Singularly Perturbed Lie Bracket Approximation

Hans-Bernd Dürr, Miroslav Krstić, Alexander Scheinker, and Christian Ebenbauer

***Abstract*—We consider the interconnection of two dynamical systems where one has an input-affine vector field. By employing a singular perturbation and a Lie bracket analysis technique, we show how the trajectories can be approximated by two decoupled systems. From this trajectory approximation result and the stability properties of the decoupled systems, we derive stability properties of the overall system.**

***Index Terms*—Extremum seeking, Lie brackets, singular perturbations.**

I. Introduction

In this note, we consider systems of the form

$$\dot{x} = \mu f(\mu t, x, z, \omega) \tag{1a}$$

$$\dot{z} = g(x, z) \tag{1b}$$

$x(t_0) = x_0 \in \mathbb{R}^n$, $z(t_0) = z_0 \in \mathbb{R}^m$, $n, m, N \in \mathbb{N}$, parameters $\mu, \omega \in (0, \infty)$ and with f having a particular structure, namely

$$f(\mu t, x, z, \omega) := b_0(\mu t, x, z) + \sqrt{\omega} \sum_{i=1}^{N} b_i(\mu t, x, z) u_i(\mu t, \omega \mu t). \tag{2}$$

Assuming the existence of a so-called *quasi-steady state* $h \in C^1: \mathbb{R}^n \to \mathbb{R}^m$ that satisfies $g(x, h(x)) = 0$, we perform a change of variables $x = x$ and $y = z - h(x)$ which yields the system

$$\dot{x} = \mu f\left(\mu t, x, y + h(x), \omega\right) \tag{3a}$$

$$\dot{y} = g(x, y + h(x)) - \mu \frac{\partial h(x)}{\partial x} f\left(\mu t, x, y + h(x), \omega\right) \tag{3b}$$

with $x(t_0) = x_0 \in \mathbb{R}^n$, $y(t_0) = y_0 := z_0 - h(x_0) \in \mathbb{R}^m$. We show that for periodic perturbations $u_i(\mu t, \cdot)$, $i = 1, \ldots, N$ under suitable assumptions and by choosing ω sufficiently large and μ sufficiently small relative to ω, practical stability of (1) can be deduced from the stability properties of a so-called *boundary layer model* and a so-called *Lie bracket system*. The boundary layer model is given by

$$\dot{y} = g\left(x, y + h(x)\right). \tag{4}$$

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with $y(t_0) = y_0 \in \mathbb{R}^m$. It is obtained from (3b) by letting $\mu \to 0$. In order to calculate the Lie bracket system, we introduce the so-called *reduced model*

$$\dot{\tilde{x}} = \mu b_0\left(\mu t, \tilde{x}, h(\tilde{x})\right) + \mu\sqrt{\omega} \sum_{i=1}^{N} b_i\left(\mu t, \tilde{x}, h(\tilde{x})\right) u_i(\mu t, \omega \mu t) \tag{5}$$

with $\tilde{x}(t_0) = \tilde{x}_0 \in \mathbb{R}^n$. It is obtained from (3a) by imposing that $z = h(x)$, i.e., $y = 0$. Then, the Lie bracket system without μ $(\mu = 1)$ is given by

$$\dot{\bar{x}} = b_0\left(t, \bar{x}, h(\bar{x})\right) + \sum_{\substack{i=1 \\ j=i+1}}^{N} [b_i, b_j]\left(t, \bar{x}, h(\bar{x})\right) \nu_{ji}(t) \tag{6}$$

with $\bar{x}(t_0) = x_0$, $\nu_{ji}(t) = (1/T) \int_0^T u_j(t, \theta) \int_0^\theta u_i(t, s) ds d\theta$ and where $[b_1, b_2]$ denotes the Lie bracket of the vector fields b_1 and b_2. Using results from [2], [8], [10], [12], the trajectories of the Lie bracket system (6) approximate those of the reduced model. Furthermore, using arguments from singular perturbation theory (see, e.g., [5]–[7], [14]), we show that the trajectories of the reduced model together with the boundary layer model approximate those of the transformed system (3). Based on this two-step trajectory approximation procedure, the stability properties of (1) can be deduced by regarding the Lie bracket system and the boundary layer model separately, i.e., two decoupled systems. In contrast to the results from literature, we do not assume that the reduced model is uniformly asymptotically stable but that the Lie bracket is uniformly asymptotically stable. One motivation to consider systems of the form (1) is their potential application in extremum seeking [2] and vibrational stabilization [9], [11].

The paper is organized as follows. In Section II we introduce our *singularly perturbed* notion of practical stability. In Section III, we present our main results that consist of two theorems establishing singular semi-global practical uniform asymptotic stability of (1).

Notation: The set of natural numbers is denoted by $\mathbb{N}$. The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$. The δ-neighborhood of a set $\mathcal{S} \subseteq \mathbb{R}^n$ is denoted by $U_\delta^{\mathcal{S}} = \{x \in \mathbb{R}^n : \inf_{e \in \mathcal{S}} \|x - e\| < \delta\}$. The closure of a set $\mathcal{S} \subseteq \mathbb{R}^n$ is denoted by $\bar{\mathcal{S}}$. The solution of a differential equation $\dot{x} = f(x, t)$ with initial condition $x(t_0) = x_0$ is denoted by $x(\cdot; t_0, x_0): \mathbb{R} \to \mathbb{R}^n$. Note that $x(t_0; t_0, x_0) = x_0$. The Jacobian matrix of a continuously differentiable function $b \in C^1: \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $(\partial b(x)/\partial x)$. The Lie bracket of two vector fields $b_1, b_2 \in C^1: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and a function $h \in C^1: \mathbb{R}^n \to \mathbb{R}^m$ is defined as $[b_1, b_2](t, x, h(x)) = (\partial b_2(t, x, h(x))/\partial x) b_1(t, x, h(x)) - (\partial b_1(t, x, h(x))/\partial x) b_2(t, x, h(x))$.

II. Preliminaries

In the following, we introduce the stability definitions that will be used in the theorems. Consider the coupled system (1) with initial conditions $x(t_0) = x_0 \in \mathbb{R}^n$, $z(t_0) = z_0 \in \mathbb{R}^m$, where $n, m \in \mathbb{N} \cup \{0\}$ and assume the existence of a quasi-steady state $h: \mathbb{R}^n \to \mathbb{R}^m$. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a compact set.

Definition 1: The set $\mathcal{S}$ is said to be **singularly practically uniformly stable** for (1) if for all $\epsilon_x, \epsilon_z \in (0, \infty)$ there exist $\delta_x, \delta_z \in$

$(0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $\omega \in (\omega_0,\infty)$ there exists a $\mu_0 \in (0,\infty)$ such that for all $\mu \in (0,\mu_0)$ and for all $t_0 \in \mathbb{R}$

$$\begin{aligned} x_0 \in U_{\delta_x}^{\mathcal{S}} \text{ and } z_0 - h(x_0) \in U_{\delta_z}^0 \Rightarrow \\ x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}} \text{ and} \\ z(t;t_0,z_0) - h\left(x(t;t_0,x_0)\right) \in U_{\epsilon_z}^0, t \in [t_0,\infty). \end{aligned} \tag{7}$$

Definition 2: The set $\mathcal{S}$ is said to be **singularly practically uniformly attractive** for (1) if for all $\delta_x, \delta_z \in (0,\infty)$ and all $\epsilon_x, \epsilon_z \in (0,\infty)$ there exists a $t_f \in [0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $\omega \in (\omega_0,\infty)$ there exists a $\mu_0 \in (0,\infty)$ such that for all $\mu \in (0,\mu_0)$ and for all $t_0 \in \mathbb{R}$

$$\begin{aligned} x_0 \in U_{\delta_x}^{\mathcal{S}} \text{ and } z_0 - h(x_0) \in U_{\delta_z}^0 \Rightarrow \\ x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}}, t \in \left[t_0 + \frac{t_f}{\mu}, \infty\right) \text{ and} \\ z(t;t_0,z_0) - h\left(x(t;t_0,x_0)\right) \in U_{\epsilon_z}^0, t \in [t_0+t_f,\infty). \end{aligned} \tag{8}$$

Definition 3: The solutions of (1) are said to be **singularly practically uniformly bounded with respect to** $\mathcal{S}$ if for all $\delta_x, \delta_z \in (0,\infty)$ there exist $\epsilon_x, \epsilon_z \in (0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $\omega \in (\omega_0,\infty)$ there exists a $\mu_0 \in (0,\infty)$ such that for all $\mu \in (0,\mu_0)$ and for all $t_0 \in \mathbb{R}$

$$\begin{aligned} x_0 \in U_{\delta_x}^{\mathcal{S}} \text{ and } z_0 - h(x_0) \in U_{\delta_z}^0 \Rightarrow \\ x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}} \text{ and} \\ z(t;t_0,z_0) - h\left(x(t;t_0,x_0)\right) \in U_{\epsilon_z}^0, t \in [t_0,\infty). \end{aligned} \tag{9}$$

Definition 4: The set $\mathcal{S}$ is said to be **singularly semi-globally practically uniformly asymptotically stable (sSPUAS)** for (1) if the set $\mathcal{S}$ is singularly practically uniformly stable, singularly practically uniformly attractive and the solutions of (1) are singularly practically uniformly bounded with respect to $\mathcal{S}$.

Definition 5: If the set $\mathcal{S}$ is sSPUAS and the dependence of μ_0 on ω can explicitly be given in Definitions 1–3, we say the set $\mathcal{S}$ is said to be **singularly semi-globally practically uniformly asymptotically stable (sSPUAS) with** $\mu_0 = \mu_0(\omega)$.

Definition 6: The set $\mathcal{S}$ is said to be **globally uniformly asymptotically stable (GUAS)** if the set $\mathcal{S}$ is sSPUAS for (1) for an arbitrary $\omega \in (0,\infty)$ and arbitrary $\mu \in (0,\infty)$, which is independent of ω, e.g., when (1) is independent of ω and μ. The definition of GUAS is similar to the notion introduced in, e.g., [3].

III. Main Results

In this section, we consider (1) and relate the stability properties of system (1) with the stability properties of the Lie bracket system (6) and the boundary layer model (4).

Assumption A: The vector fields of (1) satisfy the following properties.

F1 $b_i \in C^2 : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 0,\ldots,N$.

F2 The vector fields b_i, $i = 0,\ldots,N$ are bounded in their first argument up to the second derivative, i.e., for all compact sets $\mathcal{C}_x \subseteq \mathbb{R}^n$ and $\mathcal{C}_z \subseteq \mathbb{R}^m$ there exist $A_1,\ldots,A_6 \in [0,\infty)$ such that $|b_i(t,x,z)| \le A_1$, $|\frac{\partial b_i(t,x,z)}{\partial t}| \le A_2$, $|\frac{\partial b_i(t,x,z)}{\partial x}| + |\frac{\partial b_i(t,x,z)}{\partial z}| \le A_3$, $|\frac{\partial^2 b_j(t,x,z)}{\partial t \partial x}| + |\frac{\partial^2 b_2 j(t,x,z)}{\partial t \partial z}| \le A_4$, $|\frac{\partial [b_j,b_k](t,x,z)}{\partial x}| + |\frac{\partial [b_j,b_k](t,x,z)}{\partial z}| \le A_5$, $|\frac{\partial [b_j,b_k](t,x,z)}{\partial t}| \le A_6$ for all $x \in \mathcal{C}_x$, $z \in \mathcal{C}_z$, $t \in \mathbb{R}$, $i = 0,\ldots,N$, $j = 1,\ldots,N$, $k = j,\ldots,N$.

F3 $u_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,\ldots,N$ are measurable functions. Moreover, for all $i = 1,\ldots,N$ there exist constants $L_i, M_i \in (0,\infty)$ such that $|u_i(t_1,\theta) - u_i(t_2,\theta)| \le L_i|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$ and such that $\sup_{t,\theta\in\mathbb{R}} |u_i(t,\theta)| \le M_i$.

F4 $u_i(t,\cdot)$ is T-periodic, i.e., $u_i(t,\theta + T) = u_i(t,\theta)$ and has zero average, i.e., $\int_0^T u_i(t,s)ds = 0$, with $T \in (0,\infty)$ for all $t,\theta \in \mathbb{R}$, $i = 1,\ldots,N$.

G1 $g \in C^0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is locally Lipschitz.

G2 There exists a unique $h \in C^2 : \mathbb{R}^n \to \mathbb{R}^m$ such that $z = h(x) \Leftrightarrow 0 = g(x,z)$ for all $x \in \mathbb{R}^n$.

Theorem 1: Consider (1) and suppose Assumption A is satisfied. Furthermore, suppose that a compact set $\mathcal{S}$ is GUAS for the Lie bracket system (6) and there exist $\mathcal{K}_\infty$-functions α_1, α_2, a $\mathcal{K}$-function α_3 and a function $V \in C^1 : \mathbb{R}^m \to \mathbb{R}$ such that for all $[x^\top, y^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^m$

$$\alpha_1\left(\|y\|\right) \le V(y) \le \alpha_2\left(\|y\|\right) \tag{10}$$

$$\frac{\partial V(y)}{\partial y} g\left(x, y + h(x)\right) \le -\alpha_3\left(\|y\|\right). \tag{11}$$

Then, the set $\mathcal{S}$ is sSPUAS for (1).

The proof is in the Appendix.

In the theorem, we do not explicitly speak of the *boundary layer model* (see also, e.g., [6], [7]) which is obtained by centering the fast state z by its quasi-steady state $h(x)$, i.e., $y = z - h(x)$, and additionally letting $\mu \to 0$ which then yields (4). Instead, we only require the existence of a Lyapunov function for its vector field.

Theorem 2: Consider (1) and suppose Assumption A is satisfied. Furthermore, suppose that a compact set $\mathcal{S}$ is GUAS for the Lie bracket system (6) and there exist constants $k_1, k_2, k_3, \alpha \in (0,\infty)$ and a function $V \in C^1 : \mathbb{R}^m \to \mathbb{R}$ such that for all $[x^\top, y^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^m$

$$k_1\|y\|^\alpha \le V(y) \le k_2\|y\|^\alpha \tag{12}$$

$$\frac{\partial V(y)}{\partial y} g\left(x, y + h(x)\right) \le -k_3\|y\|^\alpha. \tag{13}$$

Let $\kappa \in (0,\infty)$ be such that $\kappa > ((2+\alpha)/2)$. Then, the set $\mathcal{S}$ is sSPUAS for (1) with $\mu_0 = \omega^{-\kappa}$.

The proof of Theorem 2 goes along the same lines as the proof of Theorem 1. In contrast to Theorem 1, Theorem 2 gives the additional information of how to choose the value of μ_0 appearing in the definition of sSPUAS.

IV. Example

In the following, we present an example for the foregoing results. We consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2 \end{aligned} \tag{14a}$$

$$\dot{z} = -z + x_3 \tag{14b}$$

which is also known as Brockett integrator with an additional state z. The goal is to design an output feedback controller that stabilizes the origin for (14a) and we assume that only states x_1, x_2, and z are directly available for the use in the controller. We propose the following control inputs: $u_1 = \mu(-x_1 + \sqrt{\omega}\sin(\omega\mu t))$ and $u_2 = \mu(-(3/2)x_2 - z\sqrt{\omega}\cos(\omega\mu t))$ (see also, e.g., [15]), and employ the previously introduced methodology to analyze the closed loop system

$$\dot{x} = \mu\left(\underbrace{\begin{bmatrix} -x_1 \\ -\frac{3}{2}x_2 \\ \frac{1}{2}x_1 x_2 \end{bmatrix}}_{b_0(x,z)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}}_{b_1(x,z)} \sqrt{\omega}\underbrace{\sin(\omega\mu t)}_{\tilde{u}_1(\omega\mu t)} + \underbrace{\begin{bmatrix} 0 \\ -z \\ x_1 z \end{bmatrix}}_{b_2(x,z)} \sqrt{\omega}\underbrace{\cos(\omega\mu t)}_{\tilde{u}_2(\omega\mu t)}\right) \tag{15a}$$

$$\dot{z} = -z + x_3 \tag{15b}$$

which is in the form (1) with $g(x,z) = -z + x_3$. We obtain for the quasi-steady state $h(x) = x_3$ which yields the reduced model

$$\dot{\tilde{x}} = b_0(\tilde{x}, h(\tilde{x})) + b_1(\tilde{x}, h(\tilde{x}))\sqrt{\omega}\tilde{u}_1(\omega t) + b_2(\tilde{x}, h(\tilde{x}))\sqrt{\omega}\tilde{u}_2(\omega t). \tag{16}$$

The associated Lie bracket system (6) is in this case

$$\dot{\bar{x}} = b_0\left(\bar{x}, h(\bar{x})\right) - \frac{1}{2}[b_1, b_2]\left(\bar{x}, h(\bar{x})\right) = \begin{bmatrix} -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \end{bmatrix} \tag{17}$$

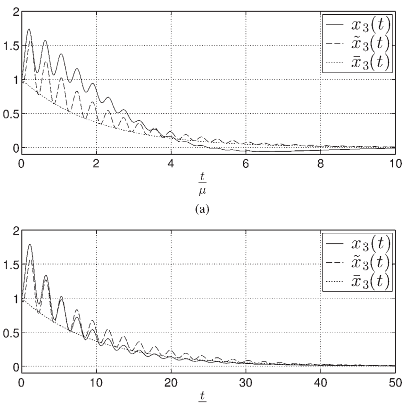

Fig. 1. Trajectories of the third state $x_3(t)$ (solid line), the third state $\tilde{x}_3(t)$ of the reduced model (dashed line) and the third state $\bar{x}_3(t)$ of the Lie bracket system (dotted line) for different parameter values. (a) $\omega = 30$, $\mu = 0.5$. (b) $\omega = 30$, $\mu = 0.1$.

and we see that the origin is GUAS for (17). The boundary layer model (4) becomes $\dot{y} = -y$. Since this is a linear system, the function $V(y) = (1/2)y^2$, satisfies (10) and (11) but also (12) and (13). We may therefore apply Theorem 1 or Theorem 2 here and conclude that the feedback laws u_1 and u_2 yield that the origin is sSPUAS for (15).

This example shows nicely the influence of the parameters ω and μ on the trajectories of the different systems. In Fig. 1, we compare the third state x_3 of (14), $\tilde{x}_3$ of the reduced model (16) and $\bar{x}_3$ of the Lie bracket system (17) for different parameter values. Notice that the axis is normalized by $(1/\mu)$. We see that for all chosen parameter values, all states converge to the origin. However, comparing Fig. 1(a) and (b) we see that for the same value of ω and different values of μ the trajectory of x_3 better approximates that of the reduced model $\tilde{x}_3$ for a smaller value of μ.

V. Summary

We proved a lemma (see Appendix) that generalizes existing trajectory approximation results based on Lie brackets to singularly perturbed systems (see, e.g., [8], [12]). Based on this, we proved two theorems that establish singular practical stability of systems of the form (1). The theorems require the knowledge of the stability properties of the Lie bracket system (6) and the boundary layer model (4). We presented an example for the theoretic results which showed the influence of the two parameters ω and μ on the system behavior.

Appendix

Proof of Theorem 1: In the proof, we employ ideas from, e.g., [10], [13] which we extend to systems depending two parameters—a large (ω) and a small (μ) one—instead of a single parameter.

Lemma 1: Consider the reduced model (5) and let the assumptions of Theorem 1 be satisfied, i.e., suppose Assumption A is satisfied, furthermore suppose that a compact set $\mathcal{S}$ is GUAS for the Lie bracket system (6) and there exist $\mathcal{K}_\infty$-functions α_1, α_2, a $\mathcal{K}$-function α_3 and a function $V \in C^1 : \mathbb{R}^m \to \mathbb{R}$ such that for all $[x^\top, y^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^m$ (10) and (11) are satisfied.

Then, there exist a $\mathcal{KL}$-function $\gamma : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for all bounded sets $B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ there exists an $\omega_0 \in (0,\infty)$ such that for all $\omega \in (\omega_0, \infty)$, for all $t_f \in (0,\infty)$ and for all $D \in (0,\infty)$ there exists a $\mu_0 \in (0,\infty)$ such that for all $\mu \in (0,\mu_0)$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ and for all $t \in [t_0, t_0 + (t_f/\mu)]$ the solutions of (3) exist, are unique and satisfy

$$\begin{aligned} &\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| < D \text{ and} \\ &\|y(t;t_0,y_0)\| < \gamma(\|y_0\|, t - t_0) + D. \end{aligned} \tag{18}$$

Proof: First step. We show existence and uniqueness of the solutions of (3) and the boundedness of the solutions of (5).

Let the bounded sets $B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be arbitrary but fixed. Observe that the vector field of (3) satisfies Assumption A, i.e., the vector field is locally Lipschitz and can be bounded on any compact set by a constant. Therefore, all conditions of [1, Theorem 2.1.1, p. 14] and [1, Theorem 2.1.3, p. 16] are satisfied, and thus we have that for all $[x_0^\top, y_0^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^m$ every $\omega, \mu \in (0,\infty)$, every $t_0 \in \mathbb{R}$ there exist a $\bar{t} \in (0,\infty)$ and a unique, absolutely continuous solution of (3) on $t \in [t_0, t_0 + \bar{t})$. Let $[t_0, t_0 + t_e)$ with $t_e \in (0,\infty)$ be the maximal interval of existence, i.e., the supremum over all $\bar{t}$ such that (5) has a unique, absolutely continuous solution on $[t_0, t_0 + t_e)$. Next, consider the reduced model (5) and let $\mu = 1$. Note that the vector field f satisfies Assumptions A.F1–A.F4. Furthermore, the set $\mathcal{S}$ is globally uniformly asymptotically stable for the Lie bracket system (6) associated to (5) with $\mu = 1$. Thus, we may use [2, Theorem 3] and conclude that (5) is SPUAS for $\mu = 1$. However, this implies that (5) is also SPUAS for any $\mu \in (0,\infty)$, since μ plays the role of a time scale and does not influence the qualitative stability properties of the system. Summarizing the first step, we can state the following claim.

Claim 1: Under the assumptions of Lemma 1, for all bounded sets $B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ there exist an $\omega_0 \in (0,\infty)$ and a bounded set $M_x \subseteq \mathbb{R}^n$ such that for all $\omega \in (\omega_0,\infty)$, for all $\mu \in (0,\infty)$, for all $t_0 \in \mathbb{R}$ and for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ there exists a maximal interval of existence $[t_0, t_0 + t_e)$ with $t_e \in (0,\infty)$, a unique absolutely continuous solution $[x(\cdot;t_0,x_0)^\top, y(\cdot;t_0,y_0)^\top]^\top$ defined on $t \in [t_0, t_0 + t_e)$ and, moreover, $\tilde{x}(t;t_0,x_0) \in M_x$, $t \in [t_0,\infty)$.

Second step. Based on Claim 1, we show that there exists a $\mathcal{KL}$-function γ and a $\mathcal{K}$-function α such that $\|y(t;t_0,y_0)\| \leq \gamma(\|y_0\|, t - t_0) + \alpha(\mu)$ as long as $x(t;t_0,x_0)$ is in a neighborhood around $\tilde{x}(t;t_0,x_0)$. We begin by defining a neighborhood $O_x(t)$ around $\tilde{x}(t;t_0,x_0)$ which contains $x(t;t_0,x_0)$ for some time-interval $[t_0, t_0 + t_D]$ and we show existence of a set C_y which contains $y(t;t_0,y_0)$ for at least the same interval.

Let $O_x(t) = \{x \in \mathbb{R}^n : \|x - \tilde{x}(t;t_0,x_0)\| < D\}$, $t \in [t_0,\infty)$. We distinguish two Cases C1) and C2). For Case C1), we may assume by continuity of solutions that for all $\mu \in (0,\infty)$, for all $t_0 \in \mathbb{R}$ and all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ there exists a time $t_D \in [0, t_e)$ when $x(t;t_0,x_0)$ leaves the set $O_x(t)$ for the first time, i.e., $\|x(t_0 + t_D;t_0,x_0) - \tilde{x}(t_0 + t_D;t_0,x_0)\| = D$. Case C2) is when $x(t;t_0,x_0) \in O_x(t)$ for $t \in [t_0, t_0 + t_e)$. We consider now the first Case C1) and treat the second Case C2) later. Define the compact set $C_x := \bar{U}_D^{M_x}$ and observe that $x(t;t_0,x_0) \in C_x$ for all $t \in [t_0, t_0 + t_D]$.

By assumption, there exists a function $V \in C^1$ satisfying (10) and (11). Consider the $\mathcal{K}_\infty$-functions α_1 and α_2 and let $c \in [0,\infty)$ be chosen such that $M_y := \{y \in \mathbb{R}^m : \alpha_2(\|y\|) \leq c\}$ contains B_y. Since B_y is bounded and α_2 is a continuous, radially unbounded function with compact level sets, such a constant exists. Define $C_y := \{y \in \mathbb{R}^m : \alpha_1(\|y\|) \leq c\}$ and note that by the definition of a $\mathcal{K}_\infty$-function, C_y is compact. Furthermore, since α_1, α_2, and V satisfy (10), it follows that:

$$B_y \subseteq M_y \subseteq \{y \in \mathbb{R}^m : V(y) \leq c\} \subseteq C_y. \tag{19}$$

We now show that there exists a $\mu_{0,1} \in (0,\infty)$ such that for all $\mu \in (0,\mu_{0,1})$ and for all $[x^\top, y^\top] \in C_x \times C_y \backslash M_y$, we have that $\dot{V} \leq 0$.

First, from (3b) it follows that:

$$\dot{V} = \frac{\partial V(y)}{\partial y} g(x, y + h(x)) - \mu \frac{\partial V(y)}{\partial y} \frac{\partial h(x)}{\partial x} f(\mu t, x, y + h(x), \omega). \tag{20}$$

Since f satisfies Assumption A.F1–F4, h satisfies Assumption A.G2, $V \in C^1$ and $C_x \times C_y$ is compact, there exists a constant $M_1 \in [0,\infty)$ that satisfies for all $[x^\top, y^\top]^\top \in C_x \times C_y$, for all $\mu \in (0,\infty)$ and for all $t \in \mathbb{R}$

$$\left\| \frac{\partial V(y)}{\partial y} \right\| \left\| \frac{\partial h(x)}{\partial x} \right\| \|f(\mu t, x, y + h(x), \omega)\| \leq M_1. \tag{21}$$

Therefore, we obtain together with (11) the following estimate, which holds for all $[x^\top, y^\top]^\top \in C_x \times C_y$ and for all $t \in \mathbb{R}$

$$\dot{V} \leq -\alpha_3(\|y\|) + M_1\mu \leq -\frac{1}{2}\alpha_3(\|y\|), \text{ with } \|y\| \geq \alpha_3^{-1}(2\mu M_1). \tag{22}$$

Let $\mu_{0,1} = ((\alpha_3(\alpha_2^{-1}(c)))/(2M_1))$, it follows from (22) that for all $\mu \in (0, \mu_{0,1})$, for all $[x^\top, y^\top]^\top \in C_x \times C_y \backslash M_y$ we have that $\dot{V} \leq 0$ and with (19) we see that the solutions $y(\cdot; t_0, y_0)$ can not leave the sub-level set $\{y \in \mathbb{R}^m : V(y) \leq c\}$ as long as $x(t; t_0, x_0) \in O_x(t)$ and therefore we have for all $t \in [t_0, t_0 + t_D]$ that $[x(t; t_0, y_0)^\top, y(t; t_0, y_0)^\top]^\top \in C_x \times C_y$. Furthermore, it follows from (22) using similar arguments as in the proof of [6, Theorem 4.18] that there exist $\mathcal{K}$-functions γ and α with $\alpha(\mu) := \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(2\mu M_1)))$ such that for all $\mu \in (0, \mu_{0,1})$ the solution $y(\cdot; t_0, y_0)$ of (3) satisfies for all $t \in [t_0, t_0 + t_D]$:

$$\|y(t; t_0, y_0)\| \leq \gamma(\|y_0\|, t - t_0) + \alpha(\mu). \tag{23}$$

In particular, the function γ is a combination of the functions α_1, α_2 and α_3 which are independent of μ. Therefore γ is also independent of μ. We furthermore let $\mu_{0,2} = \min\{\alpha^{-1}(D/2), \mu_{0,1}\}$, which yields that $\alpha(\mu) \leq (D/2)$ for all $\mu \in (0, \mu_{0,2})$. It is left to consider the Case C2) when $x(t; t_0, x_0) \in O_x(t) \subseteq C_x$ for $t \in [t_0, t_0 + t_e)$. In Case C2), one can show by the same arguments as above, that for all $\mu \in (0, \mu_{0,1})$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ and for all $t \in [t_0, t_0 + t_e)$ we have that $y(t; t_0, y_0) \in C_y$. Since $C_x \times C_y$ is compact and therefore the solutions $[x(\cdot; t_0, x_0)^\top, y(\cdot; t_0, y_0)^\top]^\top$ are bounded, it follows from [1, Theorem 2.1.4, p. 18] (or [4, Theorem 5.2, p. 29]) that $t_e = \infty$.

Summarizing the second step, we can state the following claim.

Claim 2: Under the assumptions of Lemma 1, there exists a $\mathcal{KL}$-function γ and a $\mathcal{K}$-function α such that for all bounded sets $B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ there exists an $\omega_0 \in (0, \infty)$ such that for all $\omega \in (\omega_0, \infty)$ and all $D \in (0, \infty)$ there exist compact sets $C_x \times C_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and a $\mu_{0,2} \in (0, \infty)$ such that for all $\mu \in (0, \mu_{0,2})$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ we have that $\alpha(\mu) < D$ and one of the following statements is true:

C1) There exists a $t_D \in (0, \infty)$ such that for all $t \in [t_0, t_0 + t_D]$ we have that $[x(t; t_0, x_0)^\top, y(t; t_0, y_0)^\top]^\top \in C_x \times C_y$ and

$$\|y(t; t_0, y_0)\| \leq \gamma(\|y_0\|, t - t_0) + \alpha(\mu). \tag{24}$$

Moreover, we have for all $t \in [t_0, t_0 + t_D)$ that $\|x(t; t_0, x_0) - \tilde{x}(t; t_0, x_0)\| < D$ and $\|x(t_0 + t_D; t_0, x_0) - \tilde{x}(t_0 + t_D; t_0, x_0)\| = D$.

C2) For all $t \in [t_0, \infty)$, we have that $[x(t; t_0, x_0)^\top, y(t; t_0, y_0)^\top]^\top \in C_x \times C_y$

$$\|x(t; t_0, x_0) - \tilde{x}(t; t_0, x_0)\| < D \text{ and}$$
$$\|y(t; t_0, y_0)\| \leq \gamma(\|y_0\|, t - t_0) + \alpha(\mu). \tag{25}$$

Third step. We observe that the statement of Lemma 1 follows from Case C2) of Claim 2. It is therefore left to show that the claim of the lemma also holds in Case C1). In the following, we show that the distance between $x(t; t_0, x_0)$ and $\tilde{x}(t; t_0, x_0)$ can be made arbitrary small on an arbitrary long time-interval for a sufficiently small value of μ.

The idea of the proof is as follows. We define a neighborhood U around the origin of $\mathbb{R}^m$ and show that for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$, for all $t_0 \in \mathbb{R}$ and for a sufficiently small μ the solutions $y(\cdot; t_0, y_0)$ enter U after a time $t_U \in [0, \infty)$ which is smaller than t_D, i.e., $t_U < t_D$. Intuitively, this becomes clear due to (3) and (24). For small values of $y(t; t_0, x_0)$, the solutions $x(\cdot; t_0, x_0)$ behave like the solutions $\tilde{x}(\cdot; t_0, x_0)$. This behavior allows us to show that by choosing μ sufficiently small, the time t_D can be made arbitrarily large.

Suppose that Case C1) of Claim 2 holds. We define the neighborhood U. For this purpose, we introduce a Lipschitz constant $L_f \in (0, \infty)$ and a bound $M \in (0, \infty)$ of f satisfying for all $x, x_1, x_2 \in C_x$, for all $y \in C_y$, for all $\mu \in (0, \infty)$ and for all $t \in \mathbb{R}$

$$\|f(\mu t, x, y, \omega) - f(\mu t, x, 0, \omega)\| \leq L_f\|y\| \tag{26}$$

$$\|f(\mu t, x_1, 0, \omega) - f(\mu t, x_2, 0, \omega)\| \leq L_f\|x_1 - x_2\| \tag{27}$$

$$\max\{\|f(\mu t, x, y, \omega)\|, \|f(\mu t, x, 0, \omega)\|\} \leq M. \tag{28}$$

Note that $\omega \in (\omega_0, \infty)$ is arbitrary but fixed. We observe that L_f and M exist, since C_x and C_y are compact, f satisfies Assumptions A.F1–A.F2 and in particular they are independent of μ, since by Assumptions A.F1–A.F4 the vector field f is bounded in its first argument independently of μ.

Furthermore, we choose a time $t_U \in (0, \infty)$ such that $\gamma(\|y_0\|, t_U) \leq (D/(8Mt_f e^{L_f t_f}))$ for all $y_0 \in \bar{B}_y$, i.e., a time such that for all initial conditions $y_0 \in \bar{B}_y$ and for all $t \in [t_0 + t_U, \infty)$, we have that $\gamma(\|y_0\|, t - t_0) \leq (D/(8L_f t_f e^{L_f t_f}))$. Now let

$$\mu_0 = \min\left\{\mu_{0,2}, \alpha^{-1}\left(\frac{D}{8L_f t_f e^{L_f t_f}}\right), \frac{D}{4Mt_U}, \frac{D}{8Mt_U e^{L_f t_f}}\right\} \tag{29}$$

where $\mu_{0,2}$ is from Claim 2. We observe that t_U is independent of μ, since γ, D, L_f, and t_f are independent of μ.

Having these definitions at hand, we define the neighborhood $U := \{y \in \mathbb{R}^m : \|y\| \leq (D/(4L_f t_f e^{L_f t_f}))\}$ and observe with (24) that if $t_U < t_D$, we have for all $\mu \in (0, \mu_0)$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ that for all $t \in [t_0 + t_U, t_0 + t_D]$

$$y(t; t_0, y_0) \in U. \tag{30}$$

In this paragraph, we show that $t_U < t_D$, i.e., $y(t; t_0, y_0)$ converges to U faster than $x(t; t_0, x_0)$ may leave $O_x(t)$. In order to show a contradiction, we assume that $t_U \geq t_D$. Consider the distance between the solution of (3a) and (5) is

$$\|x(t; t_0, x_0) - \tilde{x}(t; t_0, x_0)\| = \left\|\int_{t_0}^{t} \mu f(\mu s, x(s), y(s), \omega) - \mu f(\mu s, \tilde{x}(s), 0, \omega) ds\right\|. \tag{31}$$

Note also that we write for the sake of brevity $x(s)$, $\tilde{x}(s)$, $y(s)$ instead of $x(s; t_0, x_0)$, $\tilde{x}(s; t_0, x_0)$, $y(s; t_0, y_0)$. With the definition of M in (28), we then obtain from (31) that

$$\|x(t; t_0, x_0) - \tilde{x}(t; t_0, x_0)\| \leq \int_{t_0}^{t_0 + t_D} 2\mu M ds = 2\mu M t_D \leq 2\mu M t_U. \tag{32}$$

Recall, by the definition of μ_0 in (29), we have that $\mu_0 \leq (D/(4Mt_U))$ and thus, we observe that for all $\mu \in (0, \mu_0)$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0, t_0 + t_D]$ we have that $\|x(t; t_0, x_0) - \tilde{x}(t; t_0, x_0)\| \leq (D/2)$. This contradicts $\|x(t_D; t_0, x_0) - \tilde{x}(t_D; t_0, x_0)\| = D$ and thus the assumption $t_U \geq t_D$ cannot be true. Therefore, we have $t_U < t_D$.

Finally, we show by contradiction that for a sufficiently small μ we have that $t_D \geq (t_f/\mu)$. We assume that there are bounded sets $B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$, a $t_f \in [0, \infty)$ and a $D \in (0, \infty)$ such that for all $\mu_0 \in (0, \infty)$ there exists a $\mu \in (0, \mu_0)$, a $t_0 \in \mathbb{R}$ and a $[x_0^\top, y_0^\top] \in B_x \times B_y$ such that $t_D < (t_f/\mu)$.

The distance between the solution of (3a) and (5) on the interval $t \in [t_0, t_0+t_D]$ $\|x(t;t_0,x_0)-\tilde{x}(t;t_0,x_0)\| = \|\int_{t_0}^{t} \mu f(\mu s, x(s), y(s), \omega) - \mu f(\mu s, \tilde{x}(s), 0, \omega) ds\|$. Since $t_U < t_D$ and $t \in [t_0, t_0 + t_D]$

$$\begin{aligned}\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| \\ \leq \mu \int_{t_0}^{t_0+t_U} \|f(\mu s, x(s), y(s), \omega) - f(\mu s, \tilde{x}(s), 0, \omega)\| ds \\ + \mu \int_{t_0+t_U}^{t_0+t_D} \|f(\mu s, x(s), y(s), \omega) - f(\mu s, \tilde{x}(s), 0, \omega)\| ds. \quad (33)\end{aligned}$$

Recall, by the definition of μ_0 in (29), we have that $\mu_0 \leq (D/(8Mt_U e^{L_f t_f}))$ and thus, we observe that similar as in (32), we obtain for the first integral in (33) that for all $\mu \in (0, \mu_0)$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$, for all $t_0 \in \mathbb{R}$ that $\mu \int_{t_0}^{t_0+t_U} \|f(\mu s, x(s), y(s), \omega) - f(\mu s, \tilde{x}(s), 0, \omega)\| ds \leq 2\mu M t_U \leq (D/(4e^{L_f t_f}))$. For the second integral in (33), we proceed as follows. We add and subtract $\mu f(\mu s, x(s), 0, \omega)$ which yields $\mu \int_{t_0+t_U}^{t_0+t_D} \|f(\mu s, x(s), y(s), \omega) - f(\mu s, \tilde{x}(s), 0, \omega)\| ds = \mu \int_{t_0+t_U}^{t_0+t_D} \|f(\mu s, x(s), y(s), \omega) - f(\mu s, x(s), 0, \omega) + f(\mu s, x(s), 0, \omega) - f(\mu s, \tilde{x}(s), 0, \omega)\| ds$. Next, we exploit the Lipschitz constant L_f satisfying (26), (27) and obtain $\mu \int_{t_0+t_U}^{t_0+t_D} \|f(\mu s, x(s), y(s), \omega) - \mu f(\mu s, \tilde{x}(s), 0, \omega)\| ds \leq \mu \int_{t_0+t_U}^{t_0+t_D} L_f \|y(s)\| ds + L_f \|x(s) - \tilde{x}(s)\| ds$. We observe with (30) that the following inequality holds for all $\mu \in (0, \mu_0)$ and all $t \in [t_0 + t_U, t_0 + t_D]$:

$$\|y(t;t_0,y_0)\| \leq \frac{D}{4 t_f L_f e^{L_f t_f}}. \quad (34)$$

Using these observations for (33) and combining it with (34), we obtain with Gronwall's Lemma $\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| \leq ((D/(4e^{L_f t_f})) + \mu (D/(4 t_f e^{L_f t_f})) t_D) e^{L_f \mu t_D}$. Recall that we assumed $t_D < (t_f/\mu)$ in order to reach a contradiction and therefore

$$\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| \leq \frac{D}{2} \quad (35)$$

which holds for all $\mu \in (0, \mu_0)$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ and for all $t \in [t_0, t_0 + t_D]$. This contradicts the definition of t_D being the time when $x(t;t_0,x_0)$ leaves $O_x(t)$ and thus $t_D < (t_f/\mu)$ cannot be true. Thus, we conclude that $t_D \geq (t_f/\mu)$ which shows that the claim of the lemma also holds in Case 1) of Claim 2. □

Lemma 2: Let the assumptions of Theorem 1 be satisfied, i.e., suppose Assumption A is satisfied, furthermore suppose that a compact set $\mathcal{S}$ is GUAS for the Lie bracket system (6) and there exist $\mathcal{K}_\infty$-functions α_1, α_2, a $\mathcal{K}$-function α_3 and a function $V \in C^1 : \mathbb{R}^m \to \mathbb{R}$ such that for all $[x^\top, y^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^m$ (10) and (11) are satisfied. Then, the set $\mathcal{S}$ is sSPUAS for (3).

Proof: The proof goes along similar lines as in [2] and [10]. We also refer to the stability definitions in [2]. The set $\mathcal{S}$ is GUAS for the Lie bracket system (6) associated to (5) with $\mu = 1$. Therefore, as in the proof of Lemma 1, we may use [2, Theorem 3] and conclude that (5) is SPUAS for $\mu = 1$. This implies that (5) is also SPUAS for any $\mu \in (0, \infty)$, since μ plays the role of a time scaling and does not influence the stability properties of the system. In the following, we prove that the set $\mathcal{S}$ is sSPUAS for (3) by showing that the conditions in Definitions 1–3 are satisfied. The proof for each definition goes along similar lines but is outlined in detail for the sake of completeness.

Singular Practical Uniform Stability: Let $\epsilon_x, \epsilon_y \in (0, \infty)$ be given. Let $C_{1,x} \in (0, \epsilon_x)$. Observe that since the set $\mathcal{S}$ is practically uniformly stable for the reduced model (5), there exists a $\delta_x \in (0, \infty)$ and a $\omega_{0,1} \in (0, \infty)$ such that for all $\omega \in (\omega_{0,1}, \infty)$, all $\mu \in (0, \infty)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0, \infty)$

$$x_0 \in U_{\delta_x}^{\mathcal{S}} \Rightarrow \tilde{x}(t;t_0,x_0) \in U_{C_{1,x}}^{\mathcal{S}}. \quad (36)$$

Furthermore, since the set $\mathcal{S}$ is practically uniformly attractive for (5) (see [2], [10]) and due to the time-scale μ, we have that for all $C_{2,x} \in (0, \delta_x)$ there exists a time $t_{f,x} \in (0, \infty)$ and an $\omega_{0,2} \in (0, \infty)$ such that for all $\omega \in (\omega_{0,2}, \infty)$, for all $\mu \in (0, \infty)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0 + (t_{f,x}/\mu), \infty)$

$$x_0 \in U_{\delta_x}^{\mathcal{S}} \Rightarrow \tilde{x}(t;t_0,x_0) \in U_{C_{2,x}}^{\mathcal{S}}. \quad (37)$$

Consider now Lemma 1. Note that all the assumptions of Lemma 1 are satisfied and let γ be the $\mathcal{KL}$-function from Lemma 1. Choose some $C_{3,y} \in (0, \epsilon_y)$ and let $\delta_y \in (0, \infty)$ be chosen such that

$$\gamma(\delta_y, 0) \leq C_{3,y}. \quad (38)$$

Choose some $C_{4,y} \in (0, \delta_y)$ and let furthermore $t_{f,y} \in (0, \infty)$ be the time such that

$$\gamma(\delta_y, t_{f,y}) \leq C_{4,y}. \quad (39)$$

We now show that a proper choice of the constants D, t_f and t_f ensures with Lemma 1 that if $x_0 \in U_{\delta_x}^{\mathcal{S}}$ and $y_0 \in U_{\delta_y}$ then $x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}}$ and $y(t;t_0,y_0) \in U_{\epsilon_y}$. Moreover, $x(t_f, t_0, x_0) \in U_{\delta_x}^{\mathcal{S}}$ and $y(t_f;t_0,y_0) \in U_{\delta_y}$ which allows to repeatedly apply this argument in order to guarantee that $x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}}$ and $y(t;t_0,y_0) \in U_{\epsilon_y}$.

Let $B_x = U_{\delta_x}^{\mathcal{S}}$, $B_y = U_{\delta_y}$, $D = \min\{\epsilon_x - C_{1,x}, \delta_x - C_{2,x}, \epsilon_y - C_{3,y}, \delta_y - C_{4,y}\}$ and $t_f = \max\{t_{f,x}, t_{f,y}\}$. With Lemma 1, we have that there exists an $\omega_{0,3} \in (0, \infty)$ such that for all $\omega \in (\omega_{0,3}, \infty)$ there exists a $\mu_{0,1} \in (0, \infty)$ such that for all $\mu \in (0, \mu_{0,1})$, for all $t_0 \in \mathbb{R}$, for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$ we have that for all $t \in [t_0, t_0 + (t_f/\mu)]$

$$\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| < D \text{ and} \quad (40)$$

$$\|y(t;t_0,y_0)\| < \gamma(\|y_0\|, t - t_0) + D. \quad (41)$$

Choosing $\omega_0 = \max\{\omega_{0,1}, \omega_{0,2}, \omega_{0,3}\}$, combining (36) with (40) and (37) with (40), we have that for all $\omega \in (\omega_0, \infty)$ there exists a $\mu_0 \in (0, \min\{1, \mu_{0,1}\})$ such that for all $\mu \in (0, \mu_0)$, for all $t_0 \in \mathbb{R}$, for all $t \in [t_0, t_0 + (t_f/\mu)]$ we have that $x_0 \in U_{\delta_x}^{\mathcal{S}} \Rightarrow x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}}$ and $x(t_0 + (t_f/\mu); t_0, x_0) \in U_{\delta_x}^{\mathcal{S}}$. Since $0 < \mu < \mu_0 < 1$ we know that $t_{f,y} < (t_f/\mu)$. This yields with (41) that for all $t \in [t_0, t_0 + (t_f/\mu)]$, we have that $y(t;t_0,y_0) \in U_{\epsilon_y}$ and $y(t_0 + (t_f/\mu); t_0, y_0) \in U_{\delta_y}$. Since $x(t_0;t_0,x_0) \in U_{\delta_x}^{\mathcal{S}}$ and $x(t_0 + (t_f/\mu); t_0, x_0) \in U_{\delta_x}^{\mathcal{S}}$ as well as $y(t_0;t_0,y_0) \in U_{\delta_y}$ and $y(t_0 + (t_f/\mu); t_0, y_0) \in U_{\delta_y}$, a repeated application of the argument above with another solution $\tilde{x}$ of (5) through $x(t_0 + (t_f/\mu); t_0, x_0)$, i.e., $\tilde{x}(t; t_0 + (t_f/\mu), x(t_0 + (t_f/\mu); t_0, x_0))$ as well as another solution of y through $y(t_0 + (t_f/\mu); t_0, x_0)$, i.e., $y(t; t_0 + (t_f/\mu), y(t_0 + (t_f/\mu); t_0, x_0))$ (and with the same choice of D, B_x, B_y and t_f as above) yields for all $\mu \in (0, \mu_0)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0, \infty)$ $x_0 \in U_{\delta_x}^{\mathcal{S}}$ and $y_0 \in U_{\delta_y} \Rightarrow x(t;t_0,x_0) \in U_{\epsilon_x}^{\mathcal{S}}$ and $y(t;t_0,y_0) \in U_{\epsilon_y}$ and thus we proved singular practical uniform stability.

Singular Practical Uniform Attractivity: Let $\delta_x, \delta_y, \epsilon_x, \epsilon_y \in (0,\infty)$ be given. By singular practical uniform stability proven above, there exist an $\omega_{0,1} \in (0,\infty)$ and $C_{1,x}, C_{1,y} \in (0,\infty)$ such that for all $\omega \in (\omega_{0,1},\infty)$ there exists an $\mu_{0,1} \in (0,\infty)$ such that for all $\mu \in (0,\mu_{0,1})$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0,\infty)$

$$x_0 \in U^{\mathcal{S}}_{C_{1,x}} \text{ and } y_0 \in U_{C_{1,y}} \Rightarrow x(t;t_0,x_0) \in U^{\mathcal{S}}_{\epsilon_x} \text{ and } y(t;t_0,y_0) \in U_{\epsilon_y}. \quad (42)$$

Let $C_{2,x} \in (0,C_{1,x})$. Since the set $\mathcal{S}$ is practically uniformly attractive for (5) and due to the time scale μ, there exists a $t_{f,x} \in (0,\infty)$ and an $\omega_{0,2} \in (0,\infty)$ such that for all $\omega \in (\omega_{0,2},\infty)$, for all $\mu \in (0,\infty)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0 + (t_{f,x}/\mu),\infty)$

$$x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow \tilde{x}(t;t_0,x_0) \in U^{\mathcal{S}}_{C_{2,x}}. \quad (43)$$

Consider now Lemma 1. Note that all the assumptions of Lemma 1 are satisfied and let γ be the $\mathcal{KL}$-function from Lemma 1. Choose some $C_{2,y} \in (0,C_{1,y})$ and let $t_{f,y} \in (0,\infty)$ be the time such that $\gamma(\delta_y,t_{f,y}) \le C_{2,y}$.

By the same idea as in the case of singular practical uniform stability, we now choose the bounded sets B_x and B_y as well as the constants D and t_f of Lemma 1.

Let $B_x = U^{\mathcal{S}}_{\delta_x}$ and $B_y = U_{\delta_y}$, $D = \min\{C_{1,x} - C_{2,x}, C_{1,y} - C_{2,y}\}$ and $t_f = \max\{t_{f,x}, t_{f,y}\}$. With Lemma 1, we have that there exists an $\omega_{0,3} \in (0,\infty)$ such that for all $\omega \in (\omega_{0,3},\infty)$ there exists a $\mu_{0,2} \in (0,\infty)$ such that for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$, for all $\mu \in (0,\mu_{0,2})$, for all $t_0 \in \mathbb{R}$ we have that for all $t \in [t_0, t_0 + (t_f/\mu)]$

$$\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| < D \text{ and} \quad (44)$$
$$\|y(t;t_0,y_0)\| < \gamma(\|y_0\|, t - t_0) + D. \quad (45)$$

Choosing $\omega_0 = \max\{\omega_{0,1},\omega_{0,2},\omega_{0,3}\}$ and combining (44) with (43) we have that for all $\omega \in (\omega_0,\infty)$ there exists a $\mu_0 \in (0,\min\{1,\mu_{0,1},\mu_{0,2}\})$ such that for all $\mu \in (0,\mu_0)$ and for all $t_0 \in \mathbb{R}$

$$x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow x\left(t_0 + \frac{t_f}{\mu}; t_0, x_0\right) \in U^{\mathcal{S}}_{C_{1,x}}. \quad (46)$$

Combining (46) with (44) and with (42) leads for all $\mu \in (0,\mu_0)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0 + (t_f/\mu),\infty)$ to $x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow x(t;t_0,x_0) \in U^{\mathcal{S}}_{\epsilon_x}$. Since $0 < \mu < \mu_0 < 1$ we know that $t_{f,y} < (t_f/\mu)$. This yields with (45) that $\|y(t_0 + (t_f/\mu);t_0,y_0)\| < \gamma(\|y_0\|,(t_f/\mu)) + D \le C_{2,y} + D \le C_{1y}$. Combining this with (42) we obtain that for all $t \in [t_0 + (t_f/\mu),\infty)$ $y_0 \in U_{\delta_y} \Rightarrow y(t;t_0,y_0) \in U_{\epsilon_y}$, which is the desired result.

Singular Practical Uniform Boundedness: Let $\delta_x, \delta_y \in (0,\infty)$ be given. Since the solutions of (5) are practically uniformly bounded, there exists a $C_{2,x} \in (0,\infty)$ and an $\omega_{0,1} \in (0,\infty)$ such that for all $t \in [t_0,\infty)$

$$x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow \tilde{x}(t;t_0,x_0) \in U^{\mathcal{S}}_{C_{2,x}}. \quad (47)$$

Now choose $C_{1,x} \in (0,\delta_x)$. Since the set $\mathcal{S}$ is practically uniformly attractive for (5) there exist a $t_{f,x} \in (0,\infty)$ and an $\omega_{0,2} \in (0,\infty)$ such that for all $\mu \in (0,\infty)$ and for all $t_0 \in \mathbb{R}$ and all $\omega \in (\omega_{0,3},\infty)$ with $\omega_{0,3} = \max\{\omega_{0,1},\omega_{0,2}\}$ and for all $t \in [t_0 + (t_{f,x}/\mu),\infty)$

$$x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow \tilde{x}(t;t_0,x_0) \in U^{\mathcal{S}}_{C_{1,x}}. \quad (48)$$

Let $\epsilon_x \in (C_{2,x},\infty)$. Consider now Lemma 1. Note that all the assumptions of Lemma 1 are satisfied and let γ be the $\mathcal{KL}$-function from Lemma 1. Let $\epsilon_y \in (0,\infty)$ be such that $\gamma(\delta_y,0) + D \le \epsilon_y$ and $t_{f,y} \in (0,\infty)$ be the time such that $\gamma(\delta_y,t_{f,y}) \le C_{3,y}$. Let furthermore $C_{3,y} \in (0,\delta_y)$.

By the same idea as in the case of singular practical uniform stability, we now choose the bounded sets B_x and B_y as well as the constants D and t_f of Lemma 1. Choose $B_x = U^{\mathcal{S}}_{\delta_x}$, $B_y = U_{\delta_y}$, $D = \min\{\delta_x - C_{1,x}, \epsilon_x - C_{2,x}, \delta_y - C_{3,y}\}$ and $t_f = \max\{t_{fx}, t_{fy}\}$. With Lemma 1, we have that there exists an $\omega_{0,4} \in (0,\infty)$ such that for all $\omega \in (\omega_{0,4},\infty)$ there exists a $\mu_{0,1} \in (0,\infty)$ such that for all $[x_0^\top, y_0^\top]^\top \in B_x \times B_y$, for all $\mu \in (0,\mu_{0,1})$, for all $t_0 \in \mathbb{R}$ we have that for all $t \in [t_0, t_0 + (t_f/\mu)]$

$$\|x(t;t_0,x_0) - \tilde{x}(t;t_0,x_0)\| < D \text{ and} \quad (49)$$
$$\|y(t;t_0,x_0)\| < \gamma(\|y_0\|, t - t_0) + D. \quad (50)$$

Choosing $\omega_0 = \max\{\omega_{0,3},\omega_{0,4}\}$ and combining (49) with (47) and (48), we have that for all $\omega \in (\omega_0,\infty)$ there exists a $\mu_0 \in (0,\min\{1,\mu_{0,1}\})$ such that for all $\mu \in (0,\mu_0)$ and for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0, t_0 + (t_f/\mu)]$ we have that $x_0 \in U^{\mathcal{S}}_{\delta_x} \Rightarrow x(t;t_0,x_0) \in U^{\mathcal{S}}_{\epsilon_x}$ and $x(t_0 + (t_f/\mu);t_0,x_0) \in U^{\mathcal{S}}_{\delta_x}$. Since $0 < \mu < \mu_0 < 1$ we know that $t_{f,y} < (t_f/\mu)$. This yields with (50) that for all $t \in [t_0, t_0 + (t_f/\mu)]$ we obtain $y(t;t_0,y_0) \in U_{\epsilon_y}$ and $y(t_0 + (t_f/\mu);t_0,y_0) \in U_{\delta_y}$. Since $x(t_0;t_0,x_0) \in U^{\mathcal{S}}_{\delta_x}$ and $x(t_0 + (t_f/\mu);t_0,x_0) \in U^{\mathcal{S}}_{\delta_x}$ as well as $y(t_0;t_0,y_0) \in U_{\delta_y}$ and $y(t_0 + (t_f/\mu);t_0,y_0) \in U_{\delta_y}$, a repeated application of the argument above with another solution $\tilde{x}$ of (5) through $x(t_0 + (t_f/\mu);t_0,x_0)$, i.e., $\tilde{x}(t;t_0 + (t_f/\mu), x(t_0 + (t_f/\mu);t_0,x_0))$ as well as another solution of y through $y(t_0 + (t_f/\mu);t_0,x_0)$, i.e., $y(t;t_0 + (t_f/\mu), y(t_0 + (t_f/\mu);t_0,x_0))$ (and with the same choice of D, B_x, B_y and t_f as above) yields for all $\mu \in (0,\mu_0)$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0,\infty)$ $x_0 \in U^{\mathcal{S}}_{\delta_x}$ and $y_0 \in U_{\delta_y} \Rightarrow x(t;t_0,x_0) \in U^{\mathcal{S}}_{\epsilon_x}$ and $y(t;t_0,y_0) \in U_{\epsilon_y}$ and thus we proved singular practical uniform boundedness. This is the last property we had to prove. □

The result of Theorem 1 follows directly from Lemma 2 and the fact that the coordinate change $(x,z) \to (x,y)$ introduced above is a diffeomorphism. Since $\mathcal{S}$ is sSPUAS for (3) the set $\mathcal{S}$ is sSPUAS for (1), which is the claim of Theorem 1.

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