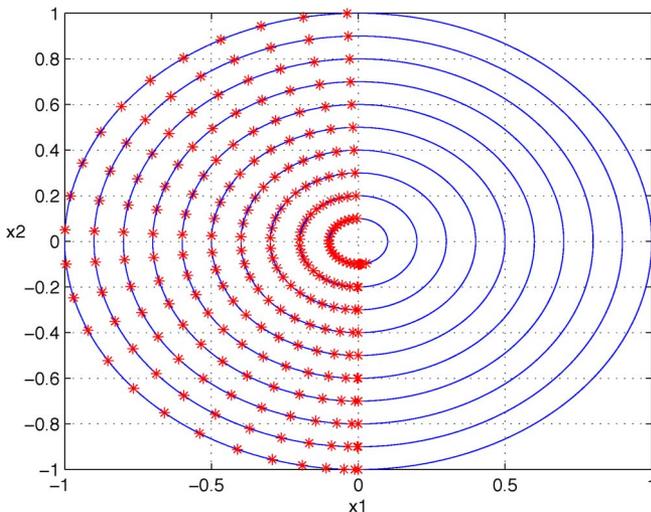

Fig. 4.   Projection of the phase portrait on the (x_1, x_2)-plane for $p = 2$, $q = 3$, $k_1 = k_2 = 1$, $\Psi(s) = |s|$ and $\eta = 0$ and the set of initial conditions $\{(x_1(0), x_2(0), x_3(0)) : x_1(0)^2 + x_2(0)^2 \leq 1, \; x_3(0) = 1\}$. The final states are represented by stars.

convergence in the mean and the "lim inf" convergence. These theorems are derived by a relaxation of Matrosov and Small-gain Theorems, and they are based on a "lim inf" Barbalat's Lemma (Lemma 5 and 7). Additional technical assumptions to have "lim" convergence are given. The "lim inf"/"lim" relation and the role of some of the assumptions are illustrated by means of examples.

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