1. A short Barbalat

In 1959, Barbalat formalized the intuitive principle that a function whose integral up to infinity exists and whose oscillation is bounded needs to be small at infinity.

**Theorem 1.** (Barbalat’s Lemma) Assume that \( f : [0, \infty) \to \mathbb{R} \) is uniformly continuous and that \( \lim_{t \to \infty} f(t) = 0 \) holds.

Barbalat’s original proof and also its usual reproductions, e.g., in [1] p. 211, are by contradiction. As a courtesy for the reader we feel obliged to mention that the proof in the latter book contains a small mistake: For the final estimate the restriction on an infinite subsequence of intervals such that \( f \) is either positive or negative on all of them is missing. In the sequel we discuss recent versions of Barbalat’s Lemma and the corresponding proofs.

In [3] Lemma 1 Tao pointed out an important alternative to the statement above and showed that \( \lim_{t \to \infty} f(t) = 0 \) holds, whenever \( f \in L^2(0, \infty) \) and \( f' \in L^\infty(0, \infty) \). Here, \( f' \) can be interpreted in the sense of distributions or, equivalently, in the sense that \( f \) is absolutely continuous with the almost everywhere existing derivative being essentially bounded. Tao’s version is important, since its assumptions are in practice much more handy than those of Barbalat. Moreover, his result extends the classical statement—that for \( f : [0, \infty) \to \mathbb{R} \) is uniformly continuous and that \( \lim_{t \to \infty} f(t) = 0 \) holds.

\[ f(x) - f(y) = \int_x^y f'(s)ds \]
(valid for almost every \( x, y \in [0, \infty) \)) we conclude that \( f \) can be identified with a continuous function satisfying

\[ |f(x) - f(y)| \leq \int_x^y |f'(s)|ds \leq |x - y|^{\frac{1}{p}} \|f'\|_q, \]

Here we used Hölder’s inequality for \( q, q' \) with \( \frac{1}{q} + \frac{1}{q'} = 1 \). So that \( f \) is indeed Hölder continuous with exponent \( \frac{1}{p} = \frac{p-1}{q} \).

To prove boundedness, we may assume \( p < \infty \), otherwise there is nothing to show. Let \( r := \frac{1}{q_1} \). Then \( r > 0 \) and we have

\[ \frac{dr}{r^2} |f(x)|^{r+1} = (r+1) |f(x)|^r f'(x) \operatorname{sgn} f(x). \]

Hence for \( x \in [0, \infty) \)

\[ |f(x)|^{r+1} = |f(0)|^{r+1} + (r+1) \int_0^x |f(s)|^r f'(s) \operatorname{sgn} f(s)ds \leq |f(0)|^{r+1} + (r+1) \|f\|_p \|f'\|_q, \]

the last step being again an application of Hölder’s inequality for \( q, q' \) with \( \frac{1}{q} + \frac{1}{q'} = 1 \).

**Theorem 3.** Let \( p \in [1, \infty) \) and \( q \in (1, \infty) \). Every function \( f \in W^{1,p,q}(0, \infty) \) tends to 0 at infinity.

**Proof.** By Lemma 2 the function \( f \) is bounded and uniformly continuous, hence so is \( |f|^p \). By assumption we can apply Barbalat’s Lemma and obtain the statement.

**Lemma 2.** Let \( q \in (1, \infty) \) and \( p \in [1, \infty) \) be arbitrary. A function \( f \in W^{1,p,q}(0, \infty) \) is bounded and uniformly continuous. More precisely \( f \) is \( \frac{q}{q-1} \)-Hölder continuous.

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lim_{t \to \infty} f(t) = 0. Keeping this in mind, Theorem 1 can be proved by just applying the original Barbălat Lemma to $M \circ f$, since the latter is uniformly continuous under the assumptions made on $f$ and $M$. For Theorem 2, which treats an a priori unbounded $f$ and compensates this by requiring that $M : \mathbb{R}^n \to \mathbb{R}$ is assumed to be bounded away from zero outside some $B_r(0)$, it seems not to be possible just to apply Barbălat’s Lemma since then a priori also $M \circ f$ is unbounded. However, the idea behind Theorem 2 is just a variation of Barbălat’s proof as Hou et al. point out, cf. [4, p. 546].

To conclude this section, we illustrate that there are counterexamples which fail the assumptions of Tao’s alternative. The proof of Theorem 5 also works in vector-valued—if not uniformly continuously differentiable—situations. In the setting of Hou et al. our method also yields results which how-ever require knowledge on the modulus of continuity of $M \circ f$ in order to estimate the decay rate explicitly.

Example 4. Let $f(x) = 0$ for $x \in [0, 2)$ and $f(x) = (-1)^n f_n(x)$ for $x \in [n, n + 1)$ with $n \geq 2$ and

$$ f_n(x) = \begin{cases} \frac{n}{x-n}, & x \in [n, n + \frac{1}{2}n^\frac{1}{2}], \\ \frac{n^{-\frac{1}{2}}}{n-x}, & x \in \left(n + \frac{1}{2}n^{-\frac{1}{2}}, n + n^{-\frac{1}{2}}\right), \\ 0, & x \in \left[n + n^{-\frac{1}{2}}, n + 1\right), \end{cases} $$

i.e., $f$ looks as follows.

It is straightforward to check that $\lim_{n \to \infty} \int_0^1 f(x) \, dx = 8\sqrt{2 \sum_{n=1}^\infty (-1)^n \frac{n}{\sqrt{n}}}$ exists and that $f$ is uniformly continuous. On the other hand $f \not\in L^2(0, \infty)$ and $f' \not\in L^\infty(a, \infty)$ for any $a > 0$.

Notice that for given $1 \leq p < \infty$ the function $f$ in Example 4 can easily be modified such that $f \not\in L^p(a, \infty)$ holds. With some more work it is also possible to construct a single $f$ such that $f \not\in L^p(a, \infty)$ is true for all $1 \leq p < \infty$. Finally, in all these cases $f$ can also be changed into a $C^\infty$–function; $|f'|$ is then bounded on any finite interval, but unbounded at infinity.

2. A Quantitative Barbălat

Similarly to Tao’s proof for the $W^{1,2}$ case, one can prove the original Barbălat Lemma by avoiding the indirect argumentation, and hence make the result in some sense quantitative.

Let $f : [0, \infty) \to \mathbb{R}$ be uniformly continuous. Then there exists a function $\omega : [0, \infty) \to [0, \infty)$ such that $|f(t) - f(\tau)| \leq \omega(|t - \tau|)$ holds for all $t, \tau \in [0, \infty)$ and $\lim_{t \to \infty} \omega(t) = \omega(0) = 0$. A function $\omega$ with the latter properties is said to be a modulus of continuity for $f$.

Theorem 5. Let $f$ be as in Barbălat’s Lemma, let $\omega$ be a modulus of continuity for $f$ and consider $S : [0, \infty) \to \mathbb{R}$, $S(t) = \sup_{\tau \leq t} \left| \int_{\tau}^{t+s} f(\tau) \, d\tau \right|$. Then we have $\left| f(t) \right| \leq 2 S(t)^{1/2} + \omega(S(t)^{1/2})$ for $t \geq 0$. Proof. We have $\lim_{t \to \infty} S(t) = 0$. If $S(t) = 0$ for some $t$, then $f(s) = 0$ for each $s \geq t$. Otherwise we fix $t$, put $s = S(t)^{1/2} > 0$ and compute

$$ |f(t)| = \frac{1}{s} \left| \int_{\tau}^{t+s} f(\tau) \, d\tau \right| \leq \frac{1}{s} \left| \int_{\tau}^{t+s} f(\tau) \, d\tau \right| + \frac{1}{s} \left| \int_{t}^{t+s} f(t) - f(\tau) \, d\tau \right| \leq \frac{1}{s} \left| \int_{0}^{s} f(\tau) \, d\tau \right| + \omega(s) \leq 2 S(t) + \omega(s) = 2 S(t)^{1/2} + \omega(S(t)^{1/2}) $$

as desired. □

Theorem 6. Let $f$ be as in Theorem 5 and assume in addition that $f$ is Hölder continuous of order $\alpha \in [0, 1]$, i.e., we can take $\omega(\tau) = |\tau|^\alpha$ for a constant $c > 0$. Let $S$ be defined as in Theorem 5. Then we have $|f(t)| \leq (2 + c) S(t)^{\alpha/(1 + \alpha)}$ for $t \geq 0$.

Proof. It is enough to repeat the proof of Theorem 5 but with $s = S(t)^{1/2}$. □

Corollary 7. Let $f \in W^{1, p, q}(0, \infty)$ for some $p \in [1, \infty)$ and $q \in (1, \infty]$. Then we have $|f(t)| \leq (2 + p \|f\|_{L^p(q)} \|f'\|_{L^q((q-1)/2q-1)})$ for $t \geq 0$, where $S(t) = \int_{t}^{\infty} |f(\tau)|^{p} \, d\tau$.

Proof. By Lemma 4 under our assumptions $f$ is bounded, and $\omega(\tau) = p \|f\|_{L^p(q)} \|f'\|_{L^q((q-1)/2q-1)}$ is a modulus of continuity for $|f'|$. It is therefore enough to apply Theorem 5 to the latter function to obtain the assertion. □

For $p = 1$ and $q = \infty$ we obtain the following.

Corollary 8. Let $f \in W^{1, 1, 1}(0, \infty)$. Then we have $|f(t)| \leq (\|f\|_{L^\infty} + 2) S(t)^{1/2}$ where $S(t) = \int_{t}^{\infty} |f(\tau)| \, d\tau$.

The proof of Theorem’s 5 and 6 also works in vector-valued—even $\infty$–dimensional—situations. In the setting of Hou et al. our method also yields results which how-ever require knowledge on the modulus of continuity of $M \circ f$ in order to estimate the decay rate explicitly.

References


