Robustness of ISS systems to inputs with limited moving average: Application to spacecraft formations

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SUMMARY

We provide a theoretical framework that fits realistic challenges related to spacecraft formation with disturbances. We show that the input-to-state stability of such systems guarantees some robustness with respect to a class of signals with bounded average-energy, which encompasses the typical disturbances acting on spacecraft formations. Solutions are shown to converge to the desired formation, up to an offset, which is somewhat proportional to the considered moving average of disturbances. In the presence of fast peaking perturbations, the approach provides a tighter evaluation of the disturbances' influence, which allows for the use of more parsimonious control gains. Copyright © 2015 John Wiley & Sons, Ltd.

1. INTRODUCTION

Spacecraft formation control is a relatively new and active field of research. Formations, characterized by the ability to maintain relative positions without real-time ground commands, are motivated by the aim of placing measuring equipment further apart than what is possible on a single spacecraft. This is desirable as the resolution of measurements is often proportional to the baseline length, meaning that either a large monolithic spacecraft or a formation of smaller, but accurately controlled, spacecraft may be used. Monolithic spacecraft architecture that satisfy the demand of resolution are often both impractical and costly to develop and to launch. On the other hand, smaller spacecraft may be standardized and have lower development cost. In addition, they may be of a lower collective weight and/or of smaller collective size such that cheaper launch vehicles can be used. There is also the possibility for them to share launch vehicle with other spacecraft. These advantages come at the cost of an increased complexity. From a control design perspective, a crucial challenge is to maintain a predefined relative trajectory, even in presence of disturbances. Most of these disturbances are hard to model in a precise manner. Only statistical or averaged characteristics of the perturbing signals (e.g., amplitude, energy, and average energy) are typically available. These perturbing signals may have diverse origins:

Intervehicle interference. In close formation or spacecraft rendezvous, thruster firings and exhaust gases may influence other spacecraft [1].
Solar wind and radiation. Particles and radiation expelled from the sun influence the spacecraft and are highly dependent on the solar activity [2], which is difficult to predict [3].

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Small debris. While large debris would typically mean the end of the mission, some space trash, including paint flakes, dust, coolant, and even small needles[‡], is small enough to 'only' deteriorate the performance; see [5].

Micrometeoroids. The damages caused by micrometeoroids may be limited because of their tiny size, but constant high-velocity impacts also degrade the performance of the spacecraft through momentum transfer [6].

Gravitational disturbances. Even gravitational models, including higher-order zonal harmonics, can only achieve a limited level of accuracy due to the shape and inhomogeneity of the Earth. In addition comes the gravitational perturbation due to other gravitating bodies such as the Sun and the Moon.

Actuator mismatch. There is commonly a mismatch between the actuation computed by the control algorithm, and the actual actuation that the thrusters can provide. This mismatch is particularly present if the control algorithm is based on continuous dynamics, without taking into account the pulse-based functioning of thrusters.

Nonlinear control theory provides instruments to guarantee a prescribed precision in spite of these disturbances. Input-to-state stability (ISS) is a concept introduced in [7], which has been thoroughly treated in the literature: see for instance the survey [8] or the more recent survey [9] and references therein. Roughly speaking, this robustness property ensures asymptotic stability, up to a term that is 'proportional' to the *amplitude* of the disturbing signal. Similarly, its integral extension, iISS [10], links the convergence of the state to a measure of the *energy* that is fed by the disturbance into the system. However, in the original works on ISS and iISS, both these notions require that these indicators (amplitude or energy) be finite to guarantee some robustness. In particular, while this concept has proved useful in many control applications, ISS may yield very conservative estimates when the disturbing signals come with high amplitude even if their *moving average* is reasonable.

These limitations have already been pointed out and partially addressed in the literature. In [11], the notions of 'Power ISS' and 'Power iISS' were introduced to estimate more tightly the influence of the power or moving average of the exogenous input on the *power* of the state. Under the assumption of local stability for the zero-input system, these properties were shown to be actually equivalent to ISS and iISS, respectively. Nonetheless, for a generic class of input signals, no hard bound on the state norm can be derived for this work. Other works have focused on quantitative aspects of ISS, such as [12, 13] and [14]. These three papers solve the problem by introducing a 'memory fading' effect in the input term of the ISS formulation. In [12], the perturbation is first fed into a linear scalar system whose output then enters the right-hand side of the ISS estimate. The resulting property is referred to as exp-ISS and is shown to be equivalent to ISS. In [13] and [14], the concept of input-to-state dynamical stability is introduced and exploited. In the input-to-state dynamical stability state estimate, the value of the perturbation at each time instant is used as the initial value of a one-dimensional system, thus generalizing the original idea of [12]. The quantitative knowledge of how past values of the input signal influence the system allows, in particular, to guarantee an explicit decay rate of the state for vanishing perturbations.

Many papers have dealt with vanishing perturbations, and have defined appropriate classes to investigate their stability properties; see for instance [15] and [16]. Notable work on persistent perturbations, except for the aforementioned ISS type formulations, is [17] where they consider a class of signals similar to ours. It can be shown that their class is strictly more general than the one introduced in this paper. In [18, p. 101], stability with respect to persistent disturbances that are bounded in the mean is defined. The said definition characterizes a local property and does not give an explicit bound on the solutions. Those are the key differences to the stability definitions of this paper.

The new stability definitions introduced in this paper will be used to analyse the stability properties of a spacecraft formation in leader-follower configuration. The ISS framework was first used for formation control in [19, 20]. Since then, their work has been extended in various ways, for example, to formation control of non-holonomic vehicles [21, 22], control of complex formation

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[‡]Project West Ford was a test carried out in the early 1960s, where 480 million needles were placed in orbit, with the aim to create an artificial ionosphere above the Earth to allow global radio communication, [4].

topologies [23], and formation control under varying communication topologies and delay [24]. A wide range of applications has been considered, for instance, autonomous underwater vehicles [24], spacecraft [25] and unmanned aerial vehicles [26].

In this paper, our objective is to guarantee *hard bound* on the state norm for ISS systems in the presence of signals with possibly unbounded amplitude and/or energy. We enlarge the class of signals to which ISS systems are robust, by simply conducting a tighter analysis on these systems. In the spirit of [11], and in contrast to most previous works on ISS and iISS, the considered class of disturbances is defined based on their moving average. We show that any ISS system is robust to such a class of perturbations. When an explicitly Lyapunov function is known, we explicitly estimate the maximum disturbances' moving average that can be tolerated for a given precision. These results are presented in Section 2. We then apply this new analysis result to the control of spacecraft formations. To this end, we exploit the Lyapunov function available for such systems to identify the class of signals to which the formation is robust. This class includes all kind of perturbing effects described earlier. This study is detailed, and illustrated by simulations, in Section 3.

Notation and terminology

A continuous function $\alpha : \mathbb{R}_{\geqslant} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}(\alpha \in \mathcal{K})$, if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $\alpha(s) \to \infty$ as $s \to \infty$, then α is of class $\mathcal{K}_\infty(\alpha \in \mathcal{K}_\infty)$. A continuous function $\sigma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{L}$ if it is non-increasing and tends to zero as its argument tends to infinity. A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class $\mathcal{KL}$ if, $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \in \mathbb{R}_{\geqslant 0}$, and $\beta(s, \cdot) \in \mathcal{L}$ for any fixed $s \in \mathbb{R}_{\geqslant 0}$. The solutions of the differential equation $\dot{x} = f(x, u)$ with initial condition $x_0 \in \mathbb{R}^n$ is denoted by $x(\cdot; x_0, u)$. We use an arrow, for example, $\vec{a}$, to distinguish geometric vectors from coordinate vectors. We use $|\cdot|$ for the Euclidean norm of vectors and the induced norm of matrices. The closed ball in $\mathbb{R}^n$ of radius $\delta \geqslant 0$ centered at the origin is denoted by $\mathcal{B}_\delta$, that is, $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| \leqslant \delta\}$. $|\cdot|_\delta$ denotes the distance to the ball $\mathcal{B}_\delta$, that is $|x|_\delta := \inf_{z \in \mathcal{B}_\delta} |x - z|$. $\mathcal{U}$ denotes the set of all measurable locally essentially bounded signals $u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^p$. For a signal $u \in \mathcal{U}$, $\|u\|_\infty := \text{ess sup}_{t \geqslant 0} |u(t)|$. The maximum and minimum eigenvalues of a symmetric matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. I_n and 0_n denote the identity and null matrices of $\mathbb{R}^{n \times n}$, respectively.

2. ISS SYSTEMS AND SIGNALS WITH LOW MOVING AVERAGE

2.1. Preliminaries

We start by recalling some classical definitions related to the stability and robustness of nonlinear systems of the form

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathcal{U}$ and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0, 0) = 0$.

Definition 1
Let δ be a nonnegative constant. The ball $\mathcal{B}_\delta$ is said to be *globally asymptotically stable (GAS)* for (1) if there exists a class $\mathcal{KL}$ function β such that the solution of (1), from any initial state $x_0 \in \mathbb{R}^n$ and with any input $u \in U \subset \mathcal{U}$, satisfies

$$|x(t; x_0, u)| \leqslant \delta + \beta(|x_0|, t), \quad \forall t \geqslant 0. \tag{2}$$

The ball $\mathcal{B}_\delta$ is said to be *globally exponentially stable (GES)* for (1) if the conditions hold with $\beta(r, s) = k_1 r e^{-k_2 s}$ for some positive constants k_1 and k_2.

The set U used in the aforementioned definition will typically be made of signals whose amplitude or energy is below a given value. When constructing the class of considered input, the function β in (2) will thus be the same for all input, $u \in U$. We next recall the definition of ISS, originally introduced in [7].

Definition 2
The system $\dot{x} = f(x, u)$ is said to be (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$, the solution of (1) satisfies

$$|x(t; x_0, u)| \leqslant \beta(|x_0|, t) + \gamma(\|u\|_\infty), \quad \forall t \geqslant 0. \tag{3}$$

Input-to-state stable thus imposes an asymptotic decay of the norm of the state up to a function of the *amplitude* $\|u\|_\infty$ of the input signal.

We also recall the following well-known Lyapunov characterization of ISS, originally established in [12] and thus extending the original characterization proposed by Sontag in [27].

Proposition 1
The system (1) is ISS if and only if there exist $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$, and $\kappa > 0$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^p$,

$$\underline{\alpha}(|x|) \leqslant V(x) \leqslant \overline{\alpha}(|x|) \tag{4}$$

$$\frac{\partial V}{\partial x}(x) f(x, u) \leqslant -\kappa V(x) + \gamma(|u|). \tag{5}$$

γ is then called a *supply rate* for (1).

Remark 1
Because ISS implies iISS (cf. [10]), it can be shown that the solutions of any ISS system with supply rate γ satisfy, for all $x_0 \in \mathbb{R}^n$,

$$|x(t; x_0, u)| \leqslant \beta(|x_0|, t) + \eta \left(\int_0^t \gamma(|u(\tau)|) d\tau \right), \quad \forall t \geqslant 0, \tag{6}$$

where $\beta \in \mathcal{KL}$ and $\eta \in \mathcal{K}_\infty$. The aforementioned integral can be seen as a measure, through the function γ, of the *energy* of the input signal u over the whole time interval $[0; t]$. Notice that, while $\|u\|_\infty$ in (3) should be finite in order to provide any information about bounds on the state norm, a stronger assumption (namely the boundedness of the integral of $\gamma(|u|)$) is required in order for (6) to provide a meaningful estimate of the state norm.

The aforementioned remark establishes a link between a measure of the energy fed into the system and the norm of the state: for ISS (and iISS) systems, if this input energy is small, then the state will eventually be small. However, Inequalities (3) and (6) do not provide any information on the behavior of the system when the amplitude (for (3)) and/or the energy (for (6)) of the input signal is not finite.

From an applicative viewpoint, the precision guaranteed by (3) and (6) involve the *maximum value* and the *total energy* of the input. These estimates may be conservative and thus lead to the design of greedy control laws, with negative consequences on the energy consumption and actuators solicitation. This issue is particularly relevant for spacecraft formations in view of the inherent fuel limitation and limited power of the thrusters.

As mentioned in the Introduction, [11] has started to tackle this problem by introducing ISS and iISS-like properties for input signals with limited *power*, thus not necessarily bounded in amplitude nor in energy. For systems that are stable when no input is applied, the authors show that ISS (resp. iISS) is equivalent to 'power ISS' (resp. 'power iISS') and 'moving average ISS' (resp. 'moving average iISS'). In general terms, these properties evaluate the influence of the *amplitude* (resp. the *energy*) of the input signal on the *power* or *moving average* of the state. However, as stressed by the authors themselves, these estimates do not guarantee in general any *hard bound* on the state norm.

Here, we consider a slightly more restrictive class of input signals under which such a hard bound can be guaranteed. Namely, we consider input signals with bounded moving average.

Definition 3
Given some constants $E, T > 0$ and some function $\gamma \in \mathcal{K}_\infty$, the set $\mathcal{W}_\gamma(E, T)$ denotes the set of all signals $u \in \mathcal{U}$ satisfying

$$\int_t^{t+T} \gamma(|u(s)|)ds \leqslant E, \quad \forall t \in \mathbb{R}_{\geqslant 0}.$$

As anticipated in the introduction, it can easily be shown that the aforementioned property is strictly more conservative than [17, A.2]. The main concern here is the measure E of the maximum energy that can be fed into the system over a moving time window of given length T. These quantities are the only information on the disturbances that will be taken into account in the control design. More parsimonious control laws than those based on the disturbances' amplitude or energy can therefore be expected. We stress that signals of this class are not necessarily globally essentially bounded, nor are they required to have a finite energy, as illustrated by the following examples. Robustness to this class of signals thus constitutes an extension of the typical properties of ISS systems.

Example 1

(i) Unbounded signals: given any $T > 0$ and any $\gamma \in \mathcal{K}_\infty$, the following signal belongs to $\mathcal{W}_\gamma(1, T)$ and satisfies $\limsup_{t \to \infty} |u(t)| = +\infty$:

$$u(t) := \begin{cases} 2k & \text{if} \quad t \in \left[2kT; 2kT + \frac{1}{2k}\right], \ k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

The signal for $T = 1$ is illustrated in Figure 1.

(ii) Essentially bounded signals: given any $T > 0$ and any $\gamma \in \mathcal{K}_\infty$, if $\|u\|_\infty$ is finite then it holds that $u \in \mathcal{W}_\gamma(T\gamma(\|u\|_\infty), T)$. We stress that this includes signals with infinite energy (think for instance of constant non-zero signals).

2.2. Robustness of ISS systems to signals in the class $\mathcal{W}$

The following result establishes that the impact of an exogenous signal on the qualitative behavior of an ISS systems is negligible if the moving average of this signal is sufficiently low.

Theorem 1
Assume that the system $\dot{x} = f(x, u)$ is ISS. Then there exists a function $\gamma \in \mathcal{K}_\infty$ and, given any precision $\delta > 0$ and any time window $T > 0$, there exists a positive average energy $E(T, \delta)$ such that the ball $\mathcal{B}_\delta \subset \mathbb{R}^n$ is GAS for any $u \in \mathcal{W}_\gamma(E, T)$.
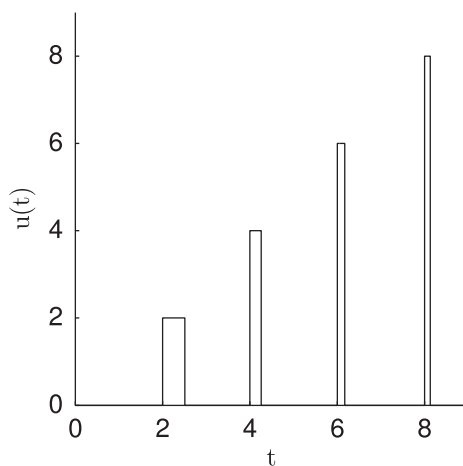

Figure 1. An example of unbounded signal with bounded moving average.

The aforementioned result, proved in Section 4.1, adds another brick in the wall of nice properties induced by ISS, namely, [8, 9], and references therein. It ensures that, provided that a steady-state error δ can be tolerated, every ISS system is robust to a class of disturbances with sufficiently small moving average.

If an ISS Lyapunov function is known for the system, then an explicit bound on the tolerable average excitation can be provided based on the proof lines of Theorem 1. More precisely, we state the following result.

Corollary 1
Assume there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, class $\mathcal{K}_\infty$ functions $\gamma, \underline{\alpha}$, and $\overline{\alpha}$ and a positive constant κ such that (4) and (5) hold for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^p$. Given any precision $\delta > 0$ and any time window $T > 0$, let E denote any average energy satisfying

$$E(T, \delta) \leqslant \frac{\underline{\alpha}(\delta)}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}.$$ (7)

Then the ball $\mathcal{B}_\delta \subset \mathbb{R}^n$ is GAS for $\dot{x} = f(x, u)$ for any $u \in \mathcal{W}_\gamma(E, T)$.

The aforementioned statement shows that, by knowing a Lyapunov function associated to the ISS of a system, and in particular its dissipation rate γ, one is able to explicitly identify the class $\mathcal{W}_\gamma(E, T)$ to which it is robust up to the prescribed precision δ. The proof of Theorem 1, provided in Section 4.1, consists in first generating a Lyapunov function satisfying the assumptions of Corollary 1 (namely an exponential decay along the system's solutions: see (47)) and to use it to estimate the tolerable average energy (see (51)). Consequently, the proof of Corollary 1 follows directly along the lines of that of Theorem 1.

In a similar way, we can state sufficient conditions for global *exponential* stability of some neighborhood of the origin. This result follows also trivially from the proof of Theorem 1.

Corollary 2
If the conditions of Corollary 1 are satisfied with $\underline{\alpha}(s) = \underline{c}s^p$ and $\overline{\alpha}(s) = \overline{c}s^p$, with $\underline{c}, \overline{c}, p$ positive constants, then, given any $T, \delta > 0$, the ball $\mathcal{B}_\delta \subset \mathbb{R}^n$ is GES for (1) with any signal $u \in \mathcal{W}_\gamma(E, T)$ provided that

$$E(T, \delta) \leqslant \frac{\underline{c}\delta^p}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}.$$ (8)

This result is an immediate consequence of Corollary 1 for the case when the lower and upper bounds on the Lyapunov function ($\underline{\alpha}$ and $\overline{\alpha}$) are monomial functions of degree p.

3. APPLICATION TO SPACECRAFT FORMATION CONTROL

We now exploit the results developed in Section 2 to demonstrate the robustness of a formation control for spacecraft in leader-follower configuration, when only position is measured. Results presented in the succeeding text are formulated for formation of two spacecraft only, but can easily be extended to formations involving more spacecraft. For instance, the results presented here can immediately be applied to any number of follower spacecraft in a parallel formation, that is when all the followers follow the same leader. A cascaded formation where the spacecraft make up a directed acyclic graph can also be handled, although the analysis become more cumbersome. For extensions to more complex formation topologies, the interested reader is referred to for instance [20] or [23]. In [20], the notion of *leader-to-formation stability* is introduced, which is based on ISS, and its invariance properties under cascading. A more general result on formation stability is given in [23], where an ISS small-gain result is used to allow for cycles in the interconnection graph of the ISS error dynamics.

The spacecraft model that we use is similar to the one derived in [28], with some small modifications to the representation; see [29] for details. Similar models have been used in [30], [31] and [32]. The dynamics of the spacecraft is described with respect to an elliptic reference orbit, which is defined in the following way: the unit vector $\vec{o}_1$ points anti-nadir, the unit vector $\vec{o}_3$ points in the direction of the orbit normal, and finally $\vec{o}_2 := \vec{o}_3 \times \vec{o}_1$ completes the right-handed orthogonal frame. Location of the origin of the orbital frame (relative to the center of Earth), denoted $\vec{r}_o$, is an elliptic solution to the following two-body problem with Earth as the central body:

$$\ddot{\vec{r}}_o = -\frac{\mu}{\left|\vec{r}_o\right|^3}\vec{r}_o\,,$$

where μ is the gravitational constant of Earth. We denote by ν_o the true-anomaly of this reference frame and make the following assumption, which is naturally satisfied when the reference frame follows a Keplerian orbit.

Assumption 1
The true anomaly rate $\dot{\nu}_o$ and true anomaly rate-of-change $\ddot{\nu}_o$ of the reference frame satisfy $\|\dot{\nu}_o\|_\infty \leqslant \beta_{\dot{\nu}_o}$ and $\|\ddot{\nu}_o\|_\infty \leqslant \beta_{\ddot{\nu}_o}$, for some positive constants $\beta_{\dot{\nu}_o}$ and $\beta_{\ddot{\nu}_o}$.

In this reference frame, the dynamics of a spacecraft is given by the following equation [29]

$$\ddot{p} + C\left(\dot{\nu}_o\right)\dot{p} + D\left(\dot{\nu}_o, \ddot{\nu}_o\right)p + n\left(r_o, p\right) = \frac{1}{m}\left(u + d\right), \tag{9}$$

where p and $\dot{p}$ are the coordinate vectors describing the position and velocity of the spacecraft, m is the spacecraft mass, u and d are the control inputs and exogenous perturbations, respectively, $C\left(\dot{\nu}_o\right) := 2\dot{\nu}_o\mathcal{C}$, $D\left(\dot{\nu}_o, \ddot{\nu}_o\right) := \dot{\nu}_o^2\mathcal{D} + \ddot{\nu}_o\mathcal{C}$, with

$$\mathcal{C} := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{D} := \mathrm{diag}(-1, -1, 0),$$

and $n\left(r_o, p\right) := \mu\left(\frac{r_o+p}{|r_o+p|^3} - \frac{r_o}{|r_o|^3}\right)$. For notational compactness, we will, in the following, write $D = D\left(\dot{\nu}_o, \ddot{\nu}_o\right)$ and $C = C\left(\dot{\nu}_o\right)$, that is, leave out the arguments of the matrices as they remain the same (although time-varying) throughout the document.

We assume that only position measurements are available for the spacecraft and their control objectives are different in the following sense: the leader spacecraft[§] has to follow a given trajectory $p_{dl} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^3$ relative to the reference frame, while the follower spacecraft has to track a desired trajectory $\rho_{df} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^3$ relative to the leader.

We remark that even though the dynamics of the spacecraft is nonlinear in terms of positions, it is linear in terms of the velocities. In the next section, we exploit this property to design an observer that ensures global convergence properties of the velocity estimates. Next, we appeal to the same property for the controller design using a *separation principle*; that is, first, we design controllers using full state measurements (both positions and velocities of the spacecraft) and then implement these controllers using estimates obtained by the observers. Global exponential stability (in absence of external disturbances) and ISS of the overall system are ensured under mild assumptions on the observer and controller gains. Finally, we show that our controller also guarantees robustness of the closed loop system in the presence of disturbances with limited moving average.

3.1. Observer design

The nonlinear observation scheme proposed in this section will be used later to estimate velocities for both spacecraft. Similar to the observer design approach of [33], we define the observer

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[§]In the sequel, we will use subscripts 'l' and 'f' to distinguish leader and follower coordinates and dynamics.

estimation error as $\tilde{p} = p - \hat{p}$, introduce the 'sliding' observation error ([33, 34]) $s_o = \dot{\tilde{p}} + \lambda_o \tilde{p}$, where $\lambda_o > 0$ is a constant, and define our observer in the form similar to that of [33]

$$\dot{\hat{p}} = z + k_d \tilde{p} \tag{10}$$

$$\dot{z} = \frac{u}{m} - C\left(\dot{\hat{p}} - \lambda_o \tilde{p}\right) - Dp - n(r_o, p) + k_p \tilde{p}, \tag{11}$$

where k_p, k_d, are positive constants. The observer error dynamics can be easily obtained from (9), (10), and (11) and is given by

$$\dot{s}_o + C s_o + c_1 s_o + c_2 \tilde{p} = \frac{d}{m}, \tag{12}$$

where $c_1 = k_d - \lambda_o$ and $c_2 = k_p - \lambda_o c_1$.

The linearity property in terms of velocities ensures exponential convergence of the observation errors in the absence of external disturbances, and in particular the following result:

Proposition 2
Consider the spacecraft dynamics (9) together with the observer (10) and (11), and assume that the observer gains satisfy $k_d > \lambda_o, k_p > \lambda_o(k_d - \lambda_o)$. Then, the observer error dynamics (12) is ISS and it is GES in the absence of external disturbances (i.e., $d \equiv 0$).

Proof
Consider Lyapunov function $V_o(s_o, \tilde{p}) = \frac{1}{2}s_o^\top s_o + \frac{c_2}{2}\tilde{p}^\top \tilde{p}$, which satisfies the following quadratic bounds

$$\alpha_{o1}|x_o|^2 \leqslant V_o(s_o, \tilde{p}) \leqslant \alpha_{o2}|x_o|^2, \tag{13}$$

where $x_o^\top = \left(s_o^\top, \tilde{p}^\top\right)$, $\alpha_{o1} = \frac{1}{2}\min\{1, c_2\}$ and $\alpha_{o2} = \frac{1}{2}\max\{1, c_2\}$. Taking its derivative along trajectories of the system (12), we obtain that

$$\dot{V}_o(s_o, \tilde{p}) = -s_o^\top\left(C s_o + c_1 s_o + c_2 \tilde{p} - \frac{1}{m}d\right) + c_2 \tilde{p}^\top(s_o - \lambda_o \tilde{p})$$

$$\leqslant -c_1|s_o|^2 - c_2\lambda_o|\tilde{p}|^2 + \frac{1}{m}s_o^\top d \leqslant -\frac{c_1}{2}|s_o|^2 - c_2\lambda_o|\tilde{p}|^2 + \frac{1}{2c_1 m^2}|d|^2, \tag{14}$$

$$\leqslant -\alpha_{o3}|x_o|^2 + \gamma_o|d|^2,$$

where $\alpha_{o3} = \min\left\{\frac{c_1}{2}, c_2\lambda_o\right\}, \gamma_o = \frac{1}{2c_1 m^2}$, and we have used the skew symmetric property of C in the first inequality. ISS of (12) follows immediately, and in absence of disturbances GES of the origin follows trivially, because both $V_o(s_o, \tilde{p})$ and its derivative are quadratic forms of x_o. □

By Corollary 2, we can establish the following result stating that in case of perturbations with limited moving average, the observation errors $\tilde{p}, \dot{\tilde{p}}$ are bounded.

Corollary 3
Given any $T, \delta > 0$, the ball $\mathcal{B}_\delta \subset \mathbb{R}^6$ is GES for (12) with any external disturbance $d \in \mathcal{W}_\gamma(E, T)$ with $\gamma = \gamma_o s^2$, provided that $k_d > \lambda_o, k_p > \lambda_o(k_d - \lambda_o)$, and that

$$E(T, \delta) \leqslant \frac{\alpha_{o1}\delta^2}{2} \frac{e^{\frac{\alpha_{o3}}{\alpha_{o2}}T} - 1}{2e^{\frac{\alpha_{o3}}{\alpha_{o2}}T} - 1}.$$

Proof
We consider the Lyapunov function $V_o(s_o, \tilde{p}) = \frac{1}{2}s_o^\top s_o + \frac{c_2}{2}\tilde{p}^\top \tilde{p}$ just as in the proof of Proposition 2. From (13) and (14), we see that the conditions of Corollary 2 are satisfied with $p = 2$, $\underline{c} = \alpha_{o1}, \overline{c} = \alpha_{o2}$ and $\kappa = \frac{\alpha_{o3}}{\alpha_{o2}}$. □

3.2. Control design

We recall that the leader spacecraft control objective is to follow a given reference trajectory p_{dl} : $\mathbb{R}_{\geq 0} \to \mathbb{R}^3$ relative to the reference frame, while the follower spacecraft has to track a desired trajectory $\rho_{dl} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ relative to the leader.

The dynamics of the leader is described be (9), and we use subscript l in all parameters of the model to underline this fact

$$\ddot{p}_l + C\dot{p}_l + Dp_l + n(r_o, p_l) = \frac{1}{m_l}(u_l + d_l). \tag{15}$$

We propose a full state feedback controller $u_l = u_l^{\text{sf}}$ that ensures convergence to zero of tracking errors $e_l = p_l - p_{dl}$ and $\dot{e}_l = \dot{p}_l - \dot{p}_{dl}$, where

$$u_l^{\text{sf}} := m_l\left(\ddot{p}_{dl} + C(\dot{p}_{dl} - \lambda_{cl}e_l) + Dp_l + n(r_o, p_l) - K_{dl}\dot{e}_l - K_{pl}e_l\right), \tag{16}$$

K_{pl}, K_{dl} are positive constants, and the superscript 'sf' stands for *state feedback*. Substituting this controller in the equation for the dynamics for the leader spacecraft (15) and using the 'sliding' tracking error $s_{cl} = \dot{e}_l + \lambda_{cl}e_l$, we obtain closed loop dynamics

$$\dot{s}_{cl} + Cs_{cl} + K_{1l}s_{cl} + K_{2l}e_l = \frac{1}{m_l}d_l, \tag{17}$$

where $K_{1l} = K_{dl} - \lambda_{cl}$, $K_{2l} = K_{pl} - \lambda_{cl}K_{dl} + \lambda_{cl}^2$. It is easy to show that with the proper choice of the gains K_{pl}, K_{dl}, controller (16) ensures ISS of the closed loop system and also GES of the origin in the absence of external disturbances.

Claim 1
Consider the spacecraft dynamics (9) in closed loop with the controller (16), where controller gains K_p, K_d, $\lambda_{cl} > 0$ satisfy inequalities $K_{1l} = K_{dl} - \lambda_{cl} > 0$ and $K_{2l} = K_{pl} - \lambda_{cl}K_{dl} + \lambda_{cl}^2 > 0$. Then, the dynamics of the closed loop system (17) is ISS, and it is GES in absence of external disturbances.

Proof of the claim follows along the lines of the proof of Proposition 2. Consider a Lyapunov function $V_{cl}(s_{cl}, e_l) = \frac{1}{2}s_{cl}^\top s_{cl} + \frac{K_{2l}}{2}e_l^\top e_l$ which satisfies the following quadratic bounds

$$\alpha_{cl1}|x_{cl}|^2 \leq V_{cl}(s_{cl}, e_l) \leq \alpha_{cl2}|x_{cl}|^2, \tag{18}$$

where $x_{cl}^\top = (s_{cl}^\top, e_l^\top)$, $\alpha_{cl1} = \frac{1}{2}\min\{1, K_{2l}\}$ and $\alpha_{cl2} = \frac{1}{2}\max\{1, K_{2l}\}$. Taking its derivative along trajectories of the system (17), we obtain

$$\dot{V}_{cl}(s_{cl}, e_l) = s_{cl}^\top\left(-Cs_{cl} - K_{1l}s_{cl} - K_{2l}e_l + \frac{1}{m_l}d_l\right) + K_{2l}e_l^\top\dot{e}_l \leq -K_{1l}|s_{cl}|^2 - \lambda_{cl}K_{2l}|e_l|^2$$
$$+ \frac{1}{m_l}|s_{cl}||d_l| \leq -\frac{K_{1l}}{2}|s_{cl}|^2 - \lambda_{cl}K_{2l}|e_l|^2 + \gamma_{cl}|d_l|^2 \leq -\alpha_{c3}|x_{cl}|^2 + \gamma_{cl}|d_l|^2, \tag{19}$$

where $\alpha_{cl3} = \min\left\{\frac{K_{1l}}{2}, \lambda_{cl}K_{2l}\right\}$, $\gamma_{cl} = 1/2m_l^2K_{1l}$. ISS and GES of the system in absence of disturbances follow trivially from (18) and (19).

Next, we show that the separation principle can be applied for the controller design without velocity measurements. We propose to use the same controller, except that the velocity measurements are replaced by their estimates delivered by the observer (10) and (11). The controller $u_l = u_l^{\text{of}}$, where

$$u_l^{\text{of}} := m_l\left(\ddot{p}_{dl} + C(\dot{p}_{dl} - \lambda_{cl}e_l) + Dp_l + n(r_o, p_l) - K_{pl}e_l - K_{dl}\left(\dot{\hat{p}}_l - \dot{p}_{dl}\right)\right) \tag{20}$$

$$vspace * -12pt \dot{\hat{p}}_l = z_l + k_d\tilde{p}_l \tag{21}$$

$$\dot{z}_l = \frac{1}{m}u_l - C\left(\dot{\tilde{p}}_l - \lambda_{ol}\tilde{p}_l\right) - Dp_l - n(r_o, p_l) + k_p\tilde{p}_l, \tag{22}$$

with z_l, $\tilde{p}_l$ as defined before, and where the superscript 'of' stands for *output feedback*.

The following theorem shows that the designed controller preserves the stability and input/output properties of the system that it had in the closed loop with the full state controller.

Proposition 3

Consider the dynamics of the leader spacecraft given by (15) together with the observer-based controller (20)-(22). Let controller and observer gains K_{dl}, K_{pl}, k_{dl}, k_{pl}, λ_{ol} satisfy the same bounds as in Proposition 2 and Claim 1. Then, the closed loop system is ISS, and in absence of external disturbances both tacking errors e_l, $\dot{e}_l$ and observation errors $\tilde{p}_l$, $\dot{\tilde{p}}_l$ go to zero exponentially fast.

Proof

The proof is based on the construction of a composite Lyapunov function in the form of a weighted sum of Lyapunov functions V_{cl} and V_{ol} designed separately for the state feedback controller and velocity estimation. We choose it in the form

$$V_l(x_{cl}, x_{ol}) = V_{cl}(s_{cl}, e_l) + \frac{2a_{cl}}{\alpha_{ol3}}V_{ol}(s_{ol}, \tilde{p}_l) = \frac{1}{2}|s_{cl}|^2 + \frac{K_{2l}}{2}|e_l|^2 + \frac{a_{cl}}{\alpha_{ol3}}|s_{ol}|^2 + \frac{a_{cl}c_{2l}}{\alpha_{ol3}}|\tilde{p}_l|^2, \tag{23}$$

where $a_{cl} = \max\left\{1, \lambda_{ol}^2\right\}\frac{K_{dl}^2}{K_{1l}}$, $\alpha_{ol3} = \min\left\{\frac{c_{1l}}{2}, c_{2l}\lambda_{ol}\right\}$, $c_{1l} = k_{dl} - \lambda_{ol}$, $c_{2l} = k_{pl} - \lambda_{ol}c_{1l}$ and $x_{ol}^\top = \left(s_{ol}^\top, \tilde{p}_l^\top\right)$. It satisfies the following quadratic bounds

$$\alpha_{l1}|x_l|^2 \leqslant V_l(x_{cl}, x_{ol}) \leqslant \alpha_{l2}|x_l|^2, \tag{24}$$

where $x_l^\top = \left(x_{cl}^\top, x_{ol}^\top\right)$, $\alpha_{l1} = \min\left\{\frac{1}{2}, \frac{K_{2l}}{2}, \frac{a_{cl}}{\alpha_{ol3}}, \frac{a_{cl}c_{2l}}{\alpha_{ol3}}\right\}$ and $\alpha_{l2} = \max\left\{\frac{1}{2}, \frac{K_{2l}}{2}, \frac{a_{cl}}{\alpha_{ol3}}, \frac{a_{cl}c_{2l}}{\alpha_{ol3}}\right\}$.

Next, note that the controller (20) is essentially the same as (16) and can be rewritten in the form

$$u_l^{\text{of}} = u_l^{\text{sf}} + m_l K_{dl}\left(\dot{p}_l - \dot{\hat{p}}_l\right) = u_l^{\text{sf}} + m_l K_{dl}s_{ol} - m_l\lambda_{ol}K_{dl}\tilde{p}_l, \tag{25}$$

where we recall that $s_{ol} = \dot{\tilde{p}}_l + \lambda_{ol}\tilde{p}_l$. Then, similar to (19), we obtain that derivative of $V_{cl}(s_{cl}, e_l)$ taken along trajectories of the closed loop system can be bounded in the following way

$$\begin{aligned}
\dot{V}_{cl}(s_{cl}, e_l) &\leqslant -\frac{K_{1l}}{2}|s_{cl}|^2 - \lambda_{cl}K_{2l}|e_l|^2 + K_{dl}|s_{cl}||s_{ol}| + \lambda_{ol}K_{dl}|s_{cl}||\tilde{p}_l| + \gamma_{cl}|d_l|^2 \\
&\leqslant -\frac{K_{1l}}{4}|s_{cl}|^2 - \frac{\lambda_{cl}}{2}K_{2l}|e_l|^2 + \frac{1}{K_{1l}}\lambda_{ol}^2 K_{dl}^2|\tilde{p}_l|^2 + \frac{1}{K_{1l}}K_{dl}^2|s_{ol}|^2 + \gamma_{cl}|d_l|^2 \\
&\leqslant -\frac{\alpha_{cl3}}{2}|x_{cl}|^2 + a_{cl}|x_{ol}|^2 + \gamma_{cl}|d_l|^2,
\end{aligned} \tag{26}$$

where α_{cl3} and γ_{cl} are the same as in the Claim 1. Now, taking the derivative of the composite Lyapunov function $V_l(x_{cl}, x_{ol})$ and using (14) and (26), we obtain that

$$\begin{aligned}
\dot{V}_l(x_{cl}, x_{ol}) &\leqslant -\frac{\alpha_{cl3}}{2}|x_{cl}|^2 + a_{cl}|x_{ol}|^2 + \gamma_{cl}|d_l|^2 + \frac{2a_{cl}}{\alpha_{ol3}}\left(-\alpha_{ol3}|x_{ol}|^2 + \gamma_{ol}|d_l|^2\right) \\
&\leqslant -\frac{\alpha_{cl3}}{2}|x_{cl}|^2 - a_{cl}|x_{ol}|^2 + \left(\gamma_{cl} + \frac{2a_{cl}}{\alpha_{ol3}}\gamma_{ol}\right)|d_l|^2 \\
&\leqslant -\alpha_{l3}|x_l|^2 + \gamma_l|d_l|^2,
\end{aligned} \tag{27}$$

where $\alpha_{l3} = \min\left\{\frac{1}{2}\alpha_{cl3}, a_{cl}\right\}$, and $\gamma_l = \gamma_{cl} + \frac{2a_{cl}}{\alpha_{ol3}}\gamma_{ol}$. ISS of (15) together with the observer-based controller (20)–(22) follows immediately, and in absence of disturbances GES of the origin follows trivially because both $V_l(x_{xl}, x_{ol})$ and its derivative are quadratic forms of x_l. $\square$

Corollary 4
Given any $T, \delta > 0$, the ball $\mathcal{B}_\delta \subset \mathbb{R}^{12}$ is GES for (15) together with the observer-based controller (20)–(22) with any external disturbance $d_l \in \mathcal{W}_\gamma(E, T)$ with $\gamma(s) = \gamma_l s^2$, provided that the controller and observer gains $K_{dl}, K_{pl}, k_{dl}, k_{pl}, \lambda_{ol}$ satisfy the same bounds as in Proposition 2 and Claim 1, and that

$$E(T, \delta) \leqslant \frac{\alpha_{l1}\delta^2}{2} \frac{e^{\frac{\alpha_{l3}}{\alpha_{l2}}T-1}}{2e^{\frac{\alpha_{l3}}{\alpha_{l2}}T-1}}. \tag{28}$$

Proof
We consider the Lyapunov function $V_l(x_{cl}, x_{ol})$ just as in the proof of Proposition 3. From (24) and (27), we see that the conditions of Corollary 2 are satisfied with $p = 2, \underline{c} = \alpha_{l1}, \overline{c} = \alpha_{l2}$ and $\kappa = \frac{\alpha_{l3}}{\alpha_{l2}}$. □

We next propose a controller to make the follower spacecraft track a desired trajectory and velocity $\rho_d, \dot{\rho}_d : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^3$ relative to the leader. We use the follower dynamics rewritten in terms of relative coordinates $\rho_f = p_f - p_l$, namely

$$\ddot{\rho}_f + C\dot{\rho}_f + D\rho_f + n_{rf}(r_o, p_f, p_l) = u_{rf} + d_{rf} \tag{29}$$

where $n_{rf}(r_o, p_f, p_l) = n(r_o, p_f) - n(r_o, p_l), u_{rf} = \frac{1}{m_f}u_f - \frac{1}{m_l}u_l$, and $d_{rf} = \frac{1}{m_f}d_f - \frac{1}{m_l}d_l$.

Similar with the leader spacecraft in (20), we choose a state feedback controller $u_{rf} = u_{rf}^{sf}$ for the follower spacecraft in the following form:

$$u_{rf}^{sf} := \ddot{\rho}_{df} + C\left(\dot{\rho}_{df} - \lambda_{cf}e_f\right) + D\rho_f + n_{rf}(r_o, p_f, p_l) - K_{pf}e_f - K_{df}\left(\dot{\rho}_f - \dot{\rho}_{df}\right) \tag{30}$$
$$\dot{\hat{\rho}}_f = z_f + k_d\tilde{\rho}_f$$

$$\dot{z}_f = u_{rf} - C\left(\dot{\hat{\rho}}_f - \lambda_{of}\tilde{\rho}_f\right) - D\rho_f - n_{rf}(r_o, p_f, p_l) + k_p\tilde{\rho}_f, \tag{31}$$

where $e_f = \rho_f - \rho_{df}$. Notice that with this choice of the controller, the closed loop system for the follower spacecraft will have exactly the same form as the one for the leader spacecraft, and therefore the system will have exactly the same stability and robustness properties. However, realizing this controller using u_f requires knowledge of u_l or estimates of the leaders' velocities because $u_f = m_f u_{rf} + \frac{m_f}{m_l}u_l$. Hence, this would either require the leader spacecraft to broadcast its control inputs or its estimated velocity to the follower, or that the follower has a dedicated observer for the state of the leader spacecraft. In order to avoid this extra communications or calculations, we propose ler $u_f = u_f^{of}$ in the following form:

$$u_f^{of} := m_f\left(\ddot{\rho}_{df} + C\left(\dot{\rho}_{df} - \lambda_{cf}e_f\right) + D\rho_f + n_{rf}(r_o, p_f, p_l) - K_{pf}e_f - K_{df}\left(\dot{\hat{p}}_f - \dot{\rho}_{df} - \dot{p}_{dl}\right)\right)$$
$$+ \frac{m_f}{m_l}\left(\ddot{p}_{dl} + C\left(\dot{p}_{dl} - \lambda_{cl}e_l\right) + Dp_l + n(r_o, p_l) - K_{pl}e_l\right) \tag{32}$$

$$\dot{\hat{p}}_f = z_f + k_d\tilde{p}_f \tag{33}$$

$$\dot{z}_f = u_f - C\left(\dot{\hat{p}}_f - \lambda_{of}\tilde{p}_f\right) - Dp_f - n_{rf}(r_o, p_f, p_l) + k_p\tilde{p}_f, \tag{34}$$

where none of the terms in the controller depends on the leaders estimated velocity, and we estimate the actual velocity of the follower spacecraft instead of the relative velocity.

We are now ready to state the following result, which establishes stability and robustness of the overall controlled formation.

Proposition 4

Consider the dynamics of the formation given by (15), (29) together with the observer-based controllers (20)–(22), and (32)–(34). Let the controller and observer gains $\lambda_{oi}, \lambda_{ci}, K_{di}, K_{pi}, k_{di}, k_{pi}, (i \in \{l, f\})$ satisfy the same bounds as in Proposition 2 and Claim 1. Then, the closed loop system is ISS, and in the absence of external disturbances, all tracking errors e_i, $\dot{e}_i$ and observation errors $\tilde{p}_i$, $\dot{\tilde{p}}_i$ go to zero exponentially fast.

Proof

As in the proof of Proposition 3, we construct a composite Lyapunov function in the form of a weighted sum of Lyapunov functions V_l and V_f designed separately for stability analysis of the leader and the follower spacecraft. We choose it in the following form:

$$V(x_l, x_f) = V_f(x_{cf}, x_{of}) + \frac{2b_f}{\alpha_{l3}} V_l(x_{cl}, x_{ol}), \tag{35}$$

where $x_l^\top = (x_{cl}^\top, x_{ol}^\top)$, $x_f^\top = \left(x_{cf}^\top, x_{of}^\top\right) = \left(s_{cf}^\top, e_f^\top, s_{of}^\top, \tilde{p}_f^\top\right)$, $\alpha_{l3} = \min\left\{\frac{\alpha_{cl3}}{2}, a_{cl}\right\}$, $b_f = \max\left\{K_{dl}^2, K_{dl}^2\lambda_{ol}^2, K_{df}^2 + K_{dl}^2, \left(K_{df}^2 + K_{dl}^2\right)\lambda_{cl}^2\right\}\frac{3}{K_{1l}}$, $V_l(x_{cl}, x_{ol})$ was defined in (23) and

$$V_f(x_{cf}, x_{of}) = V_{cf}(s_{cf}, e_f) + \frac{2a_{cf}}{\alpha_{of3}} V_{of}\left(s_{of}, \tilde{p}_f\right) \tag{36}$$

$$= \frac{1}{2}|s_{cf}|^2 + \frac{K_{2f}}{2}|e_f|^2 + \frac{a_{cf}}{\alpha_{of3}}|s_{of}|^2 + \frac{a_{cf}c_{2l}}{\alpha_{of3}}\left|\tilde{p}_f\right|^2, \tag{37}$$

where $a_{cf} = \max\left\{1, \lambda_{of}^2\right\}\frac{3K_{df}^2}{K_{1f}}$. It can be noted that the Lyapunov function satisfies

$$\alpha_1|x|^2 \leqslant V(x_l, x_f) \leqslant \alpha_2|x|^2, \tag{38}$$

where $\alpha_1 = \min\left\{\frac{1}{2}, \frac{K_{2l}}{2}, \frac{a_{cl}}{\alpha_{ol3}}, \frac{a_{cl}c_{2l}}{\alpha_{ol3}}\right\} + \frac{2b_f}{\alpha_{l3}}\min\left\{\frac{1}{2}, \frac{K_{2f}}{2}, \frac{a_{cf}}{\alpha_{of3}}, \frac{a_{cf}c_{2f}}{\alpha_{of3}}\right\}$ and $\alpha_2 = \max\left\{\frac{1}{2}, \frac{K_{2l}}{2}, \frac{a_{cl}}{\alpha_{ol3}}, \frac{a_{cl}c_{2l}}{\alpha_{ol3}}\right\} + \frac{2b_f}{\alpha_{l3}}\max\left\{\frac{1}{2}, \frac{K_{2f}}{2}, \frac{a_{cf}}{\alpha_{of3}}, \frac{a_{cf}c_{2f}}{\alpha_{of3}}\right\}$. □

Substituting the controller (32) in the equation for the dynamics of the follower spacecraft, (29), and using the 'sliding' tracking error $s_f = \dot{e}_f + \lambda_{cf}e_f$, we obtain the closed loop dynamics

$$\dot{s}_{cf} + Cs_{cf} + K_1 s_{cf} + K_2 e_f = K_{df}\left(s_{of} - \lambda_{of}\tilde{p}_f\right) - K_{dl}\left(s_{ol} - \lambda_{ol}\tilde{p}_l\right) \\ + (K_{df} + K_{dl})(s_{cl} - \lambda_{cl}e_l) + d_{rf}. \tag{39}$$

We find that the derivative of $V_{cf}(s_{cf}, e_f)$ taken along trajectories of the closed loop system can be bounded in the following way:

$$\begin{aligned} \dot{V}_{cf}(s_{cf}, e_f) \leqslant & -\frac{K_{1f}}{2}|s_{cf}|^2 - \lambda_{cf}K_{2f}|e_f|^2 \\ & + K_{df}|s_{cf}|\left(|s_{of}| + \lambda_{of}\left|\tilde{p}_f\right| + |s_{cl}| + \lambda_{cl}|e_l|\right) \\ & + K_{dl}|s_{cf}|\left(|s_{ol}| + \lambda_{ol}|\tilde{p}_l| + |s_{cl}| + \lambda_{cl}|e_l|\right) + |s_{cf}|\left(\frac{|d_l|}{m_l} + \frac{|d_f|}{m_f}\right) \\ \leqslant & -\frac{K_{1f}}{4}|s_{cf}|^2 - \frac{\lambda_{cf}}{2}K_{2f}|e_f|^2 \\ & + \frac{10K_{df}^2}{K_{1f}}\left(|s_{of}|^2 + \lambda_{of}^2|\tilde{p}_f|^2 + |s_{cl}|^2 + \lambda_{cl}^2|e_l|^2\right) \\ & + \frac{10K_{dl}^2}{K_{1f}}\left(|s_{ol}|^2 + \lambda_{ol}^2|\tilde{p}_l|^2 + |s_{cl}|^2 + \lambda_{cl}^2|e_l|^2\right) + \gamma_{cf}\left(|d_l|^2 + |d_f|^2\right) \\ \leqslant & -\frac{\alpha_{cf3}}{2}|x_{cf}|^2 + a_{cf}|x_{of}|^2 + b_f|x_l|^2 + \gamma_{cf}\left(|d_l|^2 + |d_f|^2\right), \end{aligned} \tag{40}$$

where $\alpha_{cf3} = \min\left\{\frac{K_{1f}}{2}, \lambda_{cf}K_{2f}\right\}$, $\gamma_{cf} = \max\left\{\frac{1}{m_f^2}, \frac{1}{m_l^2}\right\}\frac{10}{K_{1f}}$. Now taking the derivative of the composite Lyapunov function $V_f(x_{cl}, x_{ol})$ and using (14) and (40), we obtain that

$$
\begin{aligned}
\dot{V}_f(x_{cf}, x_{of}) &\leqslant -\frac{\alpha_{cf3}}{2}|x_{cf}|^2 + a_{cf}|x_{of}|^2 + b_f|x_l|^2 + \gamma_{cf}\left(|d_l|^2 + |d_f|^2\right) \\
&\quad + \frac{2a_{cf}}{\alpha_{of3}}\left(-\alpha_{of3}|x_{of}|^2 + \gamma_{of}|d_f|^2\right) \\
&\leqslant -\alpha_{f3}|x_f|^2 + b_f|x_l|^2 + \gamma_{cf}|d_l|^2 + \left(\gamma_{cf} + \frac{2a_{cf}}{\alpha_{of3}}\gamma_{of}\right)|d_f|^2,
\end{aligned}
\tag{41}
$$

where $\alpha_{f3} = \min\left\{\frac{\alpha_{cf3}}{2}, a_{cf}\right\}$. Finally, taking the time derivative of $V(x_l, x_f)$, and using (41) and (27) with $\alpha_{l3} = \min\left\{\frac{\alpha_{cl3}}{2}, a_{cl}\right\}$, we find that

$$
\begin{aligned}
\dot{V}(x_l, x_f) &\leqslant -\alpha_{f3}|x_f|^2 + b_f|x_l|^2 + \gamma_{cf}|d_l|^2 + \left(\gamma_{cf} + \frac{2a_{cf}}{\alpha_{of3}}\gamma_{of}\right)|d_f|^2 \\
&\quad + \frac{2b_f}{\alpha_{l3}}\left(-\alpha_{l3}|x_l|^2 + \left(\gamma_{cl} + \frac{2a_{cl}}{\alpha_{ol3}}\gamma_{ol}\right)|d_l|^2\right) \\
&\leqslant -\alpha_{f3}|x_f|^2 - b_f|x_l|^2 + \left(\gamma_{cf} + \frac{2a_{cf}}{\alpha_{of3}}\gamma_{of}\right)|d_f|^2 \\
&\quad + \left(\gamma_{cf} + \frac{2b_f}{\alpha_{l3}}\left(\gamma_{cl} + \frac{2a_{cl}}{\alpha_{ol3}}\gamma_{ol}\right)\right)|d_l|^2 \\
&\leqslant -\alpha_3|x|^2 + \tilde{\gamma}|d|^2,
\end{aligned}
\tag{42}
$$

where $\alpha_3 = \min\{\alpha_{f3}, b_f\}$ and $\tilde{\gamma} = \gamma_{cf} + \max\left\{\frac{2a_{cf}}{\alpha_{of3}}\gamma_{of}, \frac{2b_f}{\alpha_{l3}}\left(\gamma_{cl} + \frac{2a_{cl}}{\alpha_{ol3}}\gamma_{ol}\right)\right\}$. ISS of (15), (29) together with the observer-based controllers (20)–(22), and (32)–(34) follows immediately, and in the absence of disturbances GES of the origin follows trivially because both $V(x_l, x_f)$ and its derivative are quadratic forms of x.

By the aforementioned proof and Corollary 2, we can establish an explicit bound on the tolerable average excitation:

Corollary 5
Consider the dynamics given by (15), (29) together with the observer-based controllers (20)–(22), and (32)–(34). Let the controller and observer gains $\lambda_{oi}, \lambda_{ci}, K_{di}, K_{pi}, k_{di}, k_{pi}, (i \in \{l, f\})$ satisfy the same bounds as in Proposition 2 and claim 1. Then, the ball $\mathcal{B}_\delta \subset \mathbb{R}^{24}$ is GES with any external disturbance $d \in \mathcal{W}_\gamma(E, T)$ with $\gamma(s) = \tilde{\gamma}s^2$, provided that

$$
E(T, \delta) \leqslant \frac{\alpha_1\delta^2}{2} \frac{e^{\frac{\alpha_3}{\alpha_2}T} - 1}{2e^{\frac{\alpha_3}{\alpha_2}T} - 1}.
\tag{43}
$$

Proof
We consider the Lyapunov function $V(x_l, x_f)$ just as in the proof of Proposition 4. From (38) and (42), we see that the conditions of Corollary 2 are satisfied with $p = 2, \underline{c} = \alpha_1, \bar{c} = \alpha_2$ and $\kappa = \frac{\alpha_3}{\alpha_2}$.   □

3.3. Simulations

Let the reference orbit be an eccentric orbit with radius of perigee $r_p = 10^7 m$ and radius of apogee $r_a = 3 \times 10^7 m$, which can be generated by numerical integration of

$$
\ddot{r}_o = -\frac{\mu}{|r_o|^3}r_o,
\tag{44}
$$

with $r_o(0) = (r_p, 0, 0)$ and $\dot{r}_o(0) = (0, v_p, 0)$, and where

$$v_p = \sqrt{2\mu\left(\frac{1}{r_p} - \frac{1}{(r_p + r_a)}\right)}.$$

The true anomaly ν_o of the reference frame can be obtained by numerical integration of the equation

$$\ddot{\nu}_o(t) = \frac{-2\mu e_o(1 + e_o \cos \nu_o(t))^3 \sin \nu_o(t)}{\left(\frac{1}{2}(r_p + r_a)(1 - e_o^2)\right)^3}.$$

From this expression, and the eccentricity which can be calculated from r_a and r_p to be $e_o = 0.5$, we see that the constant $\beta_{\ddot{\nu}_o}$ in Assumption 1 can be chosen as $\beta_{\ddot{\nu}_o} = 4 \times 10^{-7}$. From the analytical equivalent for $\dot{\nu}_o$,

$$\dot{\nu}_o(t) = \frac{\sqrt{\mu}(1 + e_o \cos \nu_o(t))^2}{\left(\frac{1}{2}(r_a + r_p)(1 - e_o^2)\right)^{3/2}},$$

we see that the constant $\beta_{\dot{\nu}_o}$ in Assumption 1 can be chosen as $\beta_{\dot{\nu}_o} = 8 \times 10^{-4}$. Because the reference frame is initially at perigee, $\nu_o(0) = 0$ and $\dot{\nu}_o(0) = v_p/r_p$. For simplicity, we choose the desired trajectory of the leader spacecraft to coincide with the reference orbit, that is, $p_{dl}(t) \equiv (0,0,0)^\top \ \forall t \geq 0$. The initial values of the leader spacecraft are $p(0) = (2,-2,3)^\top$ and $\dot{p}(0) = (0.4,-0.8,-0.2)^\top$. The initial values of the observer are chosen as $\hat{p}(0) = (0,0,0)^\top$ and $z_l(0) = (0,0,0)^\top$.

The reference trajectory of the follower spacecraft are chosen as the solutions of a special case of the Clohessy–Wiltshire equations, compared with [35]. We use

$$\rho_{df}(t) = \begin{bmatrix} 10\cos\nu_o(t) \\ -20\sin\nu_o(t) \\ 0 \end{bmatrix}. \tag{45}$$

This choice imposes that the two spacecraft evolve in the same orbital plane, and that the follower spacecraft makes a full rotation about the leader spacecraft at each orbit around the Earth. The initial values of the follower spacecraft are $p_f(0) = (13,-1,2)^\top$ and $\dot{p}_f(0) = (0.5,0.2,0.6)^\top$. The initial values of the observer are chosen to be $\hat{p}_f(0) = (15,0,0)^\top$ and $z_f(0) = (0,0,0)^\top$. The controller parameters are $K_{pl} = K_{pf} = 1.2, K_{dl} = K_{df} = 1$ and $\lambda_{cl} = \lambda_{cf} = 0.3$, and the observer parameters are $k_{pl} = k_{pf} = 8, k_{dl} = k_{df} = 10$ and $\lambda_{ol} = \lambda_{of} = 0.5$. We use $m_f = m_l = 25$ kg both in the model and the control structure. Over a 10-s interval (that is, $T = 10$), the average excitation must satisfy $E(T,\delta) \leq 0.043\delta^2$, based on the aforementioned chosen controller and observer gains, according to (4). We consider two types of disturbances acting on the spacecraft: 'impacts' and continuous disturbances. The 'impacts' have random amplitude, but with maximum of 1.5 N in each direction of the Cartesian frame. For simplicity, we assume that at most one impact can occur over each 10-s interval, and we assume that the duration of each impact is at most 0.1s. The continuous disturbances are taken as sinusoids, also acting in each direction of the Cartesian frame, and are chosen to be $(0.1\sin 0.01t, 0.25\sin 0.03t, 0.3\sin 0.04t)^\top$ for both spacecraft. It can easily be shown that the disturbances satisfy the following:

$$\int_t^{t+10} |d(\tau)|^2 d\tau \leq 1.42, \quad \forall t \geq 0.$$

Figure 2(a), 2(b), and 2(c) show the position tracking error, position estimation error, and control history of the leader spacecraft, whereas Figure 3(a), 3(b), and 3(c) are the equivalent figures for the follower spacecraft. Figure 2(d) and 3(d) show the effect of d_l and d_f acting on the formation. Because $E = 1.42$, and should satisfy $E \leq 0.043\delta^2$ based on (8) and our choice of controller and observer gains, this gives a very large tolerance δ. As can be seen from Figure 2(a), 2(b), 3(a), 3(b) the actual precision reached is much better than the theoretical expectations. The control gains have
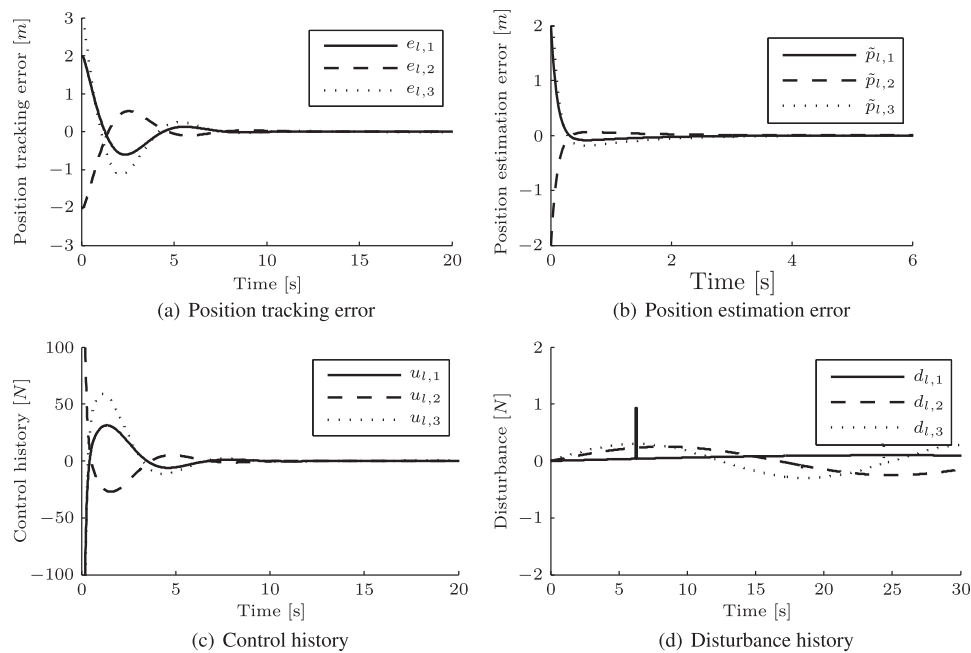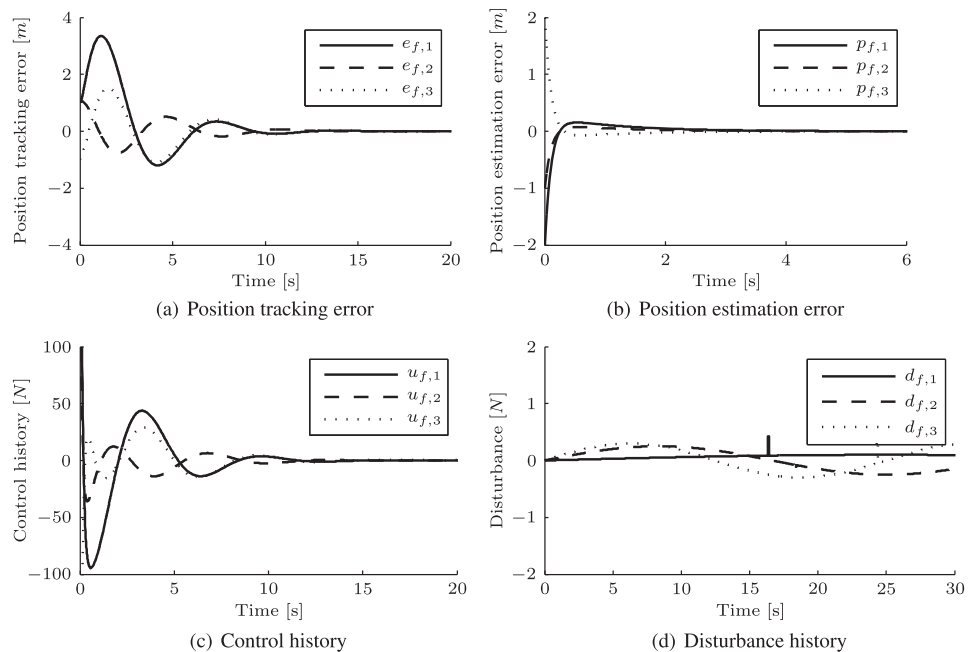
Figure 2. Leader spacecraft.

Figure 3. Follower spacecraft.

been chosen based on the Lyapunov analysis. This yields in general very conservative constraints on the choice of control gains, and also conservative estimates of the disturbances the control system is able to handle. As shown in Figure 2(c) and Figure 3(c), this leads to large transients in the actuation. However, we stress that the control gains proposed by this approach are still much smaller that those obtained through a classical ISS approach (i.e., relying on the disturbance magnitude rather than its moving average). Tighter gains could possibly have been achieved by choosing them based

on some optimization problem that maximizes the average energy of the disturbance signal, while maintaining the constraints on the gains from the Lyapunov analysis, [36].

4. PROOFS

4.1. Proof of Theorem 1

In view of [12, Lemma 11] and [37, Remark 2.4], there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and γ, and a positive constant κ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{46}$$

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq -\kappa V(x) + \gamma(|u|). \tag{47}$$

Let $w(t) := V(x(t; x_0, u))$. Then, it holds in view of (47) that

$$\begin{aligned} \dot{w}(t) &= \dot{V}(x(t; x_0, u)) \\ &\leq -\kappa V(x(t; x_0, u)) + \gamma(|u(t)|) \\ &\leq -\kappa w(t) + \gamma(|u(t)|). \end{aligned}$$

In particular, it holds that, for all $t \geq 0$,

$$w(t) \leq w(0)e^{-\kappa t} + \int_0^t \gamma(|u(s)|)ds. \tag{48}$$

Assuming that u belongs to the class $\mathcal{W}_\gamma(E, T)$, for some arbitrary constants $E, T > 0$, it follows that

$$w(T) \leq w(0)e^{-\kappa T} + \int_0^T \gamma(|u(s)|)ds \leq w(0)e^{-\kappa T} + E.$$

Considering this inequality recursively, it follows that, for each $\ell \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned} w(\ell T) &\leq w(0)e^{-\ell\kappa T} + E \sum_{j=0}^{\ell-1} e^{-j\kappa T} \\ &\leq w(0)e^{-\ell\kappa T} + E \sum_{j\geq 0} e^{-j\kappa T} \\ &\leq w(0)e^{-\ell\kappa T} + E \frac{e^{\kappa T}}{e^{\kappa T} - 1}. \end{aligned} \tag{49}$$

Given any $t \geq 0$, pick ℓ as $\lfloor t/T \rfloor$ and define $t' := t - \ell T$. Note that $t' \in [0, T]$. It follows from (48) that

$$w(t) \leq w(\ell T)e^{-\kappa t'} + \int_{\ell T}^t \gamma(|u(s)|)ds \leq w(\ell T)e^{-\kappa t'} + E,$$

which, in view of (49), implies that

$$w(t) \leqslant \left(w(0)e^{-\ell\kappa T} + E\frac{e^{\kappa T}}{e^{\kappa T}-1}\right)e^{-t'} + E$$

$$\leqslant w(0)e^{-k(\ell T+t')} + E\left(1 + \frac{e^{\kappa T}}{e^{\kappa T}-1}\right)$$

$$\leqslant w(0)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,.$$

Recalling that $w(t) = V(x(t;x_0,u))$, it follows that

$$V(x(t;x_0,u)) \leqslant V(x_0)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,$$

which implies, in view of (46), that

$$\underline{\alpha}(|x(t;x_0,u)|) \leqslant \overline{\alpha}(|x_0|)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,$$

Recalling that $\underline{\alpha}^{-1}(a+b) \leqslant \underline{\alpha}^{-1}(2a) + \underline{\alpha}^{-1}(2b)$ as $\underline{\alpha} \in \mathcal{K}_{\infty}$, we finally obtain that, given any $x_0 \in \mathbb{R}^n$, any $u \in \mathcal{W}_{\gamma}(E,T)$ and any $t \geqslant 0$,

$$|x(t;x_0,u)| \leqslant \underline{\alpha}^{-1}\left(2\overline{\alpha}(|x_0|)e^{-\kappa t}\right) + \underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right). \tag{50}$$

Given any $T, \delta \geqslant 0$, the following choice of E:

$$E(T,\delta) \leqslant \frac{\alpha(\delta)}{2}\frac{e^{\kappa T}-1}{2e^{\kappa T}-1}\,, \tag{51}$$

ensures that

$$\underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right) \leqslant \delta$$

and the conclusion follows in view of (50) with the $\mathcal{KL}$ function

$$\beta(s,t) := \underline{\alpha}^{-1}\left(2\overline{\alpha}(s)e^{-\kappa t}\right)\,, \quad \forall s,t \geqslant 0\,.$$

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