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On zero-input stability inheritance for time-varying systems with decaying-to-zero input power

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ABSTRACT

Stability results for time-varying systems with inputs are relatively scarce, as opposed to the abundant literature available for time-invariant systems. This paper extends to time-varying systems existing results that ensure that if the input converges to zero in some specific sense, then the state trajectory will inherit stability properties from the corresponding zero-input system. This extension is non-trivial, in the sense that the proof technique is completely novel, and allows to recover the existing results under weaker assumptions in a unifying way.

1. Introduction

Stability properties for systems with inputs find natural application in control systems. Input-to-state stability (ISS) [1–3], integral ISS (iISS) [4,5], converging-input converging-state (CICS) [6,7], uniformly bounded-energy input bounded state (UBEBS) [8], bounded-energy-input convergent-state (BEICS) [9,10] and L^p-input converging-state [11] are examples of such properties. Most of the existing analyses and characterizations of these properties apply to time-invariant systems. Analogous results for time-varying systems are very scarce. There exist some characterizations of the ISS property [12–14] and a recent result by the authors characterizing the iISS property [15]. In a more general setting, some asymptotic behaviour results exist for asymptotically autonomous differential equations [16,17], and some also dealing with weak invariance principles [18]. An asymptotically autonomous differential equation is one such that the function f_0 defining its dynamics $\dot{x} = f_0(t, x)$ approaches a time-invariant function g, i.e. $f_0(t, x) \to g(x)$ as $t \to \infty$, in some suitable sense.

A time-invariant system $\dot{x} = \bar{f}(x, u)$, with an input u that converges to zero can be interpreted as an asymptotically autonomous system $[f_0(t, x) := \bar{f}(x, u(t)) \to g(x) := \bar{f}(x, 0)]$ under reasonable assumptions. By contrast, time-varying systems of the form $\dot{x} = f(t, x, u)$ do not in general allow such a possibility. An interesting result in the latter case is provided in [18], where the concept of weakly asymptotically autonomous system is introduced, which, loosely speaking, means that $\dot{x} = f_0(t, x)$ approaches the differential inclusion $\dot{x} \in F(x)$ as $t \to \infty$ in some appropriate sense. The latter can be employed in the time-varying case with $f_0(t, x) := f(t, x, u(t))$.

An iISS system has, inter alia, the property that inputs with bounded energy, where energy is measured according to the iISS gain, produce state trajectories that asymptotically converge to zero. The latter is the BEICS property [9]. The function that weighs the input in order to measure input energy, i.e. the iISS gain in the iISS setting, is extremely important in the sense that a system may be iISS for some iISS gains but not for others. Interesting examples of some perhaps counter-intuitive facts are given in [19] and [20], where globally asymptotically stable systems (exponentially in [20]) are destabilized by additive inputs of arbitrarily small energy (exponentially decaying in [20]). The main point we make is that the ensuing stability or instability depends on how input energy is measured.

This work relates to the CICS and BEICS properties. Roughly speaking, these properties entail that if the system input converges to zero in some specific manner, then the state will also converge to zero. These properties are of importance in stability analysis for cascade systems and also in ensuring stability robustness under certain types of disturbances. We consider time-varying systems with inputs and pinpoint specific input power 'measures' (see Section 2.3) so that solutions corresponding to inputs with decaying-to-zero power may inherit specific properties from the corresponding zero-input system. More precisely, suppose that the zero-input system has a uniformly locally asymptotically stable compactum C within an open set $\mathcal{G}$ contained in the "region of attraction" (see [21] for the latter concept in time-varying systems,

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and Section 2). Let x be a forward complete solution of $\dot{x} = f(t, x, u)$ corresponding to an input u having decaying-to-zero power. Then, one of the results that we prove is that if the ω-limit set of x has nonempty intersection with $\mathcal{G}$, then x approaches C.

In this context, our contribution is the following. First, we provide a convergence result for time-varying systems with inputs under very mild assumptions on the function f defining the system dynamics. Worthy of mention is that we do not require $f(t, x, u)$ to be continuous in t, nor locally Lipschitz in x. As a consequence, solutions are not necessarily unique. Second, we pinpoint input power 'measures' for which such convergence is possible. These 'measures' relate to specific bounds on f. Third, we extend some of the main results in [6,9] and [11] to time-varying systems, under weaker assumptions and in a unifying way. We emphasize that these extensions are novel and nontrivial, since existing results for time-invariant systems, such as those in [9] and [11], cannot be adapted to the current setting (the corresponding proofs rely on converse Lyapunov theorems that do not remain valid).

The remainder of this paper is organized as follows. In Section 2 we introduce the notation, definitions and main assumptions required. Our main result and explanations of how our result subsumes other existing results are contained in Section 3. Section 4 contains some secondary technical results and conclusions are drawn in Section 5.

2. Preliminaries

2.1. Notation and preliminary definitions

The reals, nonnegative reals, naturals and nonnegative integers are denoted $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$ and $\mathbb{N}_0$, respectively. For $\xi \in \mathbb{R}^n$, $|\xi|$ denotes its Euclidean norm. For a given nonempty subset $A \subset \mathbb{R}^n$, $|\xi|_A$ denotes the distance from $\xi \in \mathbb{R}^n$ to A, that is $|\xi|_A = \inf\{|\xi - \zeta| : \zeta \in A\}$. Given $r \geq 0$, $A_r = \{\xi \in \mathbb{R}^n : |\xi|_A \leq r\}$ and $B_r(\xi) = \{\xi\}_r$ for every $\xi \in \mathbb{R}^n$. Thus, if $\xi \in \mathbb{R}^n$ and $r \geq 0$, the statements $\xi \in A_r$ and $|\xi|_A \leq r$ are equivalent. For $p \geq 1$ and $m \in \mathbb{N}$, $L^p_{m,loc}$ (L^p_m) denotes the set of all the Lebesgue measurable functions $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ such that $|v|^p$ is integrable on each finite interval $I \subset \mathbb{R}_{\geq 0}$ ($|v|^p$ is integrable on $\mathbb{R}_{\geq 0}$). When $m = 1$ we just write L^p_{loc} and L^p. For a Lebesgue measurable set $J \subset \mathbb{R}$, $|J|$ will denote its Lebesgue measure. Given a metric space (U, d) and an interval $I \subset \mathbb{R}$, we say that $v : I \to U$ is piecewise constant if there exists a partition $I_1, \ldots, I_m$ of I such that I_i is an interval for every i and v is constant on I_i. The function $u : I \to U$ is Lebesgue measurable if there exists a sequence of piecewise-constant functions $u_k : I \to U$ such that $\lim_{k\to\infty} u_k(t) = u(t)$ for almost all $t \in I$, that is $|\{t \in I : \lim_{k\to\infty} u_k(t) \neq u(t)\}| = 0$. When U is separable, $u : I \to U$ is measurable if and only if $u^{-1}(V)$ is Lebesgue measurable for every open subset V of U (see Remark C.1.1. in [22]). A function $\omega : U \to \mathbb{R}$ is proper if for all $r \in \mathbb{R}$ the sublevel set $\omega^{-1}((-\infty, r])$ is compact. We write $\sigma \in \mathcal{K}$ if $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing, and $\sigma(0) = 0$. We write $\sigma \in \mathcal{K}_\infty$ if $\sigma \in \mathcal{K}$ and σ is unbounded.

2.2. Problem statement

This work deals with time-varying control systems of the general form

$$\dot{x} = f(t, x, u) \tag{1}$$

where $f : \mathbb{R}_{\geq 0} \times \mathcal{X} \times U \to \mathbb{R}^n$ with $\mathcal{X}$ an open subset of $\mathbb{R}^n$ and (U, d) a metric space. An input is a Lebesgue measurable function $u : \mathbb{R}_{\geq 0} \to U$ and $\mathcal{U}$ is the set of all the inputs. We suppose that U is nonempty and there exists $0 \in U$, where "0" is nothing but some element in U that we distinguish from the rest. For an arbitrary $\mu \in U$, we define $|\mu| := d(\mu, 0)$, i.e. $|\mu|$ is the distance between μ and 0. In the case in which $U \subset \mathbb{R}^m$, 0 denotes the origin of $\mathbb{R}^m$ and d will be the metric induced by Euclidean norm. The *zero input* is the map $\mathbf{0} \in \mathcal{U}$ such that $\mathbf{0}(t) \equiv 0$. With system (1) we associate the zero-input system

$$\dot{x} = f(t, x, \mathbf{0}) =: f_0(t, x). \tag{2}$$

Assumption 1. The function $f : \mathbb{R}_{\geq 0} \times \mathcal{X} \times U \to \mathbb{R}^n$ satisfies the following conditions.

(A1) (Carathéodory) $f(\cdot, \xi, \mu)$ is Lebesgue measurable for all $(\xi, \mu) \in \mathcal{X} \times U$ and $f(t, \cdot, \cdot)$ is continuous for every $t \geq 0$.

(A2) (Zero-input Lipschitzianity) $f_0(t, \xi)$ is locally Lipschitz in ξ uniformly in t in the following sense: for every compact subset $K \subset \mathcal{X}$ there exists a nonnegative function $L_K \in L^1_{\text{loc}}$ such that $\sup_{t\geq 0} \int_t^{t+T} L_K(s)\, ds < \infty$ for all $T > 0$ and

$$|f_0(t, \xi) - f_0(t, \xi')| \leq L_K(t)|\xi - \xi'|\ \forall t \geq 0,\ \forall \xi, \xi' \in K.$$

In view of Assumption 1, for each $t_0 \geq 0$ and $\xi \in \mathcal{X}$ there is a unique maximally defined (forward) solution $x(t) = \varphi(t, t_0, \xi)$ of (2) which verifies $x(t_0) = \xi$. We will denote by $[t_0, t_{t_0,\xi})$ its maximal interval of definition. It is well-known that in the case in which $\varphi(t, t_0, \xi)$ belongs to a fixed compact subset of $\mathcal{X}$ for all $t \in [t_0, t_{t_0,\xi})$, then $t_{t_0,\xi} = \infty$.

Let $C \subset \mathcal{G} \subset \mathcal{X}$ be such that C is nonempty and compact and $\mathcal{G}$ is open. In what follows we assume that C is uniformly asymptotically stable with respect to (2) and that $\mathcal{G}$ is contained in the region of attraction of C. These statements are made precise in the following assumption.

Assumption 2 (*Zero-input stability*). There exist a nonempty compact set C and an open set $\mathcal{G}$ such that $C \subset \mathcal{G} \subset \mathcal{X}$ and

(B1) (uniform Lyapunov stability) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_0 \geq 0$ and $\xi \in C_\delta$, $\varphi(t, t_0, \xi) \in C_\varepsilon$ for all $t \geq t_0$;

(B2) (uniform boundedness of solutions) for every compact set $K \subset \mathcal{G}$ there exists a compact set $\Gamma \subset \mathcal{X}$ such that for all $t_0 \geq 0$ and $\xi \in K$ we have that $\varphi(t, t_0, \xi) \in \Gamma$ for all $t \geq t_0$;

(B3) (uniform attractiveness) for every compact set $K \subset \mathcal{G}$ and every $\varepsilon > 0$ there exists $T = T(K, \varepsilon) \geq 0$ such that for all $t_0 \geq 0$ and $\xi \in K$ we have that $\varphi(t, t_0, \xi) \in C_\varepsilon$ for all $t \geq t_0 + T$.

Note that under the uniform Lyapunov stability in (B1) above, it follows that C is forward invariant under (2), i.e. for all $t_0 \geq 0$ and $\xi \in C$, $\varphi(t, t_0, \xi) \in C$ for all $t \geq t_0$. When $\mathcal{G} = \mathcal{X} = \mathbb{R}^n$, then C is globally uniformly asymptotically stable with respect to (2).

Remark 1. When the zero-input system (2) is time-invariant, i.e. $f_0(t, \xi) \equiv f_0^*(\xi)$, Assumption 2 is satisfied with any compact set $C \subset \mathcal{X}$ which is asymptotically stable with respect to (2) [that is (i) C is stable: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\xi \in C_\delta$, $\varphi(t, 0, \xi) \in C_\varepsilon$ for all $t \geq 0$ and (ii) C is attractive: there exists $\delta_0 > 0$ such that $|\varphi(t, 0, \xi)|_C \to 0$ for all $\xi \in C_{\delta_0}$] and with $\mathcal{G} = \mathcal{A}$, where $\mathcal{A} = \{\xi \in \mathcal{X} : |\varphi(t, 0, \xi)|_C \to 0\}$ is the region of attraction of C. □

The problem we address in this paper is the following:

Give conditions under which the property of convergence to C that applies to solutions of the zero-input system (2) is inherited by (i.e. also applies to) solutions of (1).

Remark 2. Some solutions to this problem are given for time-invariant systems in [6,11] and [9]. The results in this paper extend these in different directions, as will be explained in more detail in Section 3. □

2.3. Admissible inputs and further assumptions

One of the conditions that we will give towards solving the considered problem is that the input u should converge to 0 in some appropriate sense. To make this notion more precise, we require the following assumption.

Assumption 3. The function f in (1) satisfies the following two conditions:

(C1) There exists a continuous function $\gamma : U \to \mathbb{R}_{\geq 0}$ so that: (i) $\gamma(0) = 0$ and $\gamma_r := \inf_{|\mu|\geq r} \gamma(\mu) > 0$ for all $r > 0$, and (ii) for every compact set $K \subset \mathcal{X}$ there exists $M = M(K) \geq 0$ such that $|f(t, \xi, \mu)| \leq M(1 + \gamma(\mu))$ for all $t \geq 0$, all $\xi \in K$ and all $\mu \in U$.

(C2) For every compact set $K \subset \mathcal{X}$ and $\varepsilon > 0$ there exists $\delta = \delta(K, \varepsilon) > 0$ such that for all $t \geq 0$, $|f(t, \xi, \mu) - f(t, \xi, 0)| < \varepsilon$ if $\xi \in K$ and $|\mu| \leq \delta$.

Condition (C1) gives a specific bound on the growth of $|f(t, \xi, \mu)|$ which is uniform over all $t \geq 0$ and over ξ in compact sets. Condition (C2) requires that $f(t, \xi, \cdot)$ be continuous at $(t, \xi, 0)$, uniformly over $t \geq 0$ and over ξ in compact sets.

For some results we will consider the following somewhat weaker conditions on f, which are equivalent to those of Assumption 3 when U is locally compact and separable (see Lemma 2.1).

Assumption 4. The function f in (1) satisfies (C2) and:

(D1) f is bounded on $\mathbb{R}_{\geq 0} \times K \times B$ for every pair of compact sets $K \subset \mathcal{X}$ and $B \subset U$.

The proof of the following lemma is given in Section 4.

Lemma 2.1. *Suppose that f satisfies condition (D1) and that U is a separable and locally compact metric space. Then, f satisfies condition (C1).*

Remark 3. Several characterizations of stability properties for time-invariant systems with inputs, of the form $\dot{x} = \bar{f}(x, u)$, are made possible by employing (i) knowledge of the stability properties of the zero-input system, and (ii) some local Lipschitz continuity assumption on $\bar{f}$ (see e.g. [2,5]). For example, the proof of the characterization of the integral input-to-state stability (iISS) property in Theorem 1 of [5] employs the fact that the 0-input system $\dot{x} = \bar{f}(x, 0)$ is globally asymptotically stable (Proposition II.5 and Lemma IV.10 of [5]) and the fact that $\bar{f}$ is locally Lipschitz continuous. In this work, we do not require any additional Lipschitz continuity assumption other than that in Assumption 1. □

Associated with the function γ in (C1), we define the set of *admissible* inputs $\mathcal{U}_\gamma := \{u \in \mathcal{U} : \gamma \circ u \in L^1_{loc}\}$. From (A1) and (C1), well-known results of the theory of ordinary differential equations (e.g. Theorem I.5.1 in [23]) ensure that for every $t_0 \geq 0$, $\xi \in \mathcal{X}$ and $u \in \mathcal{U}_\gamma$ there exists at least one maximally defined (forward) solution $x : [t_0, t_x) \to \mathcal{X}$ to (1) which verifies $x(t_0) = \xi$ and that $t_x = \infty$ if $x(t)$ belongs to some fixed compact subset of $\mathcal{X}$ for all $t \in [t_0, t_x)$. We emphasize that Assumptions 1 and 3, and the fact $u \in \mathcal{U}_\gamma$ are not sufficient to ensure the uniqueness of the corresponding solutions of (1), and that uniqueness of solutions is not required along this paper.

For a given $T > 0$, consider the positive semidefinite functional $\|\cdot\|_T : \mathcal{U}_\gamma \to [0, \infty]$, defined by

$$\|u\|_T := \sup_{t\geq 0} \int_t^{t+T} \gamma(u(s))\, ds. \tag{3}$$

Given $u \in \mathcal{U}_\gamma$ and $T > 0$, the quantity $\|u\|_T$ can be interpreted as a measure of the maximum energy that u contains in any interval of length T. Hence, $\|u\|_T/T$ is a measure of maximum average power. For the sake of simplicity, and to distinguish $\|u\|_T$ from other measures of input energy, we will refer to $\|u\|_T$ as the *power* of the admissible input u.

Proposition 2.2. *Let $T_1, T_2 > 0$. Then, there exists $k = k(T_1, T_2)$ such that $\|u\|_{T_2} \leq k\|u\|_{T_1}$ for all $u \in \mathcal{U}_\gamma$.*

Proof. Let k denote the least integer not less than $\frac{T_2}{T_1}$. Then, $k \geq 1$ and $T_2 \leq kT_1$. By direct application of (3) and simple properties of integrals and suprema, it can be shown that $\|u\|_{T_2} \leq k\|u\|_{T_1}$. ■

Definition 2.3. We say that $u \in \mathcal{U}_\gamma$ converges to 0, and write $u \to 0$, if $\lim_{\tau\to\infty} \|u(\cdot + \tau)\|_T = 0$. We define $\mathcal{U}^0_\gamma := \{u \in \mathcal{U}_\gamma : u \to 0\}$ as the set of all the admissible inputs that converge to 0.

By Proposition 2.2, it follows that if $\lim_{\tau\to\infty} \|u(\cdot + \tau)\|_T = 0$ for some $T > 0$ then $\lim_{\tau\to\infty} \|u(\cdot + \tau)\|_T = 0$ for all $T > 0$. Therefore, neither the set $\mathcal{U}^0_\gamma$ nor the convergence of u to 0 depend on the number T in their definition.

3. Zero-input stability inheritance

In Section 3.1, we state and prove our main result, namely Theorem 3.2. Some particular cases are addressed in Sections 3.2 and 3.3. Specifically, Section 3.2 contains results for finite-energy inputs and Section 3.3 for essentially bounded ones. Along this section, we show that Theorem 3.2 subsumes and extends many existing results under weaker assumptions.

3.1. Main result

We require the following definition.

Definition 3.1. A maximally defined forward solution $x : [t_0, t_x) \to \mathcal{X}$ of (1) is said to be forward complete if $t_x = \infty$. A forward complete solution $x : [t_0, \infty) \to \mathcal{X}$ of (1) converges to C, denoted by $x \to C$, if $\lim_{t\to\infty} |x(t)|_C = 0$.

We recall that the ω-limit set $\Omega(x)$ of a forward complete solution $x : [t_0, \infty) \to \mathcal{X}$ of (1) is the set of all the points $\xi \in \bar{\mathcal{X}}$ ($\bar{\mathcal{X}}$ being the closure of $\mathcal{X}$) for which there exists a sequence $\{t_k\} \subset \mathbb{R}_{\geq 0}$ such that $t_k \nearrow \infty$ and $x(t_k) \to \xi$.

The following is the main result of the paper.

Theorem 3.2. *Let Assumptions 1–3 hold. Let x be a forward complete solution of (1) corresponding to some input $u \in \mathcal{U}^0_\gamma$ [where γ is as in (C1) in Assumption 3]. Then, the following are equivalent:*

(i) $x \to C$,
(ii) $\Omega(x) \cap \mathcal{G} \neq \emptyset$,
(iii) $\emptyset \neq \Omega(x) \subset C$.

Remark 4. Item (iii) in Theorem 3.2 could be replaced by the statement "x approaches a connected component of C". The equivalence is explained as follows. If x approaches a connected component of C, then clearly (i) holds and by Theorem 3.2, also (iii) holds. Conversely, if (iii) above holds, then properties of ω-limit sets (see, e.g. Proposition 2.1 in [11]) allow the following reasoning. From (iii) above and $\Omega(x)$ being closed, it is hence compact. Since x is continuous and $\Omega(x)$ is nonempty and compact, then x is bounded and, *a posteriori*, $\Omega(x)$ is connected and contained in a connected component of C. The fact that x approaches $\Omega(x)$ implies then that x approaches a connected component of C. □

The proof of Theorem 3.2 requires Lemmas 3.3 and 3.4. Lemma 3.3 bounds the difference between solutions of (1) and (2) having the same initial condition. The bound given by this lemma is useful only for small values of $t-t_0$ because it already assumes that solutions lie in the compact set Γ in the time interval of interest. The proof of Lemma 3.3 is similar to part of the proof of Lemma 3 in [15] but is included here for the reader's convenience.

Lemma 3.3. *Let Assumptions 1 and 3 hold. Let $\Gamma \subset \mathcal{X}$ be a compact set and let $L_\Gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be as in (A2) of Assumption 1 with $K = \Gamma$. Then, for every $\eta > 0$ there exists a positive constant $\kappa = \kappa(\Gamma, \eta)$ such that the following holds: if x is a solution of (1) corresponding to $u \in \mathcal{U}_\gamma$, z is a solution of (2) such that $x(t)$ and $z(t)$ belong to Γ for all $t \in [t_0, t_0+T]$, and $x(t_0) = z(t_0)$ then for all $t \in [t_0, t_0+T]$,*

$$|x(t) - z(t)| \leq \left[\eta(t-t_0) + \kappa \int_{t_0}^{t} \gamma(u(\tau))\, d\tau\right] e^{\int_{t_0}^{t} L_\Gamma(s)\, ds}.$$

Proof. Let $\Gamma \subset \mathcal{X}$ be compact and $\eta > 0$.

Claim: there exists $\kappa = \kappa(\Gamma, \eta) > 0$ such that for all $t \geq 0$, $\xi \in \Gamma$ and $\mu \in U$,

$$|f(t,\xi,\mu) - f(t,\xi,0)| \leq \eta + \kappa\gamma(\mu). \tag{4}$$

From (C2) there exists $0 < \delta = \delta(\Gamma, \eta) < 1$ such that for all $t \geq 0$, all $\xi \in \Gamma$ and all $\mu \in U$ such that $|\mu| < \delta$, it happens that $|f(t,\xi,\mu) - f(t,\xi,0)| < \eta$. If $\xi \in \Gamma$ and $|\mu| \geq \delta$, using (C1) it follows that $|f(t,\xi,\mu) - f(t,\xi,0)| \leq |f(t,\xi,\mu)| + |f(t,\xi,0)| \leq 2M(1+\gamma(\mu))$ and hence $|f(t,\xi,\mu) - f(t,\xi,0)|/\gamma(\mu) \leq 2M[1/\gamma_\delta + 1] =: \kappa$. In consequence

$$|f(t,\xi,\mu) - f(t,\xi,0)| \leq \kappa\gamma(\mu) \quad \forall \xi \in \Gamma, |\mu| \geq \delta.$$

Combining the inequalities obtained, the claim is established.

Let x be a solution of (1) corresponding to some $u \in \mathcal{U}_\gamma$ and let z be a solution of (2) such that $x(t)$ and $z(t)$ lie in Γ for all $t \in [t_0, t_0+T]$ for some $t_0 \geq 0$ and $T > 0$. Fix $t \in [t_0, t_0+T]$. Then, for all $t_0 \leq \tau \leq t$, we have

$$\begin{aligned}
|x(\tau) - z(\tau)| &\leq \int_{t_0}^{\tau} |f(s, x(s), u(s)) - f(s, z(s), 0)| ds \\
&\leq \int_{t_0}^{\tau} |f(s, x(s), u(s)) - f(s, x(s), 0)| ds \\
&\quad + \int_{t_0}^{\tau} |f(s, x(s), 0) - f(s, z(s), 0)| ds \\
&\leq \int_{t_0}^{\tau} [\eta + \kappa\gamma(u(s))] ds + \int_{t_0}^{\tau} L_\Gamma(s)\, |x(s) - z(s)| ds \\
&\leq \eta(t-t_0) + \kappa \int_{t_0}^{t} \gamma(u(s)) ds + \int_{t_0}^{\tau} L_\Gamma(s)\, |x(s) - z(s)| ds.
\end{aligned}$$

Using Gronwall's inequality, it follows that

$$|x(t) - z(t)| \leq \left[\eta(t-t_0) + \kappa \int_{t_0}^{t} \gamma(u(s)) ds\right] e^{\int_{t_0}^{t} L_\Gamma(s)\, ds}$$

for all $t \in [t_0, t_0+T]$. ■

Lemma 3.4 shows that solutions of (1) that begin sufficiently close to the compact set C and correspond to inputs with sufficiently small power are forward complete, and that C is uniformly Lyapunov stable under the dynamics of (1).

Lemma 3.4. *Let Assumptions 1–3 hold. For every $\varepsilon > 0$ and every $T > 0$ there exists $\delta = \delta(\varepsilon, T) > 0$ so that the following holds: if x is a solution of (1) corresponding to an input $u \in \mathcal{U}_\gamma$ such that $\|u\|_T \leq \delta$ and $|x(t_0)|_C \leq \delta$ for some $t_0 \geq 0$, then x is forward complete and $|x(t)|_C \leq \varepsilon$ for all $t \geq t_0$.*

Proof. Let $\varepsilon > 0$ and $T > 0$. We can assume, without loss of generality, that ε is small enough so that $\Gamma := C_\varepsilon \subset \mathcal{G}$. From (B1), there exists $\delta^* \in (0, \varepsilon/2)$ such that for all $t_0 \geq 0$ and all $\zeta \in C_{\delta^*}$, the unique maximal solution of the zero-input system satisfies $\varphi(t, t_0, \zeta) \in C_{\varepsilon/2}$ for all $t \geq t_0$. From (B3), there exists $T^* = T^*(C_{\delta^*}, \delta^*/2) > 0$ such that for all $t_0 \geq 0$ and all $\zeta \in C_{\delta^*}$, $\varphi(t, t_0, \zeta) \in C_{\delta^*/2}$ for all $t \geq t_0 + T^*$.

Let $L_\Gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be as in (A2) with $K = \Gamma$ and let $L^* = \sup_{t\geq 0} \int_t^{t+T^*} L_\Gamma(s)\, ds$. Consider $\eta = \frac{\delta^*}{4T^* e^{L^*}}$. Let $\kappa = \kappa(\Gamma, \eta)$ be given by Lemma 3.3. Pick $c > 0$ such that $\|u\|_{T^*} \leq c\|u\|_T$ holds for every $u \in \mathcal{U}_\gamma$. Let x be a solution of (1) corresponding to some $u \in \mathcal{U}_\gamma$ such that $\kappa c\|u\|_T e^{L^*} < \delta^*/4$ and such that $|x(t_0)|_C \leq \delta^*$ for some $t_0 \geq 0$. Define $\tau^* = \sup\{\tau \geq t_0 : x(s) \in \Gamma\ \forall s \in [t_0, \tau]\}$. Since $x(t_0) \in C_{\delta^*} \subset C_\varepsilon = \Gamma$, $\delta^* < \varepsilon$ and x is continuous, it follows that $x(s) \in \Gamma$ for all s in some interval $[t_0, t_0']$ with $t_0' > t_0$. So $\tau^* > t_0$. We claim that $\tau^* > t_0 + T^*$. Suppose on the contrary that $\tau^* \leq t_0 + T^*$. From the definition of τ^*, the continuity of x and the definition and compactness of Γ it follows that $x(t) \in \Gamma$ for all $t \in [t_0, \tau^*]$ and that $|x(\tau^*)|_C = \varepsilon$. Since $z(t) := \varphi(t, t_0, x(t_0)) \in C_{\varepsilon/2} \subset \Gamma$ for all $t \geq t_0$, by applying Lemma 3.3 it follows that for all $t \in [t_0, \tau^*]$

$$\begin{aligned}
|x(t) - z(t)| &\leq \left[\eta(t-t_0) + \kappa \int_{t_0}^{t} \gamma(u(\tau))\, d\tau\right] e^{\int_{t_0}^{t} L_\Gamma(s)\, ds} \\
&\leq \left[\eta T^* + \kappa\|u\|_{T^*}\right] e^{L^*} \leq \left[\eta T^* + \kappa c\|u\|_T\right] e^{L^*} < \frac{\delta^*}{2}.
\end{aligned} \tag{5}$$

The latter and the fact that $|z(t)|_C \leq \varepsilon/2$ for all $t \geq t_0$ yield, for all $t \in [t_0, \tau^*]$,

$$|x(t)|_C \leq |z(t)|_C + |x(t) - z(t)| \leq \frac{\varepsilon}{2} + \frac{\delta^*}{2} < \varepsilon.$$

In particular, we have that $|x(\tau^*)|_C < \varepsilon$, which contradicts the fact that $|x(\tau^*)|_C = \varepsilon$. Thus $\tau^* > t_0 + T^*$ as claimed. Besides, since $|z(t_0+T^*)|_C \leq \delta^*/2$ and (5) holds for $t = t_0 + T^*$, we also have that $|x(t_0+T^*)|_C \leq \delta^*$. Repeating the same reasoning in a recursive manner we obtain that for all $j \in \mathbb{N}_0$, $|x(t)|_C < \varepsilon$ for all $t \in [t_0 + jT^*, t_0 + (j+1)T^*]$ and $|x(t_0 + (j+1)T^*)|_C \leq \delta^*$. Taking $\delta = \min\{\delta^*, \frac{\delta^*}{4\kappa c e^{L^*}}\}$ thus establishes the first assertion. Since x is contained in the compact set $C_\varepsilon \subset \mathcal{X}$ for as long as it is defined, then x is forward complete. ■

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. (i) ⇒ (ii) Since $x \to C$ and C is compact, then $\emptyset \neq \Omega(x) \cap C \subset \Omega(x) \cap \mathcal{G}$.

(ii) ⇒ (iii) Firstly we will prove that $\Omega(x) \cap C_\delta \neq \emptyset$ for every $\delta > 0$. Let $\delta > 0$. We can assume without loss of generality that $C_\delta \subset \mathcal{G}$. Pick any $\xi \in \Omega(x) \cap \mathcal{G}$ and any $r > 0$ such that $B_r(\xi) \subset \mathcal{G}$. Due to (B2) there exists a compact set $\Gamma \subset \mathcal{X}$ so that for all $t_0 \geq 0$ and all $\zeta \in B_r(\xi)$, $\varphi(t, t_0, \zeta) \in \Gamma$ for all $t \geq t_0$. From (B3), there exists $T > 0$ so that for every $t_0 \geq 0$ and every $\zeta \in B_r(\xi)$, $\varphi(t, t_0, \zeta) \in C_{\delta/2}$ for all $t \geq t_0 + T$. Pick any $0 < \delta_1 < \delta/2$ such that $\Gamma_{\delta_1} \subset \mathcal{X}$ is compact. Let $L_{\Gamma_{\delta_1}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be as in (A2) with $K = \Gamma_{\delta_1}$ and let $L = \sup_{t\geq 0} \int_t^{t+T} L_{\Gamma_{\delta_1}}(s)\, ds$. Define $\eta = \frac{\delta_1}{4Te^L}$. Let $\kappa = \kappa(\Gamma_{\delta_1}, \eta)$ be given by Lemma 3.3.

Since $\xi \in \Omega(x) \cap \mathcal{G}$, there exists a sequence $\{t_k\}$ in $\mathbb{R}_{\geq 0}$ so that $x(t_k) \to \xi$. Then $\xi_k = x(t_k) \in B_r(\xi)$ for k large enough, say $k \geq k_0$. Since $\rho_k := \|u(\cdot + t_k)\|_T \to 0$, $\kappa\rho_k e^L < \delta_1/4$ for k large enough, say $k \geq k_1$. Let $k \geq \max\{k_0, k_1\}$ and $\tau_k = \sup\{\tau \geq t_k : x(s) \in \Gamma_{\delta_1}\ \forall s \in [t_k, \tau]\}$. Since $x(t_k) \in \Gamma$ and x is continuous, it follows that $x(s) \in \Gamma_{\delta_1}$ for all s in some interval $[t_k, t_k']$, with $t_k' > t_k$. So $\tau_k > t_k$. We claim that $\tau_k > t_k + T$. Suppose on the contrary that $\tau_k \leq t_k + T$. From the definition of τ_k, the continuity of x and the compactness of Γ_{δ_1} it follows that $x(t) \in \Gamma_{\delta_1}$ for all $t \in [t_k, \tau_k]$ and that $|x(\tau_k)|_\Gamma = \delta_1$. Since $z_k(t) := \varphi(t, t_k, \xi_k) \in \Gamma \subset \Gamma_{\delta_1}$ for all $t \geq t_k$, by applying Lemma 3.3 it follows that for all $t \in [t_k, \tau_k]$

$$|x(t) - z_k(t)| \leq \left[\eta(t - t_k) + \kappa \int_{t_k}^{t} \gamma(u(\tau))\, d\tau\right] e^{\int_{t_k}^{t} L_{\Gamma_{\delta_1}}(s)\, ds}$$
$$\leq [\eta T + \kappa \rho_k]\, e^L < \frac{\delta_1}{4} + \frac{\delta_1}{4} = \frac{\delta_1}{2}. \quad (6)$$

In consequence, the latter and the fact that $|z_k(t)|_\Gamma = 0$ for all $t \geq t_k$ yield

$$|x(\tau_k)|_\Gamma \leq |z_k(\tau_k)|_\Gamma + |x(\tau_k) - z_k(\tau_k)| < \frac{\delta_1}{2},$$

which contradicts $|x(\tau_k)|_\Gamma = \delta_1$. Thus $\tau_k > t_k + T$.

From the facts that (6) holds for $t'_k = t_k + T$ and $z_k(t_k + T) \in C_{\delta/2}$ it follows that

$$|x(t'_k)|_C \leq |z_k(t'_k)|_C + |x(t'_k) - z_k(t'_k)| < \frac{\delta}{2} + \frac{\delta_1}{2} \leq \delta.$$

We then have that for all $k \geq \max\{k_0, k_1\}$, $x(t'_k) \in C_\delta$. From the compactness of C_δ and the fact that $t'_k \nearrow \infty$, the existence of an ω-limit point of x lying in C_δ follows.

Next we will prove that $\Omega(x) \subset C$. Let $\varepsilon > 0$ and pick any $T' > 0$. Let $\delta = \delta(\varepsilon, T') > 0$ as in Lemma 3.4. Let $\zeta \in \Omega(x) \cap C_{\delta/2}$ and let $\{t_k\}$ be a sequence such that $\zeta_k = x(t_k) \to \zeta$. Then $\zeta_k \in C_\delta$ for k large enough, say $k \geq k_2$. Since $u \to 0$, then there exists k_3 such that $\|u(\cdot + t_k)\|_{T'} \leq \delta$ for all $k \geq k_3$. Let $k_4 = \max\{k_2, k_3\}$ and let $v : \mathbb{R}_{\geq 0} \to U$ be the input defined by $v(t) = 0$ if $t \in [0, t_{k_4})$ and $v(t) = u(t)$ if $t \in [t_{k_4}, \infty)$. Then the restriction of x to $[t_{k_4}, \infty)$ is a solution of (1) corresponding to the input v, $|x(t_{k_4})|_C \leq \delta$ and $\|v\|_{T'} \leq \delta$. Then, by applying Lemma 3.4 it follows that $|x(t)|_C \leq \varepsilon$ for all $t \geq t_{k_4}$. Therefore $\Omega(x) \subset C_\varepsilon$ for all $\varepsilon > 0$ and $\Omega(x) \subset C$ follows.

(iii) $\Rightarrow$ (i). Since x is continuous and $\Omega(x)$ is nonempty and compact, well-known properties of ω-limits (see, e.g. Proposition 2.1 in [11]) imply that $\Omega(x)$ is approached by x and therefore $x \to C$. ■

We next consider the case of globally defined systems for which C is globally uniformly asymptotically stable for the zero-input system (2). The following corollary extends Corollary 4.4 of [11] to time-varying systems and under weaker assumptions (see Section 3.2).

Corollary 3.5. *Let Assumptions 1–3 hold with $\mathcal{G} = \mathcal{X} = \mathbb{R}^n$. If x is a forward complete solution of (1) corresponding to some $u \in \mathcal{U}_\gamma^0$, then as $t \to \infty$ either $|x(t)|_C \to 0$ or $|x(t)|_C \to \infty$.*

Proof. Suppose that $\liminf_{t\to\infty} |x(t)|_C < \infty$. Then $\emptyset \neq \Omega(x) = \Omega(x) \cap \mathcal{G}$. By Theorem 3.2, then $|x(t)|_C \to 0$. If $\liminf_{t\to\infty} |x(t)|_C = \infty$, then $\lim_{t\to\infty} |x(t)|_C = \infty$. ■

3.2. Bounded-energy inputs

In this subsection, we consider the case in which the inputs u have finite energy, that is $\int_0^\infty \gamma(u(s))\, ds < \infty$ for some function γ as in condition (C1).

Note that any input $u \in \mathcal{U}$ such that $\int_0^\infty \gamma(u(s))\, ds < \infty$ belongs to $\mathcal{U}_\gamma^0$, and that therefore Theorem 3.2 and Corollary 3.5 remain valid if we replace the condition $u \in \mathcal{U}_\gamma^0$ by the stronger one $\int_0^\infty \gamma(u(s))\, ds < \infty$. A consequence of this simple observation is that the L^p-input converging-state results for time-invariant systems in [11], namely Theorem 4.2 and Corollary 4.4, can be easily deduced from Theorem 3.2 and Corollary 3.5, respectively. Those results straightforwardly follow by observing that: (a) the continuity of f and the zero-input local Lipschitz condition assumed in [11] imply that f satisfies Assumption 1 with $U = \mathbb{R}^m$ and condition (C2) of Assumption 3; (b) from the zero-input asymptotic stability of the compact set C assumed in [11] it follows that Assumption 2 holds with C and $\mathcal{G} = \mathcal{A}$, where $\mathcal{A}$ is domain of attraction of C (see Remark 1); (c) the continuity of f, the growth condition assumed in [11, eq. (1)] and the fact that $|\mu| \leq 1 + |\mu|^p$ for all $\mu \in \mathbb{R}^n$ and all $p \geq 1$ imply that condition (C1) holds with $\gamma(\mu) = |\mu|^p$, for every $p \geq 1$; and (d) the inputs u considered in [11] belong to L_m^p for some $p \geq 1$.

We also remark that the growth condition assumed in [11, eq. (1)] in conjunction with the continuity of f constitute conditions that are more restrictive than (C1) in Assumption 3. Also, such growth condition is a kind of Lipschitz continuity requirement on f that we do not assume (recall Remark 3).

The main proof technique in [11] requires a converse Lyapunov argument that is not valid for time-varying systems. Consequently, the proof in the current paper is completely novel even for this particular case.

Theorem 3.6 provides a result in the case $\mathcal{G} = \mathcal{X} = \mathbb{R}^n$, related to the BEICS property. Let γ be as in condition (C1). We say that (1) has the γ-BEICS property with respect to a compact subset $C \subset \mathbb{R}^n$ if every solution x of (1) corresponding to an input $u \in \mathcal{U}$ such that $\int_0^\infty \gamma(u(s))\, ds < \infty$ satisfies $x \to C$.

Theorem 3.6. *Let Assumptions 1–3 hold with $\mathcal{G} = \mathcal{X} = \mathbb{R}^n$. Suppose there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ such that*

1. *there exists $\phi \in \mathcal{K}_\infty$ so that for all $t \geq 0$ and $\xi \in \mathbb{R}^n$*
$$\phi(|\xi|_C) \leq V(t, \xi); \quad (7)$$
2. *there exists $R \geq 0$ such that for all $t \geq 0$, $\mu \in U$ and $\xi \in \mathbb{R}^n$ the following implication holds*
$$V(t, \xi) \geq R \;\Rightarrow\; \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \xi} f(t, \xi, \mu) \leq \gamma(\mu), \quad (8)$$
with γ as in (C1) in Assumption 3.

Then (1) has the γ-BEICS property with respect to C.

Proof. Let $u \in \mathcal{U}$ be such that $\int_0^\infty \gamma(u(s))\, ds < \infty$. Then, $u \in \mathcal{U}_\gamma^0$. Let x be a solution of (1) corresponding to u, maximally defined on $[t_0, t_x)$. The existence of a function V satisfying (8) implies that

$$V(t, x(t)) \leq R + V(t_0, x(t_0)) + \int_{t_0}^\infty \gamma(u(s))\, ds \quad \forall t \in [t_0, t_x).$$

Then (7) and the compactness of C imply that x is bounded on $[t_0, t_x)$ and that $t_x = \infty$. Application of Corollary 3.5 shows that $|x|_C \to 0$. ■

The BEICS part of the main result in [9] (Theorem 3.1) is a particular case of Theorem 3.6, corresponding to $C = \{0\}$ and $U = \mathbb{R}^m$. This can be seen as follows. Theorem 3.1 of [9] assumes that f in (1) is time-invariant and locally Lipschitz, and that (1) is zero-input globally asymptotically stable and dissipative with supply function $\sigma \in \mathcal{K}$. As a consequence, Assumptions 1 and 2, and (C2) of Assumption 3 are clearly satisfied. Theorem 3.1 of [9] also requires a condition named (A), which implies that (C1) of Assumption 3 is satisfied with $\gamma(\cdot) = \sigma(|\cdot|)$. Finally, the dissipativity assumption with supply function σ, implies (but it is not equivalent to) the existence of the function V as required in Theorem 3.6. Therefore, application of Theorem 3.6 recovers the σ-BEICS result of Theorem 3.1 of [9] under weaker hypotheses. The iISS part of Theorem 3.1 of [9] has already been extended in [15]. The results in the current paper do not require those of [15] (with the aforementioned exception of part of the proof of Lemma 3.3) and are proved in a different manner.

3.3. Essentially bounded inputs

By considering the case of essentially bounded inputs, we are able to relax the assumptions of Theorem 3.2 even further. We recall that an input $u \in \mathcal{U}$ is locally essentially bounded if for each $T > 0$ there exists a compact set $B_T \subset U$ such that $|\{t \in [0,T] : u(t) \notin B_T\}| = 0$. Also, $u \in \mathcal{U}$ is essentially bounded if it is locally essentially bounded and the set B_T can be selected independently of $T > 0$. We also consider the following type of 'meagreness' condition on $u \in \mathcal{U}$ (cf. [18,24]):

(M) for every $T > 0$ and $\lambda > 0$,

$$\lim_{t\to\infty} |\{s \in [t, t+T] : \ |u(s)| \geq \lambda\}| = 0.$$

Lemma 3.7 characterizes essentially bounded inputs which belong to $\mathcal{U}_\gamma^0$ in terms of Property (M).

Lemma 3.7. *Let $\gamma : U \to \mathbb{R}_{\geq 0}$ be continuous and such that $\gamma(0) = 0$ and $\inf_{|\mu|\geq r} \gamma(\mu) > 0$ for all $r > 0$. Let $u \in \mathcal{U}$ be essentially bounded. Then, the following are equivalent:*

(a) $u \in \mathcal{U}_\gamma^0$.
(b) *u satisfies condition (M).*

Proof. (a) $\Rightarrow$ (b). Suppose that u does not satisfy condition (M). Then there exist $T > 0$, $\lambda > 0$, $\varepsilon_0 > 0$ and a sequence $t_k \nearrow \infty$ such that $|\{s \in [t_k, t_k+T] : \ |u(s)| \geq \lambda\}| \geq \varepsilon_0$ for all k. Let $\gamma_\lambda = \inf_{|\mu|\geq\lambda} \gamma(\mu) > 0$. Then

$$\int_{t_k}^{t_k+T} \gamma(u(s))\, ds \geq \gamma_\lambda\, \varepsilon_0 \quad \forall k$$

which contradicts the fact that $u \in \mathcal{U}_\gamma^0$.

(b) $\Rightarrow$ (a). Since u is essentially bounded and γ is continuous it follows that $\gamma \circ u$ is essentially bounded, hence $\gamma(u(t)) \leq M$ for almost all $t \geq 0$ for some $M > 0$. Let $T > 0$ and $\varepsilon > 0$. From the continuity of γ and the fact that $\gamma(0) = 0$, there exists $\delta > 0$ such that $\gamma(\mu) < \frac{\varepsilon}{2T}$ for all $|\mu| < \delta$. For $t \geq 0$, let $J(t) = \{s \in [t, t+T] : |u(s)| \geq \delta\}$ and $I(t) = \{s \in [t, t+T] : \ |u(s)| < \delta\}$. Since u satisfies condition (M) there exists $t_0 > 0$ such that $|J(t)| < \frac{\varepsilon}{2M}$ for all $t \geq t_0$. Then, for every $t \geq t_0$ we have that

$$\int_t^{t+T} \gamma(u(s))\, ds = \int_{J(t)} \gamma(u(s))\, ds + \int_{I(t)} \gamma(u(s))\, ds \leq M|J(t)| + \frac{\varepsilon}{2T}|I(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

The following is the main result of this subsection.

Theorem 3.8. *Let Assumptions 1, 2 and 4 hold. Let x be a forward complete solution of (1) corresponding to some essentially bounded input $u \in \mathcal{U}$ which satisfies condition (M). Then the statements* (i), (ii) *and* (iii) *of Theorem 3.2 are equivalent.*

Proof. Since u is essentially bounded, there exists a compact set $B \subset U$ such that $u(t) \in B$ for almost all $t \geq 0$. Then, there exists $u_B \in \mathcal{U}$ such that $u_B(t) \in B$ for all $t \geq 0$ and $u_B(t) = u(t)$ for almost all $t \geq 0$. It is clear that x is a solution of (1) corresponding to u if and only if it also is a solution corresponding to u_B. By replacing u by u_B and U by B, and restricting f to $\mathbb{R}_{\geq 0} \times \mathcal{X} \times B$ we can suppose without loss of generality that U is a compact metric space and that f satisfies Assumptions 1, 2 and 4. By applying Lemma 2.1 it follows that f also satisfies Assumption 3. The theorem then follows from Lemma 3.7 and Theorem 3.2. $\blacksquare$

The converging-input converging-state result in Theorem 1 of [6] is a simple consequence of Theorem 3.8 and the following result.

Lemma 3.9. *Let $u \in \mathcal{U}$ be locally essentially bounded and such that $\lim_{t\to\infty}|u(t)| = 0$. Then u is essentially bounded and satisfies condition (M).*

Proof. The fact that u satisfies condition (M) is straightforward. Since u is locally essentially bounded and satisfies $\lim_{t\to\infty}|u(t)| = 0$, application of Remark C.1.3 in [22] to the sequence $\{u_i\} \subset \mathcal{U}$ with $u_i : [0,1] \to U$, $u_i(t) := u(t+i)$ for $i = 0, 1, \ldots$ shows the existence of a compact set $B \subset U$ such that $u(t) \in B$ for almost all $t \geq 0$. $\blacksquare$

Theorem 1 of [6] assumes that f in (1) is time-invariant, continuous and locally Lipschitz in x uniformly in u, when u belongs to a compact subset of U. It is also assumed that $\bar{x} \in \mathcal{X}$ is an asymptotically stable equilibrium point of the zero-input system with region of attraction $\mathcal{O}$. Then, if x is a forward complete solution of (1) corresponding to a locally essentially bounded input u such that $\lim_{t\to\infty}|u(t)| = 0$ and x satisfies a certain recurrence condition (see [6] for details), the aforementioned Theorem 1 asserts that $x \to \bar{x}$. It is clear that the assumptions made on f in [6] imply that f satisfies Assumptions 1 and 4. In addition, Assumption 2 holds with $C = \{\bar{x}\}$ and $\mathcal{G} = \mathcal{O}$. The facts that u is locally essentially bounded and $\lim_{t\to\infty}|u(t)| = 0$ imply that u is essentially bounded and satisfies condition (M) in virtue of Lemma 3.9. The assumptions of Theorem 3.8 are thus fulfilled. In consequence, being $\Omega(x) \cap \mathcal{G} \neq \emptyset$ due to the recurrence condition assumed in [6], application of Theorem 3.8 shows that $x \to \bar{x}$, as Theorem 1 of [6] asserts.

We note that our assumptions on the function f, particularized to the time-invariant case, are slightly weaker than those assumed in [6], since we do not require the uniform-in-the-input local Lipschitz condition assumed in that paper.

We close the subsection with a convergence result for bounded-input bounded-state (BIBS) systems, easily derived from Theorem 3.8. We say that system (1) is BIBS if for every maximal solution $x : [t_0, t_x) \to \mathcal{X}$ of (1) corresponding to an essentially bounded input $u \in \mathcal{U}$ there exists a compact set $K \subset \mathcal{X}$ such that $x(t) \in K$ for all $t \in [t_0, t_x)$. Then, note that x is forward complete.

Corollary 3.10. *Let Assumptions 1 and 4 hold. Suppose that Assumption 2 holds with $\mathcal{G} = \mathcal{X}$ and that (1) is BIBS. Let x be a maximal solution of (1) corresponding to some essentially bounded input $u \in \mathcal{U}$ which satisfies condition (M). Then $x \to C$.*

4. Some technical results

Lemma 4.1. *Let (S, d) be a separable and locally compact metric space and let $s_0 \in S$. Then, there exists a proper continuous function $\omega : S \to \mathbb{R}_{\geq 0}$ such that $\omega(s_0) = 0$ and $\omega(s) \geq d(s, s_0)$ for all $s \in S$.*

Proof. If S is compact just take $\omega(s) = d(s, s_0)$. Suppose that S is locally compact and separable but not compact. Then there exists a sequence $\{K_n\}_{n\geq1}$ of compact subsets of S such that $s_0 \in K_1$, $K_n \subset \mathrm{int}(K_{n+1})$ for every $n \geq 1$ and $\cup_{n\geq1}\mathrm{int}(K_n) = S$. Here $\mathrm{int}(K_l)$ denotes the interior of K_l for every $l \geq 1$. Since S is not compact, $S \setminus K_n \neq \emptyset$ for all $n \geq 1$. From Urysohn's lemma (see e.g. Chapter II of [25]), for each $n \geq 1$ there exists a continuous function $g_n : S \to [0,1]$ such that $g_n(s) = 0$ for all $\mu \in K_n$ and $g_n(s) = 1$ for all s outside $\mathrm{int}(K_{n+1})$. Let $g : U \to \mathbb{R}_{\geq 0}$ be defined via $g(s) = \sum_{n=1}^{\infty} g_n(s)$. It is an exercise to show that g is well defined, continuous, proper and $g(s_0) = 0$. The lemma follows by taking $\omega : S \to \mathbb{R}_{\geq 0}$, with $\omega(s) = \max\{d(s, s_0), g(s)\}$. In fact, ω is continuous, $\omega(s_0) = 0$ and $\omega(s) \geq d(s, s_0)$. That ω is proper follows from the fact that for every $r \in \mathbb{R}$, $\omega^{-1}((-\infty, r]) = g^{-1}((-\infty, r]) \cap \{s \in S : \ d(s, s_0) \leq r\}$ is compact since it is the intersection of the compact set $g^{-1}((-\infty, r])$ with the closed set $\{s \in S : \ d(s, s_0) \leq r\}$. $\blacksquare$

Proof of Lemma 2.1. Pick any $\xi_0 \in \mathcal{X}$. Both U and $\mathcal{X}$ are separable locally compact metric spaces, the first one by hypothesis and the second one because $\mathcal{X}$ is an open subset of a separable and locally compact metric space. Then, invoking Lemma 4.1, there exist proper continuous functions $\omega_1 : U \to \mathbb{R}_{\geq 0}$ and $\omega_2 : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that $\omega_1(0) = 0$ and $\omega_1(\mu) \geq |\mu|$ for all $\mu \in U$, and $\omega_2(\xi_0) = 0$ and $\omega_2(\xi) \geq |\xi - \xi_0|$ for all $\xi \in \mathcal{X}$. Note that for each $r > 0$, the sets $S_1(r) := \{\mu \in U : \omega_1(\mu) \leq r\}$ and $S_2(r) := \{\xi \in \mathcal{X} : \omega_2(\xi) \leq r\}$ are compact, because the functions $\omega_i, i = 1, 2$ are proper. Define, for all $r > 0$,

$$\hat{\gamma}(r) := \sup\{|f(t, \xi, \mu)| : t \geq 0, \ \mu \in S_1(r), \ \xi \in S_2(r)\}.$$

$\hat{\gamma}(r)$ is nonnegative, nondecreasing and finite for all $r > 0$ due to the compactness of $S_1(r)$ and $S_2(r)$ and the boundedness condition assumed. Let $\hat{\gamma}(0) := \lim_{r\to 0^+} \hat{\gamma}(r)$; this limit exists because $\hat{\gamma}$ is nondecreasing. Then the function $\tilde{\gamma} : [0, \infty) \to \mathbb{R}_{\geq 0}$, defined via $\tilde{\gamma}(r) = \hat{\gamma}(r) - \hat{\gamma}(0)$ is nondecreasing, continuous at 0 and $\tilde{\gamma}(0) = 0$. Therefore, there exists a continuous and strictly increasing function $\gamma^* : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma^*(0) = 0$ and $\tilde{\gamma}(r) \leq \gamma^*(r)$ for all $r > 0$. Then, for every $\xi \in \mathcal{X}$ and $\mu \in U$ and every $r > 0$ and $s > 0$ such that $\xi \in S_2(r)$ and $\mu \in S_1(s)$ we have

$$\begin{aligned} |f(t, \xi, \mu)| &\leq \hat{\gamma}(\max\{r, s\}) \leq \hat{\gamma}(r) + \hat{\gamma}(s) \\ &\leq 2\hat{\gamma}(0) + \gamma^*(r) + \gamma^*(s). \end{aligned}$$

So, for all $\xi \in \mathcal{X}$ and $\mu \in U$

$$\begin{aligned} |f(t, \xi, \mu)| &\leq \inf\big\{2\hat{\gamma}(0) + \gamma^*(r) + \gamma^*(s) : \\ &\qquad r > 0, \ s > 0, \ \omega_2(\xi) \leq r, \ \omega_1(\mu) \leq s\big\} \\ &= 2\hat{\gamma}(0) + \gamma^*(\omega_1(\mu)) + \gamma^*(\omega_2(\xi)). \end{aligned}$$

Let $K \subset \mathcal{X}$ be a compact set and $c = \max_{\xi\in K} \gamma^*(\omega_2(\xi)) + 2\hat{\gamma}(0) + 1$. Then we have that

$$|f(t, \xi, \mu)| \leq c(1 + \gamma^*(\omega_1(\mu))) \quad \forall \xi \in K, \ \forall \mu \in U.$$

If we take $\gamma = \gamma^* \circ \omega_1$, it follows that $\gamma(0) = 0$, $\inf_{|\mu|\geq r} \gamma(\mu) \geq \inf_{|\mu|\geq r} \gamma^* \circ \omega_1(\mu) \geq \inf_{|\mu|\geq r} \gamma^*(|\mu|) \geq \gamma^*(r) > 0$ for all $r > 0$, where we have used the facts that $\omega_1(\mu) \geq |\mu|$ and that γ^* is strictly increasing. Hence, condition (C1) is satisfied. ■

5. Conclusions

We have provided stability results for time-varying systems with inputs. More precisely, we have given conditions under which if the maximum average power of the input converges to zero, then the state trajectories will inherit stability properties from the corresponding zero-input system. For this property to hold, input power must be measured according to a function that bounds the growth of the function f defining the system dynamics as the input value grows. Our results generalize to time-varying systems other existing results that are valid for time-invariant systems. Even when particularized to time-invariant systems, the assumptions required are weaker than existing ones. This relaxation of the required assumptions is made possible by avoiding the use of converse Lyapunov arguments (which are not valid for time-varying systems), and by not requiring Lipschitz continuity other than for the zero-input system. In addition, our results do not require uniqueness of solutions.

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