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Scalable Input-to-state Stability for Performance Analysis of Large-scale Networks

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Abstract— **This paper investigates networks of interconnected systems and introduces the notion of "scalable input-to-state stability" (sISS). This concept is based on input-to-state stability (ISS) and can be interpreted as an extension of the well-known concept of string stability from simple line graphs to general graphs. It guarantees that the trajectories of all states are bounded at all times independently of the network's size and structure and can hence be regarded as an important performance notion. Further, sufficient conditions are derived to guarantee sISS of homogeneous networks with well-defined interconnection structures. In fact, the conditions depend on local ISS Lyapunov functions but guarantee the global condition of sISS. Hence, a first step is made towards developing suitable extensions of string stability to general networks. Two examples are discussed to illustrate the theoretical result.**

I. Introduction

Engineering systems have become increasingly complex and diverse over the last decades and the emerging distinguishing feature of such systems is their large-scale interconnection, resulting from physical interaction or information exchange. Examples include formations of unmanned vehicles, smart grids, sensor networks, and traffic networks.

The complexity caused by the large-scale interconnected nature of such networks might lead to undesired behaviour such as instability or the amplification of perturbations as they propagate through the network. The latter is particularly undesirable as it could lead to cascaded failures or instabilities as the network size grows. Therefore, the main objective of this paper is to provide a *scalable notion of network performance* that prohibits the growth of perturbations.

Stability analysis for large-scale interconnected systems has a long history (see, e.g., [20], [33]) and also network-specific control objectives such as synchronisation and consensus have received a lot of attention in the literature, see, e.g., [11], [22], [18] and the overview [16]. However, these notions have in common that they can be regarded as types of stability and convergence and, despite the long history of robust control [34], only few results are available on notions of *network performance* or *robustness*. Exceptions are given by [3], [2] in which spatially invariant systems are considered. In particular, [2] captures the effect of external disturbances on large-scale interconnected systems in a concept known as coherence. Specifically, it is shown how measures of the rigidity of a formation scale with increasing network size.

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One of the few fields in which network properties beyond stability-like notions are extensively studied is that of vehicle platooning, where formations of closely-spaced vehicles are considered (see [1] for a motivation). As the amplification of perturbations through such string of vehicles could, for large groups, lead to collisions, one of the main control objectives is to prohibit this amplification. This desired behaviour is captured in the notion of *string stability*. Roughly two formulations of string stability can be considered, which have in common that they characterise attenuation of perturbations. First, the early work [12] as well as [28], [21] give a local definition in which perturbations from a vehicle to its follower are attenuated. Second, the approach in [27], [4], [19] call for the existence of a bound on relevant error signals that is *independent* of the length of the platoon. This can be regarded as a global definition as it is based on the entire string, which has the advantage that the notion of string stability is not limited to unidirectional information flow structures as in the first approach. Also, the fact that the bound holds for any string length implies *scalability* of the platoon, as vehicles can be added or removed without requiring renewed string stability analysis.

The above definitions of string stability generally rely on the characterisation of input-output gains for suitable chosen inputs and outputs and linear systems are typically considered in these works. A formal definition on the basis of state trajectories, applicable also for non-linear systems, is given in [32]. Whereas autonomous systems are considered in this work, an extension to platoons with disturbances is given in [5]. Similar definitions involving state trajectories can be found in [17] (for a first step towards an extension to more general network topologies, see [15]). These approaches have the advantages that the effect of initial conditions and transients can explicitly be taken into account. For recent overviews of string stability properties, see [23], [31]. Finally, we stress that string stability should be regarded as a notion for *network performance* rather than stability. In fact, a vehicle formation can be stable in the classical sense even though perturbations grow as they propagate.

Motivated by the need for notions of performance for general large-scale interconnected systems and inspired by results in vehicle platooning, this paper studies scalable performance notions. It has the following two contributions.

First, we introduce scalable input-to-state stability (sISS) as a characterisation of the robustness of a large-scale interconnected system against external disturbances. Specifically, sISS requires the existence of bounds on state deviations that are *independent* of the network size. As such, this prohibits

the amplification of disturbances as they travel through the network. This notion is inspired by input-to-state stability (ISS) [29] and extends the related notion of disturbance string stability [5] to more general network structures.

Second, we provide a sufficient condition for sISS for a class of homogeneous networks that is solely based on ISS properties of the subsystems and the interconnection structure. As such, this can be regarded as a *local* test for the *global* property of sISS, which is in fact independent of the network size. The sufficient condition is based on the construction of a so-called max-separable ISS Lyapunov function for the large-scale system and it is exactly this max-separable structure that enables scalable analysis. The usefulness of such structure in scalable analysis has been recognised before, see [24]. Finally, we mention that this approach draws inspiration from so-called general small-gain conditions for networked systems as studied in [8], [7], [26].

The paper is organized as follows. The notion of sISS is introduced in Section II, whereas Section III presents the main results. Two examples are discussed in Section IV before conclusions are stated in Section V.

Notation. The field of real numbers is denoted by $\mathbb{R}$ and $\mathbb{R}_+ = [0, \infty)$. On the real vector space $\mathbb{R}^n$, the Euclidian norm is denoted as $|\cdot|$. The vector for which all elements equal 1 is written as $\mathbb{1} \in \mathbb{R}^n$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. If, in addition, α is unbounded, then it is of class $\mathcal{K}_\infty$. Moreover, function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed s, $\beta(\cdot, s) \in \mathcal{K}$ and, for each fixed r, the function $\beta(r, \cdot)$ is strictly decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$. Finally, for a bounded function $x : \mathbb{R}_+ \to \mathbb{R}^n$, we define $\|x\|_\infty = \sup_{t\in\mathbb{R}_+} |x(t)|$.

II. Scalable input-to-state stability

Consider the spatially invariant large-scale system comprised of N subsystems Σ_i of the form

$$\Sigma_i : \quad \dot{x}_i = f\big(x_i, \{x_{i+j}\}_{j\in\mathcal{N}}, d_i\big), \tag{1}$$

with subsystem state $x_i \in \mathbb{R}^n$, local disturbance $d_i \in \mathbb{R}^m$, and $i \in \mathcal{I}_N := \{1, 2, \ldots, N\}$. It is assumed that $f(0, \{0, \ldots, 0\}, 0) = 0$, f is locally Lipschitz in x_i and $\{x_{i+j}\}_{j\in\mathcal{N}}$, and continuous in d_i. In addition to its own state, the dynamics of Σ_i in (1) depend on the states of its neighbours, as captured through the set of relative neighbours $\mathcal{N} \subset \mathbb{Z}$ with $0 \notin \mathcal{N}$. In particular, the index $i+j$ in characterizing the neighbours should be understood modulo N (precisely, $i + j$ should be read as $1 + ((i + j - 1) \bmod N)$, where $k \bmod N$ denotes the remainder after Euclidean division of k by N.). Thus, as an example, $\mathcal{N} = \{-1, 1\}$ characterises a circular interconnection structure with bidirectional coupling between neighbours. Another example is given in Figure 1.

After collecting the states and disturbances as $x^{\mathrm{T}} = [\, x_1^{\mathrm{T}} \ \cdots \ x_N^{\mathrm{T}} \,]$ and $d^{\mathrm{T}} = [\, d_1^{\mathrm{T}} \ \cdots \ d_N^{\mathrm{T}} \,]$, the interconnected system Σ can be written compactly as

$$\Sigma : \quad \dot{x} = F_N(x, d), \tag{2}$$

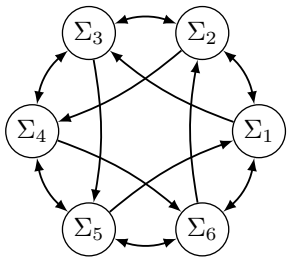

Fig. 1. Example of Σ in (2) with $\mathcal{N} = \{-2, -1, 1\}$ and $N = 6$.

where

$$F_N(x, d) = \begin{bmatrix} f\big(x_1, \{x_{1+j}\}_{j\in\mathcal{N}}, d_1\big) \\ \vdots \\ f\big(x_N, \{x_{N+j}\}_{j\in\mathcal{N}}, d_N\big) \end{bmatrix} \tag{3}$$

with $F_N(0, 0) = 0$. We remark that the system (2) is well-defined for each $N \geq N_{\min}$ with $N_{\min} = 1 + \max\{|j| \mid j \in \mathcal{N}\}$, which gives the minimal system size for which the absence of self-loops is guaranteed. In fact, it is easy to show that F_N in (3) is locally Lipschitz in x and continuous in d for any N. Thus, (2) can be regarded as a family of systems, in which each system is characterised by its number of subsystems N. We note that the number of neighbours for a given subsystem is independent of the network size.

In this paper, we are interested in a *scalable* performance notion of Σ in (2), i.e., a characterisation that is independent of the size N. This leads to the following definition.

Definition 1: The system (2) is said to be scalable input-to-state stable (sISS) if there exist functions $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{K}_\infty$ such that, for any $N \in \mathbb{N}$ such that $N \geq N_{\min}$, for any initial condition $x_i(0) \in \mathbb{R}^n$, $i \in \mathcal{I}_N$, and any disturbance function $d_i(\cdot)$, $i \in \mathcal{I}_N$, the solution $x_i(\cdot)$ satisfies

$$\max_{i\in\mathcal{I}_N} |x_i(t)| \leq \beta\left(\max_{i\in\mathcal{I}_N} |x_i(0)|, t\right) + \sigma\left(\max_{i\in\mathcal{I}_N} \|d_i\|_\infty\right) \tag{4}$$

for all $t \geq 0$.

This notion of sISS stability extends the concept of ISS originally introduced in [29]. The key distinguishing features are, first, the choice of norm on the state x in terms of the largest Euclidean norm over all subsystems and, second, the fact that the condition should hold for all $N \geq N_{\min}$. Exactly these features enable the interpretation of sISS as a *scalable performance notion* of large-scale systems. Namely, the fact that the bound (4) holds regardless of system size means that state perturbations do not grow without bound whenever subsystems are added or removed. Stated differently, sISS prohibits the amplification of perturbations as they propagate through the network.

Remark 1: Even though sISS is defined for large-scale systems of the form (1), Definition 1 is independent of the interconnected structure and does not require homogeneity of the subsystem dynamics. Consequently, sISS could be defined for more general networks, as long as the network structure is well-defined and consistent for growing size N.

Remark 2: Taking a different perspective, sISS can be interpreted as an extension of the notion of string stability,

see, e.g., [23]. Whereas string stability characterises the attenuation of disturbances in a group of vehicles which can typically be regarded as a cascaded system, Definition 1 considers more general network structures. A formal definition of string stability was given in [32] (see also [12] for an early characterization) and is extended to include disturbances on all vehicles in [5]. Definition 1 can in fact be regarded as an extension of [5, Definition 3]. The work [31] provides an alternative approach in extending string stability to more general network topologies.

III. Main results

In this section, a sufficient condition for sISS will be given in terms of ISS properties of the subsystems Σ_i. To this end, we assume that (1) is input-to-state stable with respect to the states of its neighbours $\{x_{i+j}\}_{j\in\mathcal{N}}$ and the external disturbance d_i, which is easy to verify in practice.

Assumption 1: There exists a differentiable ISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ for the subsystems (1), i.e., the following two conditions hold:

i) There exist $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_i \in \mathbb{R}^n$,
$$\nu_1(|x_i|) \le V(x_i) \le \nu_2(|x_i|). \tag{5}$$

ii) There exist functions $\alpha \in \mathcal{K}_\infty$, $\gamma_j \in \mathcal{K}_\infty$, $j \in \mathcal{N}$ and $\mu \in \mathcal{K}_\infty$, such that
$$\frac{\partial V}{\partial x}(x_i) f\big(x_i, \{x_{i+j}\}_{j\in\mathcal{N}}, d_i\big) \le -\alpha\big(V(x_i)\big) + \sum_{j\in\mathcal{N}} \gamma_j\big(V(x_{i+j})\big) + \mu(|d_i|) \tag{6}$$
for all $x_i \in \mathbb{R}^n$, all $x_{i+j} \in \mathbb{R}^n$ for $j \in \mathcal{N}$, all $d_i \in \mathbb{R}^m$.

We note that, due to the uniformity of the subsystems Σ_i in (1), the function V in Assumption 1 is independent of i.

Now, the main result of this paper can be stated as follows.

Theorem 1: Consider the large-scale system (2) and let the subsystems be such that Assumption 1 holds. If there exists a function $\rho \in \mathcal{K}_\infty$ such that
$$-\alpha(s) + \sum_{j\in\mathcal{N}} \gamma_j(s) \le -\rho(s) \tag{7}$$
for all $s \ge 0$, then (2) is scalable input-to-state stable.

Proof: To prove the result, it will be shown that the max-separable function $V_N : \mathbb{R}^{Nn} \to \mathbb{R}_+$ defined as
$$V_N(x) = \max\big\{V(x_1), \dots, V(x_N)\big\} \tag{8}$$
is an ISS Lyapunov function for Σ in (2) *for any* N.

Thereto, take any $N \in \mathbb{N}$ satisfying $N \ge N_{\min}$ and consider (8). As V_N is generally not continuously differentiable, the upper-right Dini derivative of V_N in the direction $v \in \mathbb{R}^n$ is introduced as
$$D^+V_N(x, v) = \limsup_{h\to 0^+} \frac{V_N\big(x + hv\big) - V_N(x)}{h}, \tag{9}$$
see, e.g., [25]. Given that V is differentiable according to Assumption 1, the max-separable structure (8) leads to
$$D^+V_N\big(x, F_N(x,d)\big) = \max_{j\in\mathcal{J}(x)} \frac{\partial V}{\partial x}(x_j) f\big(x_j, \{x_{j+l}\}_{l\in\mathcal{N}}, d_j\big), \tag{10}$$
where $\mathcal{J}(x) = \{j \mid V(x_j) = V_N(x)\}$ denotes the set of indices for which the maximum in (8) is obtained, see [6].

For a given $x \in \mathbb{R}^{Nn}$, let $k \in \mathcal{J}(x)$ be an index (not necessarily unique) that achieves the maximum in (10). Then,
$$\begin{aligned} D^+V_N\big(x, F_N(x,d)\big) &= \max_{j\in\mathcal{J}(x)} \frac{\partial V}{\partial x}(x_j) f\big(x_j, \{x_{j+l}\}_{l\in\mathcal{N}}, d_j\big), \\ &= \frac{\partial V}{\partial x}(x_k) f\big(x_k, \{x_{k+l}\}_{l\in\mathcal{N}}, d_k\big), \\ &\le -\alpha\big(V(x_k)\big) + \sum_{l\in\mathcal{N}} \gamma_l\big(V(x_{k+l})\big) + \mu\big(|d_k|\big), \end{aligned} \tag{11}$$
where the inequality (11) follows from ISS of the subsystems through (6) in Assumption 1 (for $i = k$). Now, as $k \in \mathcal{J}(x)$, it follows from the definition of $\mathcal{J}(x)$ that
$$V(x_i) \le V(x_k) = V_N(x) \tag{12}$$
for all $i \in \mathcal{I}_N$. Given that $\gamma_l \in \mathcal{K}_\infty$, the inequality (12) can be used to bound the right-hand-side of (11) to obtain
$$D^+V_N\big(x, F_N(x,d)\big) \le -\alpha\big(V_N(x)\big) + \sum_{l\in\mathcal{N}} \gamma_l\big(V_N(x)\big) + \mu\big(|d_k|\big), \tag{13}$$
where the substitution $V(x_k) = V_N(x)$ (recall again (12)) is used in the first term of the right hand side. A similar reasoning exploiting the fact that $\mu \in \mathcal{K}_\infty$ leads to
$$D^+V_N\big(x, F_N(x,d)\big) \le -\alpha\big(V_N(x)\big) + \sum_{l\in\mathcal{N}} \gamma_l\big(V_N(x)\big) + \mu\Big(\max_{l\in\mathcal{I}_N} |d_l|\Big), \tag{14}$$
whereas condition (7) gives
$$D^+V_N\big(x, F_N(x,d)\big) \le -\rho\big(V_N(x)\big) + \mu\Big(\max_{l\in\mathcal{I}_N} |d_l|\Big). \tag{15}$$
At this point, it is remarked that (15) holds for all $x \in \mathbb{R}^{Nn}$, all $d \in \mathbb{R}^{Nm}$, and, most importantly, for any $N \ge N_{\max}$. Moreover, (15) represents a characterization of ISS in dissipation form, albeit with non-smooth ISS Lyapunov function V_N. In order to employ results on ISS with non-differentiable ISS Lyapunov functions, we note that (15) gives rise to the so-called implication form
$$(1-\varepsilon)\rho\big(V_N(x)\big) \ge \mu\Big(\max_{l\in\mathcal{I}_N} |d_l|\Big) \implies D^+V_N\big(x, F_N(x,d)\big) \le -\varepsilon\rho\big(V_N(x)\big), \tag{16}$$
for any $\varepsilon \in (0,1)$. Then, the use of [7, Theorem 2.3] (see also [13] and [10] for extensions of ISS to the non-differentiable case) guarantees the existence of functions $\tilde\beta \in \mathcal{KL}$ and $\tilde\sigma \in \mathcal{K}_\infty$ such that
$$V_N(x(t)) \le \tilde\beta\big(V_N(x(0)), t\big) + \tilde\sigma\Big(\max_{i\in\mathcal{I}_N} \|d_i\|_\infty\Big) \tag{17}$$
for all $t \ge 0$. Here, $x(\cdot)$ is the state trajectory of (2) for $x(0) \in \mathbb{R}^{Nn}$ and bounded disturbance function $d(\cdot)$.

In order to obtain a bound of the form (4), it is noted that the properties of the Lyapunov function V in Assumption 1 (see (5)) imply, for any $i \in \mathcal{I}_N$,

$$|x_i(t)| \leq \nu_1^{-1}\big(V(x_i(t))\big) \leq \nu_1^{-1}\big(V_N(x(t))\big), \tag{18}$$

where the latter inequality is a result of the definition of V_N in (8). Note that ν_1^{-1} exists as $\nu_1 \in \mathcal{K}_\infty$ and recall that ν_1^{-1} is itself of class $\mathcal{K}_\infty$. Similarly, it holds for any $i \in \mathcal{I}_N$ that

$$V(x_i(0)) \leq \nu_2\big(|x_i(0)|\big) \leq \nu_2\left(\max_{i\in\mathcal{I}_N}|x_i(0)|\right), \tag{19}$$

where (5) is used again. Since (18) and (19) hold for any $i \in \mathcal{I}_N$ (including where the respective maxima are obtained), the use of (18) and (19) in (17) yields

$$\begin{aligned}&\max_{i\in\mathcal{I}_N}|x_i(t)|\\ &\leq \nu_1^{-1}\left(\tilde{\beta}\left(\nu_2\left(\max_{i\in\mathcal{I}_N}|x_i(0)|\right),t\right)+\tilde{\sigma}\left(\max_{i\in\mathcal{I}_N}\|d_i\|_\infty\right)\right).\end{aligned} \tag{20}$$

Then, the use of the property (e.g., [29])

$$\nu_1^{-1}(s_1+s_2) \leq \nu_1^{-1}(2s_1)+\nu_1^{-1}(2s_2), \tag{21}$$

for $\nu_1^{-1} \in \mathcal{K}_\infty$, leads to a bound of the form (4) with

$$\beta(s_1,s_2) = \nu_1^{-1}\big(2\tilde{\beta}(\nu_2(s_1),s_2)\big),\ \sigma(s) = \nu_1^{-1}\big(2\tilde{\sigma}(s)\big). \tag{22}$$

We stress again that N was chosen arbitrarily, such that (20) holds for any $N \in \mathbb{N}$ (with $N \geq N_{\min}$). More importantly, the functions ρ and σ in (15) are *independent* of N as a result of the max-separable structure of the Lyapunov function V_N in (8). As a result, also $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{K}_\infty$ in (22) are independent of N, such that the bound (4) is uniform for all $N \in \mathbb{N}$ and sISS as in Definition 1 is proven. ■

Theorem 1 provides a characterization of sISS that, first, depends on properties of the subsystems only, and, second, is independent of the network size N. As (7) is easy to verify, this theorem provides a practically relevant test for sISS.

Remark 3: The proof of Theorem 1 crucially depends on the introduction of the ISS Lyapunov function candidate (8), whose max-separable structure enables the desired scalability properties (see (18) and (19) for a relation to the signal norms used in (4)). This approach is inspired by results on ISS for networks originally developed in [8], [7] and [26]. However, whereas these works consider ISS for a given network of *fixed* size, Theorem 1 guarantees an ISS property of a family of systems with *arbitrary* network size, see Definition 1. We note that input-to-state stability of a network of fixed size (even if the ISS property holds regardless of size) does not imply sISS as uniformity of the gain functions is required in (4), cf. Example 1. Nonetheless, for a fixed network size, the condition (7) reflects the results in [26, Section 3.4].

The results in Theorem 1 can directly be exploited in the scope of cascaded systems. To this end, consider

$$\begin{aligned}\dot{x}_1 &= f(x_1,0,d_1),\\ \dot{x}_i &= f(x_i,x_{i-1},d_i), \qquad i \in \mathcal{I}_N\backslash\{1\},\end{aligned} \tag{23}$$

where $x_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}^m$ for $i \in \mathcal{I}_N$ as before. Also, f is assumed to be locally Lipschitz in the first two arguments, continuous in the third argument, and satisfies $f(0,0,0) = 0$. The cascaded system (23) can be regarded as a system Σ as in (2) with $\mathcal{N} = \{-1\}$, with the exception that the first subsystem has no incoming links.

Nonetheless, the following result is immediate.

Corollary 1: Consider the large-scale cascaded system (23) and let the subsystems be such that Assumption 1 holds for $\mathcal{N} = \{-1\}$. If there exists a function $\rho \in \mathcal{K}_\infty$ such that

$$-\alpha(s)+\gamma_{-1}(s) \leq -\rho(s) \tag{24}$$

for all $s \geq 0$, then (23) is scalable input-to-state stable.

Proof: This is a direct result of Theorem 1 after observing that (6) also holds for subsystem 1 in (23) despite the absence of the incoming link from x_N. ■

Corollary 1 highlights the interpretation of sISS as a robustness property in which effects of neighbouring systems are taken to be adversarial (we note that such perspective also forms the basis for small-gain conditions as in, e.g., [14]). Namely, the removal of any interconnection in the large-scale system (2) does not affect the result of Theorem 1, enabling the scalable analysis of (classes of) heterogeneous networks. In a similar way, homogeneity of subsystem dynamics is not required as long as the same estimates (6) hold.

Remark 4: The relevance of Corollary 1 is in the scope of scalable stability properties for vehicle platoons usually referred to as string stability. Namely, such systems are generally modelled using a unidirectional and non-cyclic interconnection topology reflecting the case in which a vehicle only exploits information of its predecessor, e.g., [30], [27], [19]. In the scope of vehicle platoons, disturbance propagation in cascaded systems of the form (23) along the lines of this paper is studied in [5].

IV. Examples

In this section, we present two simple examples to illustrate the results from the previous section. The first example illustrates Definition 1 by presenting a linear system that is ISS for each fixed $N \in \mathbb{N}$, but is *not* sISS. The second example is a non-linear system that has the sISS property.

Example 1: Consider the subsystems

$$\dot{z}_i = -2z_i + z_{i+1} + z_{i-1} + w_i, \tag{25}$$

with state $z_i \in \mathbb{R}$, disturbance $w_i \in \mathbb{R}$, and $i \in \mathcal{I}_N$. After defining $z = [\ z_1\ \ \dots\ \ z_N\ ]^{\mathrm{T}}$ and $w = [\ w_1\ \ \dots\ \ w_N\ ]^{\mathrm{T}}$, the large-scale system comprising the subsystems (25) can compactly be written as $\dot{z} = -L_N z + w$, where the matrix

$$L_N = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1\\ -1 & 2 & -1 & & & 0\\ 0 & -1 & 2 & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \ddots & 0\\ 0 & & & \ddots & 2 & -1\\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{26}$$

has a circulant structure as the indices in (25) are interpreted modulo N. In this example, we are interested in the deviations from the average state and disturbance defined as

$$x_i = z_i - \tfrac{1}{N}\mathbb{1}^{\mathrm{T}} z, \quad d_i = w_i - \tfrac{1}{N}\mathbb{1}^{\mathrm{T}} w, \tag{27}$$

respectively. It is easily shown that this satisfies the dynamics

$$\dot{x} = -L_N x + d, \tag{28}$$

with $x = [\ x_1\ \ \dots\ \ x_N\ ]^{\mathrm{T}}$ and $d = [\ d_1\ \ \dots\ \ d_N\ ]^{\mathrm{T}}$. We remark however that $\mathbb{1}^{\mathrm{T}} d(t) = 0$ for all $t \geq 0$ and that, as $\mathbb{1}^{\mathrm{T}} x(0) = 0$ by definition, trajectories of (28) satisfy $\mathbb{1}^{\mathrm{T}} x(t) = 0$ for all $t \geq 0$. This not only follows from (27), but is also apparent from (28) as $\ker \mathbb{1}^{\mathrm{T}}$ is an L_N-invariant subspace. In fact, the restriction of L_N to $\ker \mathbb{1}^{\mathrm{T}}$ is asymptotically stable, from which it is immediate that (28) is input-to-state stable. Using the equivalence of norms

$$\max_{i\in\mathcal{I}_N} |x_i| \leq |x| \leq \sqrt{N} \max_{i\in\mathcal{I}_N} |x_i|, \tag{29}$$

there thus exist functions $\beta_N \in \mathcal{KL}$ and $\sigma_N \in \mathcal{K}_\infty$ such that

$$\max_{i\in\mathcal{I}_N} |x_i(t)| \leq \beta_N\Big(\max_{i\in\mathcal{I}_N} |x_i(0)|, t\Big) + \sigma_N\Big(\max_{i\in\mathcal{I}_N} \|d_i\|_\infty\Big). \tag{30}$$

From the above, we conclude that (28) (restricted to $\ker \mathbb{1}^{\mathrm{T}}$) is ISS for each fixed N. In the remainder of this example, it will be shown that (28) is *not* sISS as in Definition 1.

To this end, observe that the smallest eigenvalue of the restriction of L_N to $\ker \mathbb{1}^{\mathrm{T}}$ (this is the second-smallest eigenvalue of L_N as $L_N\mathbb{1} = 0$) is given by

$$\lambda = 2 - 2\cos\left(\frac{2\pi}{N}\right) \tag{31}$$

with multiplicity 2, as follows from the theory of circulant matrices, see [9]. A corresponding eigenvalue $v = [\ v_1\ \ \dots\ \ v_N\ ]^{\mathrm{T}}$ has components

$$v_i = 1 - \cos\left(\frac{2\pi(i-1)}{N}\right), \quad i \in \mathcal{I}_N, \tag{32}$$

and is scaled such that $\max_{i\in\mathcal{I}_N} |v_i| = v_1 = 1$.

Now, we will construct a lower bound on σ_N in (30) by considering trajectories of (28) for the constant disturbance $d(t) = rv$, $t \geq 0$, for some fixed $r > 0$. It readily follows from (32) that $\mathbb{1}^{\mathrm{T}} d(t) = 0$ for all $t \geq 0$ as required and that $\max_{i\in\mathcal{I}_N} \|d_i\|_\infty = r$. Choosing $x(0) = 0$, the resulting trajectory of (28) satisfies

$$x(t) = \int_0^t e^{-L_N(t-s)} rv \,\mathrm{d}s = \int_0^t rv e^{-\lambda(t-s)} \,\mathrm{d}s, \tag{33}$$

where the latter equality follows from the fact that v is an eigenvalue of L_N. Consequently, $\lim_{t\to\infty} x(t) = r\lambda^{-1}v$. Thus, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\max_{i\in\mathcal{I}_N} |x_i(t)| \geq |x_1(t)| \geq \frac{r(1-\varepsilon)}{\lambda}. \tag{34}$$

for all $t > T$. Here, the latter inequality follows from the property $v_1 = 1$, see (32). By recalling $\max_{i\in\mathcal{I}_N} \|d_i\|_\infty = r$ and comparing (34) to (30), it follows that the gain function σ_N necessarily satisfies

$$\sigma_N(r) \geq \frac{r(1-\varepsilon)}{\lambda} = \frac{r(1-\varepsilon)}{2 - 2\cos\big(\frac{2\pi}{N}\big)}, \tag{35}$$

where (31) is used to obtain the equality. For a given r and ε, it is clear that (35) grows without bound as N grows. Hence, there does not exist a gain function $\sigma \in \mathcal{K}_\infty$ such that $\sigma_N(r) \leq \sigma(r)$, $r \geq 0$ for all $N \in \mathbb{N}$. Consequently, the dynamics describing the deviations from the average in (28) is *not* scalable input-to-state stable.

Remark 5: The reasoning of Example 1 relies on L_N in (26) being circulant and can be easily extended to more general circulant matrices capturing relative measurements. Then, it can be concluded that such spatially invariant systems are not sISS, i.e., the performance measure capturing the deviation from the average scales unfavourably as the network size grows. Indeed, it is known that the system (28) suffers from a lack of *coherence*, as is shown in [2]. Nonetheless, we stress that the alternative performance measure of coherence differs significantly from sISS as stochastic disturbances and the $\mathcal{H}_2$ system norm are studied in the former. Moreover, the notion of sISS is applicable to non-linear systems, whereas [2] considers linear systems.

Example 2: Consider the non-linear subsystems

$$\dot{x}_i = -x_i - x_i^3 + x_i x_{i-1}^2 + d_i \tag{36}$$

with state $x_i \in \mathbb{R}$, disturbance $d_i \in \mathbb{R}$, and $i \in \mathcal{I}_N$. Note that (36) is of the form (1) with $\mathcal{N} = \{-1\}$. To show that the system is scalable input-to-state stable as in Definition 1, we first show that Assumption 1 holds by choosing the function $V(x_i) = x_i^2$. Namely, a direct computation shows that

$$\begin{aligned}\frac{\partial V}{\partial x}(x_i) f\big(x_i, \{x_{i+j}\}_{j\in\mathcal{N}}, d_i\big) &= -2x_i^2 - 2x_i^4 + 2x_i^2 x_{i-1}^2 + 2x_i d_i, &(37)\\ &= -x_i^2 - x_i^4 + x_{i-1}^2 + d_i^2 \\ &\quad - (x_i^2 - x_{i-1}^2)^2 - (x_i - d_i)^2, &(38)\end{aligned}$$

where the result (38) is readily verified by completing the squares. Then, it follows that (36) satisfies (6) with

$$\alpha(s) = s^2 + s, \quad \gamma_{-1}(s) = s^2, \quad \mu(s) = s^2, \tag{39}$$

such that it can be verified that (7) holds with $\rho(s) = s$. Consequently, by Theorem 1, the system (36) is sISS.

Figure 2 shows a simulation of (36). It can be observed that all states remain bounded despite the influence of the disturbance and the effect of the initial conditions die away.

The same simulation is then repeated for systems with network size ranging from $N = 10$ to $N = 200$. Figure 3 illustrates that the maximum state perturbations remain bounded independent of network size, i.e., the system is sISS.

V. Conclusions

The notion of scalable input-to-state stability (sISS) is introduced as a performance notion for large-scale networks. It guarantees the boundedness of state trajectories independent

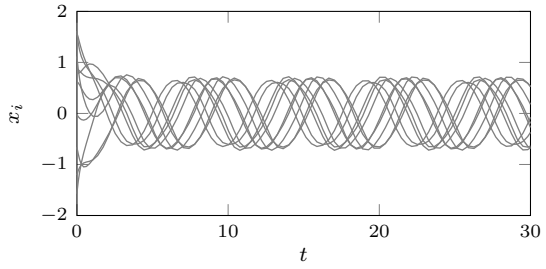

Fig. 2. Simulation results for the example system (36) with $N = 10$: state trajectories for randomly chosen initial conditions $x_i(0) \in [-2, 2]$ and $d_i(t) = \sin(t + \phi_i)$ with randomly chosen phase ϕ_i.

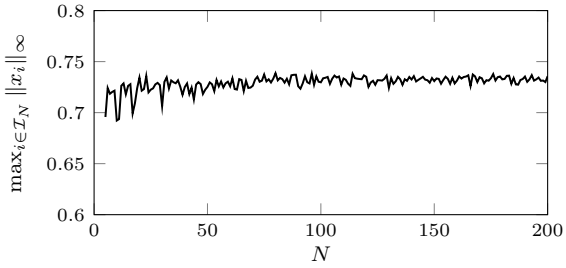

Fig. 3. Simulation results for the example system (36): maximum state perturbation $\max_{i\in\mathcal{I}_N} \|x_i\|_\infty$ in steady-state for disturbances $d_i(t) = \sin(t + \phi_i)$ with randomly chosen phase ϕ_i.

of the network size, prohibiting the amplification of perturbations as they propagate through the network. A sufficient condition for sISS is given on the basis of ISS properties of the subsystems, providing a local and easily verifiable condition for the global performance notion of sISS.

We view these results as the first steps towards scalable performance notions for arbitrary large-scale networks and future work will aim at generalisations to allow for heterogeneous dynamics or more general interconnection structures.

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