VARIATIONS ON BARBĂLAT'S LEMMA

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Abstract. We first review recent versions of Barbălat's Lemma by comparing their proofs and by discussing a concrete example. Then we present a proof which allows for a quantitative interpretation.

1. A short Barbălat

In 1959, Barbălat formalized the intuitive principle that a function whose integral up to infinity exists and whose oscillation is bounded needs to be small at infinity.

Theorem 1. *(Barbălat's Lemma [2, p. 269]) Assume that* $f\colon [0,\infty)\to\mathbb{R}$ *is uniformly continuous and that* $\lim_{t\to\infty}\int_0^t f(\tau)\mathrm{d}\tau$ *exists. Then* $\lim_{t\to\infty} f(t)=0$ *holds.*

Barbălat's original proof and also its usual reproductions, e.g., in [1, p. 211], are by contradiction. As a courtesy for the reader we feel obliged to mention that the proof in the latter book contains a small mistake: For the final estimate the restriction on an infinite subsequence of intervals such that f is either positive or negative on all of them is missing. In the sequel we discuss recent versions of Barbălat's Lemma and the corresponding proofs.

In [3, Lemma 1] Tao pointed out an important alternative to the statement above and showed that $\lim_{t\to\infty} f(t)=0$ holds, whenever $f\in L^2(0,\infty)$ and $f'\in L^\infty(0,\infty)$. Here, f' can be interpreted in the sense of distributions or, equivalently, in the sense that f is absolutely continuous with the almost everywhere existing derivative being essentially bounded. Tao's version is important, since its assumptions are in practice much more handy than those of Barbălat. Moreover, his result extends the classical statement—that for $1\leqslant p<\infty$ all functions in the Sobolev space $W^{1,p}(0,\infty)$ tend to zero for $t\to\infty$, see, e.g., Brezis [5, Corollary 8.9]—to the "mixed space"

$$W^{1,p,q}(0,\infty)=\{f\mid f\in L^p(0,\infty) \text{ and } f'\in L^q(0,\infty)\}$$

for $p=2$ and $q=\infty$. Tao's proof is a direct estimation, which has the advantage that it provides a decay rate. However, [3, Lemma 1] can alternatively be proved just by applying the original Barbălat Lemma to $|f|^2$, cf. the proof of Theorem 3 below. Tao pointed out that this is indeed the case if f is known to belong to $L^\infty(0,\infty)$, a property that is a consequence of Tao's assumptions:

Lemma 2. *Let* $q\in(1,\infty]$ *and* $p\in[1,\infty]$ *be arbitrary. A function* $f\in W^{1,p,q}(0,\infty)$ *is bounded and uniformly continuous. More precisely* f *is* $\frac{q-1}{q}$*-Hölder continuous.*

Proof. From

$$f(x)-f(y)=\int_x^y f'(s)\mathrm{d}s$$

(valid for almost every $x,y\in[0,\infty)$) we conclude that f can be identified with a continuous function satisfying

$$|f(x)-f(y)|\leq\Big|\int_x^y f'(s)\mathrm{d}s\Big|\leq|x-y|^{\frac{1}{q'}}\|f'\|_q.$$

Here we used Hölder's inequality for q,q' with $\frac{1}{q'}+\frac{1}{q}=1$. So that f is indeed Hölder continuous with exponent $\frac{1}{q'}=\frac{q-1}{q}$.
To prove boundedness, we may assume $p<\infty$, otherwise there is nothing to show. Let $r:=\frac{q}{q-1}p$. Then $r>0$ and we have

$$\tfrac{\mathrm{d}}{\mathrm{d}x}|f(x)|^{r+1}=(r+1)|f(x)|^r f'(x)\operatorname{sgn} f(x).$$

Hence for $x\in[0,\infty)$

$$\begin{aligned}|f(x)|^{r+1}&=|f(0)|^{r+1}+(r+1)\int_0^x|f(s)|^r f'(s)\operatorname{sgn} f(s)\mathrm{d}s\\&\leq|f(0)|^{r+1}+(r+1)\|f\|_p^r\cdot\|f'\|_q,\end{aligned}$$

the last step being again an application of Hölder's inequality for q,q' with $\frac{1}{q}+\frac{1}{q'}=1$. □

Theorem 3. *Let* $p\in[1,\infty)$ *and* $q\in(1,\infty]$*. Every function* $f\in W^{1,p,q}(0,\infty)$ *tends to* 0 *at infinity.*

Proof. By Lemma 2 the function f is bounded and uniformly continuous, hence so is $|f|^p$. By assumption we can apply Barbălat's Lemma and obtain the statement. □

Hou et. al. [4, Theorem 1] provided the following version of Barbălat's Lemma: If $f\colon[0,\infty)\to\mathbb{R}$ is uniformly continuous and bounded with $f([0,\infty))\subseteq B_r(0):=\{x\in\mathbb{R}^n\mid\|x\|\leqslant r\}$ for some $r\geqslant0$ and if $M\colon B_r(0)\to\mathbb{R}$ is continuous and positive [or negative] definite, then $\lim_{t\to\infty}f(t)=0$ holds whenever $M\circ f\in L^1(0,\infty)$. Here, M is positive [negative] definite, if $M(0)=0$ and $M(x)>0$ [$M(x)<0$] for $x\neq0$. The proof is an adaption of Barbălat's proof which uses the following "conservation law": If f is bounded away from zero on a disjoint union $\cup_{i=1}^\infty[t_i-\delta,t_i+\delta]$ for $t_i\nearrow\infty$ and $\delta>0$ then the same is true for the composition $M\circ f$. This idea becomes even more apparent when observing that its contraposition means that $\lim_{t\to\infty}M(f(t))=0$ implies

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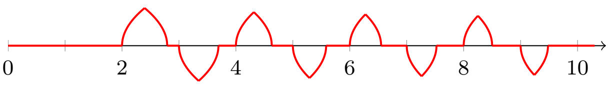
$\lim_{t\to\ infty} f(t) = 0$. Keeping this in mind, [4, Theorem 1] can be proved by just applying the original Barbălat Lemma to $M\circ f$, since the latter is uniformly continuous under the assumptions made on f and M. For [4, Theorem 2], which treats an *a priori* unbounded f and compensates this by requiring that $M\colon \mathbb{R}^n \to \mathbb{R}$ is assumed to be bounded away from zero outside some $B_r(0)$, its seems not to be possible just to apply Barbălat's Lemma since then *a priori* also $M\circ f$ is unbounded. However, the idea behind [4, Theorem 2] is just a variation of Barbălat's proof as Hou et. al. point out, cf. [4, p. 546].

To conclude this section, we illustrate that there are functions to which the original Barbălat Lemma applies, but which fail the assumptions of Tao's alternative. The following example combines two different effects: The difference between Lebesgue and improper Riemann integral on the one and that between uniform continuity and having a bounded derivative on the other hand.

Example 4. *Let $f(x) = 0$ for $x \in [0,2)$ and $f(x) = (-1)^n f_n(x)$ for $x \in [n, n+1)$ with $n \geqslant 2$ and*

$$f_n(x) = \begin{cases} (x-n)^{\frac{1}{2}}, & x \in [n, n+\frac{1}{2}n^{-\frac{1}{3}}),\\ (n+n^{-\frac{1}{3}}-x)^{\frac{1}{2}}, & x\in[n+\frac{1}{2}n^{-\frac{1}{3}}, n+n^{-\frac{1}{3}}),\\ 0, & x\in[n+n^{-\frac{1}{3}}, n+1),\end{cases}$$

i.e., f looks as follows.



It is straight forward to check that $\lim_{t\to\infty}\int_0^t f(\tau)\mathrm{d}\tau = 8\sqrt{2}\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ exists and that f is uniformly continuous. On the other hand $f \notin L^2(0,\infty)$ and $f' \notin L^\infty(a,\infty)$ for any $a \geqslant 0$.

Notice that for given $1 \leqslant p < \infty$ the function f in Example 4 can easily be modified such that $f \notin L^p(a,\infty)$ holds. With some more work it is also possible to construct a single f such that $f \notin L^p(a,\infty)$ is true for all $1 \leqslant p < \infty$. Finally, in all these cases f can also be changed into a C^∞–function; $|f'|$ is then bounded on any finite interval, but unbounded at infinity.

2. A quantitative Barbalǎt

Similarly to Tao's proof for the $W^{1,2,\infty}$ case, one can prove the original Barbalǎt Lemma by avoiding the indirect argumentation, and hence make the result in some sense quantitative.

Let $f\colon [0,\infty) \to \mathbb{R}$ be uniformly continuous. Then there exists a function $\omega\colon [0,\infty) \to [0,\infty)$ such that $|f(t) - f(\tau)| \leqslant \omega(|t-\tau|)$ holds for all t, $\tau \in [0,\infty)$ and $\lim_{t\to 0}\omega(t) = \omega(0) = 0$. A function ω with the latter properties is said to be a modulus of continuity for f.

Theorem 5. *Let f be as in Barbălat's Lemma, let ω be a modulus of continuity for f and consider $S\colon [0,\infty)\to\mathbb{R}$, $S(t) = \sup_{s\geqslant t}|\int_t^s f(\tau)\mathrm{d}\tau|$. Then we have $|f(t)| \leqslant 2\,S(t)^{1/2} + \omega(S(t)^{1/2}))$ for $t \geqslant 0$.*

Proof. We have $\lim_{t\to\infty} S(t) = 0$. If $S(t) = 0$ for some t, then $f(s) = 0$ for each $s \geqslant t$. Otherwise we fix t, put $s = S(t)^{1/2} > 0$ and compute

$$\begin{aligned} |f(t)| &= \frac{1}{s}\Big|\int_t^{t+s} f(t)\mathrm{d}\tau\Big| \\ &\leqslant \frac{1}{s}\Big|\int_t^{t+s} f(\tau)\mathrm{d}\tau\Big| + \frac{1}{s}\Big|\int_t^{t+s} f(t)-f(\tau)\mathrm{d}\tau\Big| \\ &\leqslant \frac{1}{s}\Big|\int_t^{t+s} f(\tau)\mathrm{d}\tau\Big| + \omega(s) \\ &\leqslant \frac{2}{s}S(t) + \omega(s) \;=\; 2\,S(t)^{1/2} + \omega(S(t)^{1/2}) \end{aligned}$$

as desired. □

Theorem 6. *Let f be as in Theorem 5 and assume in addition that f is Hölder continuous of order $\alpha\in(0,1]$, i.e., we can take $\omega(\tau) = c\tau^\alpha$ for a constant $c \geqslant 0$. Let S be defined as in Theorem 5. Then we have $|f(t)| \leqslant (2+c)S(t)^{\alpha/(1+\alpha)}$ for $t \geqslant 0$.*

Proof. It is enough to repeat the proof of Theorem 5 but with $s = S(t)^{1/(1+\alpha)}$. □

Corollary 7. *Let $f \in W^{1,p,q}(0,\infty)$ for some $p \in [1,\infty)$ and $q \in (1,\infty]$. Then we have $|f(t)| \leqslant (2 + p\|f\|_\infty^{p-1}\|f'\|_q)S(t)^{(q-1)/(2q-1)}$ for $t \geqslant 0$, where $S(t) = \int_t^\infty |f(\tau)|^p\mathrm{d}\tau$.*

Proof. By Lemma 2, under our assumptions f is bounded, and $\omega(\tau) = p\|f\|_\infty^{p-1}\|f'\|_q\tau^{(q-1)/q}$ is a modulus of continuity for $|f|^p$. It is therefore enough to apply Theorem 5 to the latter function to obtain the assertion. □

For $p = 1$ and $q = \infty$ we obtain the following.

Corollary 8. *Let $f \in W^{1,1,\infty}(0,\infty)$. Then we have $|f(t)| \leqslant (\|f'\|_{L^\infty} + 2)S(t)^{1/2}$ where $S(t) = \int_t^\infty |f(\tau)|\mathrm{d}\tau$.*

The proof of Theorem's 5 and 6 also works in vector-valued—even ∞-dimensional—situations. In the setting of Hou et. al. our method also yields results which however require knowledge on the modulus of continuity of $M\circ f$ in order to estimate the decay rate explicitly.

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