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# ISS of Switched Systems and Applications to Switching Adaptive Control

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*Abstract*— **In this paper we prove that a switched nonlinear system has several useful ISS-type properties under average dwell-time switching signals if each constituent dynamical system is ISS. This extends available results for switched linear systems. We apply our result to stabilization of uncertain nonlinear systems via switching supervisory control, and show that the plant states can be kept bounded in the presence of bounded disturbances when the candidate controllers provide ISS properties with respect to the estimation errors. Illustrative examples are included.**

## I. Introduction

Switched systems arise in situations where there are several dynamical subsystems and a switching signal that specifies the active subsystem at each instant of time. In general, a switched system does not inherit properties of the individual subsystems; a well-known example is that switching among globally exponentially stable subsystems could lead to instability (see, *e.g.*, [12]). Morse has shown in [15] that for *dwell-time switching signals*, a switched linear system is exponentially stable if the individual subsystems are exponentially stable. This result was later extended to a larger class of switching signals, namely *average dwell-time switching signals*, and to switched linear systems with inputs and switched nonlinear without inputs by Hespanha and Morse in [9]. For switched nonlinear systems with inputs, Xie *et. al.* showed that for dwell-time switching signals, a switched system is *input-to-state stable* (ISS) if the individual systems are ISS [21]; see also [11, Section 5]. If the individual systems are *integral input-to-state stable* (iISS), De Persis *et. al.* showed in [1] that the switched system remains iISS with state-dependent dwell-time switching signals.

This paper extends the results in [9] to switched nonlinear systems with inputs. When the individual subsystems of a switched system are ISS and their ISS-Lyapunov functions satisfy a suitable condition (which was also used in [9]), we show that for switching signals with sufficiently large average dwell-time, the switched system has *ISS*, *exponentially weighted-ISS*, and *exponentially weighted-iISS* properties. Unlike the ISS result in [21] which relies on dwell-time switching, our result only requires average dwell-time switching, which is a less stringent requirement. Compared to state-dependent dwell-time switching employed in [1] which requires the knowledge of the state, average dwell-time switching can be achieved using simple hysteresis-based switching logics [3], [9].

We apply our results in switched systems to the problem of stabilizing uncertain nonlinear systems in the presence of disturbances via *switching supervisory control* (Morse *et. al.* [5], [6], [15], [16]). In switching supervisory control, a *supervisor* orchestrates switching among a parameterized family of *candidate controllers* by appropriately filtering the estimation errors coming out of the *multi-estimator*. This control scheme with the *scale-independent hysteresis switching logic* has been applied successfully to linear systems in the presence of modeling uncertainty and disturbances [7]. For nonlinear plants with the same switching logic, it has been shown that if there are no disturbances, then switching stops in finite time and the states converge to zero [4], [8]. However, in the presence of disturbances, switching is not guaranteed to stop and the states could diverge. In this paper, we show that using switching supervisory control with the scale-independent hysteresis switching logic, the states of an uncertain nonlinear plant can be kept bounded for arbitrary initial conditions and bounded disturbances when the controllers provide ISS property with respect to the estimation errors.

## II. Preliminaries

Consider a family of systems

$$\dot{x} = f_p(x, v), \qquad p \in \mathcal{P}, \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $v \in \mathbb{R}^\ell$, and $\mathcal{P}$ is an index set. For each $p \in \mathcal{P}$, $f_p$ is locally Lipschitz and $f_p(0,0) = 0$. A *switched system* generated by the family of systems (1) and a *switching signal* $\sigma$ is

$$\dot{x} = f_\sigma(x, v), \tag{2}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant function, continuous from the right, specifying at every time the index of the active system. We assume that there are no jumps in the state $x$ at the switching instants, and that a finite number of switches occur on every bounded time interval.

The switched system (2) is *input-to-state stable (ISS)* [19] if there exist functions[1] $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$, such that

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[1] See, *e.g.*, [10, p.144] for definitions on class $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions.

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$\forall\, v \in \mathcal{V}, x_0 \in \mathbb{R}^n$ we have

$$\alpha(|x(t)|) \leqslant \beta(|x_0|, t) + \gamma\left(\|v\|_{[0,t)}\right) \quad \forall\, t \geqslant 0, \tag{3}$$

where $|\cdot|$ the Euclidean norm, and $\|\cdot\|_{\mathcal{I}}$ is the supremum norm of a signal over the interval $\mathcal{I} \subseteq [0, \infty)$. The function $\alpha$ can be taken to be the identity function without loss of generality, see [10, Lemma 4.2].

**Definition II.1** *The switched system* (2) *is* $e^{\lambda t}$-weighted input-to-state stable ($e^{\lambda t}$-weighted ISS) *for some* $\lambda > 0$ *if* $\exists\, \alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$*, such that* $\forall\, v \in \mathcal{V}, x_0 \in \mathbb{R}^n$ *we have*

$$e^{\lambda t}\alpha_1(|x(t)|) \leqslant \alpha_2(|x_0|) + \sup_{s\in[0,t)}\left\{e^{\lambda s}\gamma(|v(s)|)\right\} \ \forall\, t \geqslant 0. \tag{4}$$

*The switched system* (2) *is* $e^{\lambda t}$-weighted integral input-to-state stable ($e^{\lambda t}$-weighted iISS) *for some* $\lambda > 0$ *if* $\exists\, \alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$*, such that* $\forall\, v \in \mathcal{V}, x_0 \in \mathbb{R}^n$ *we have*

$$e^{\lambda t}\alpha_1(|x(t)|) \leqslant \alpha_2(|x_0|) + \int_0^t e^{\lambda\tau}\gamma(|v(\tau)|)d\tau \ \forall\, t \geqslant 0. \tag{5}$$

The $e^{\lambda t}$-weighted ISS and $e^{\lambda t}$-weighted iISS properties generalize ISS and iISS properties[2] in the spirit of exponentially weighted induced norms considered in [9]. While the ISS property characterizes stability in general, the $e^{\lambda t}$-weighted ISS and $e^{\lambda t}$-weighted iISS properties characterize stability with a "stability margin" $\lambda$ (similarly to stability margin of linear systems), which is useful in quantitative analysis (such as in supervisory control as we shall see later).

## III. Input-to-state Properties of Switched Systems

Recall that a switching signal $\sigma$ has an *average dwell-time* $\tau_a$ if there are two positive numbers $N_\circ$ and $\tau_a$ such that

$$N_\sigma(T,t) \leqslant N_\circ + \frac{T-t}{\tau_a} \qquad \forall\, T \geqslant t \geqslant 0, \tag{6}$$

where $N_\sigma(T,t)$ is the number of switches in the interval $(t,T)$ [9]. Average dwell-time switching, in contrast to dwell-time switching, allows switching intervals less than $\tau_a$. Note that for $N_\circ = 1$, this reduces to dwell-time switching. We have the following theorem, which is an extension of the results from [9] to switched nonlinear systems with inputs.[3]

**Theorem III.1** *Consider the switched system* (2)*. Suppose that there exist continuously differentiable functions* $V_p : \mathbb{R}^n \to [0,\infty)$*,* $p \in \mathcal{P}$*, class* $\mathcal{K}_\infty$ *functions* $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\gamma}$*, and numbers* $\lambda_\circ > 0, \mu \geqslant 1$ *such that* $\forall\, \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$*, and* $\forall\, p, q \in \mathcal{P}$*, we have*

$$\overline{\alpha}_1(|\xi|) \leqslant V_p(\xi) \leqslant \overline{\alpha}_2(|\xi|), \tag{7}$$

$$\frac{\partial V_p}{\partial \xi} f_p(\xi,\eta) \leqslant -\lambda_\circ V_p(\xi) + \overline{\gamma}(|\eta|), \tag{8}$$

$$V_p(\xi) \leqslant \mu V_q(\xi). \tag{9}$$

*Let a switching signal* $\sigma$ *having average dwell-time* $\tau_a$*. Then:*

(i) *the switched system* (2) *is ISS if* $\tau_a > \dfrac{\ln\mu}{\lambda_\circ}$*,*

(ii) *the switched system* (2) *is* $e^{\lambda t}$*-weighted ISS if*

$$\tau_a > \frac{\ln\mu}{\lambda_\circ - \lambda}, \quad \lambda \in (0, \lambda_\circ),$$

(iii) *the switched system* (2) *is* $e^{\lambda t}$*-weighted iISS if*

$$\tau_a \geqslant \frac{\ln\mu}{\lambda_\circ - \lambda}, \quad \lambda \in (0, \lambda_\circ). \tag{10}$$

*Proof:* For notational compactness, we define $G_a^b(\lambda) := \int_a^b e^{\lambda s}\overline{\gamma}(|v(s)|)ds$. Let $T > 0$ be an arbitrary time. Denote by $\tau_1, \ldots, \tau_{N_\sigma(T,0)}$ the switching instants on the interval $(0,T)$ (by convention, $\tau_0 := 0$, $\tau_{N_\sigma(T,0)+1} := T$). Consider the function

$$W(s) := e^{\lambda_\circ s} V_{\sigma(s)}(x(s)). \tag{11}$$

On each interval $[\tau_i, \tau_{i+1})$, the switching signal is constant. From (8) and (11), we obtain $\dot{W}(s) \leqslant e^{\lambda_\circ s}\overline{\gamma}(|v(s)|)\ \forall\, s \in [\tau_i, \tau_{i+1})$. Integrating both sides of the foregoing inequality from $\tau_i$ to $\tau_{i+1}^-$ and using (9), we arrive at $W(\tau_{i+1}) \leqslant \mu\left(W(\tau_i) + G_{\tau_i}^{\tau_{i+1}}(\lambda_\circ)\right)$. Iterating the foregoing inequality from $i=0$ to $N_\sigma(T,0)$, we get

$$W(T^-) \leqslant \mu^{N_\sigma(T,0)}\left(W(0) + \sum_{k=0}^{N_\sigma(T,0)} \mu^{-k} G_{\tau_k}^{\tau_{k+1}}(\lambda_\circ)\right). \tag{12}$$

From (10), for every $\delta \in [0, \lambda_\circ - \lambda - \ln\mu/\tau_a)$, we have $\tau_a \geqslant \ln\mu/(\lambda_\circ - \lambda - \delta)$, and by virtue of (6) and since $N_\sigma(T,0) - k - 1 \leqslant N_\sigma(T, \tau_{k+1})$, it follows that

$$\mu^{N_\sigma(T,0)-k} \leqslant \mu^{1+N_\circ} e^{(\lambda_\circ - \lambda - \delta)(T - \tau_{k+1})}, \tag{13}$$

for all $k = 0, \ldots, N_\sigma(T,0)$. Also, since $\lambda + \delta < \lambda_\circ$, we have

$$G_{\tau_k}^{\tau_{k+1}}(\lambda_\circ) \leqslant e^{(\lambda_\circ - \lambda - \delta)\tau_{k+1}} G_{\tau_k}^{\tau_{k+1}}(\lambda + \delta). \tag{14}$$

From (12), (13) and (14), we then arrive at

$$\overline{\alpha}_1(|x(T)|) \leqslant c e^{-(\lambda+\delta)T}(\overline{\alpha}_2(|x_0|) + G_0^T(\lambda + \delta)), \tag{15}$$

$$c := \mu^{1+N_\circ}, \tag{16}$$

by virtue of (11) and (7) and since $x(\cdot)$ is continuous. Letting $\delta = 0$ in (15), we obtain (5) with $\alpha_1 := \overline{\alpha}_1$, $\alpha_2 := c\overline{\alpha}_2$, $\gamma := c\overline{\gamma}$. We have $G_0^T(\lambda + \delta) \leqslant (c_1/c)(e^{(\lambda+\delta-\bar{\lambda})T} - 1)\sup_{\tau\in[0,T)}\left\{e^{\bar{\lambda}\tau}\overline{\gamma}(|v(\tau)|)\right\}$ for all $\bar{\lambda} \in [0, \lambda+\delta)$ where $c_1 := c/(\lambda + \delta - \bar{\lambda})$. This together with (15) yields

$$\begin{aligned}\overline{\alpha}_1(|x(T)|) &\leqslant c e^{-(\lambda+\delta)T}\overline{\alpha}_2(|x_0|) \\ &\quad + c_1 e^{-\bar{\lambda}T} \sup_{\tau\in[0,T)}\left\{e^{\bar{\lambda}\tau}\overline{\gamma}(|v(\tau)|)\right\} \quad \forall\, T \geqslant 0.\end{aligned} \tag{17}$$

Picking some $\delta$ such that $0 < \delta < \lambda_\circ - \lambda - \ln\mu/\tau_a$, and letting $\bar{\lambda} = \lambda$ in (17), we have the property (4) with $\alpha_1 := \overline{\alpha}_1$, $\alpha_2 := c\overline{\alpha}_2$, and $\gamma := c_1\overline{\gamma}$. If we let $\bar{\lambda} = 0, \delta = 0$ in (17), we have the property (3) with $\alpha := \overline{\alpha}_1$, $\beta(r,s) := c e^{-\lambda s}\overline{\alpha}_2(r)$, and $\gamma := c\overline{\gamma}/\lambda$ by the fact that $\sup_{\tau\in[0,T)}\overline{\gamma}(|v(\tau)|) \leqslant \overline{\gamma}\left(\|v\|_{[0,T)}\right)$. ∎

[2]See [18] for the original definition of iISS for nonswitched systems.

[3]It has come to the authors' attention that the ISS property of switched nonlinear systems under average dwell-time switching (but not the $e^{\lambda t}$-weighted ISS and $e^{\lambda t}$-weighted iISS properties) has been independently reported without proof in [2].

**Remark 1** If the individual subsystems in the family (1) are ISS, then for every $p \in \mathcal{P}$ there exist $\overline{\alpha}_{1,p}, \overline{\alpha}_{2,p}, \overline{\gamma}_p \in \mathcal{K}_\infty$, $\lambda_{\circ,p} > 0$, and ISS-Lyapunov functions $V_p$, satisfying $\overline{\alpha}_{1,p}(|\xi|) \leqslant V_p(\xi) \leqslant \overline{\alpha}_{2,p}(|\xi|)$, and $\frac{\partial V_p}{\partial \xi} f_p(\xi) \leqslant -\lambda_{\circ,p} V_p(\xi) + \overline{\gamma}_p(|\eta|)\ \forall\, \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$; see [17], [19]. If the set $\mathcal{P}$ is finite, then (7) and (8) are trivially satisfied. Also, if the set $\mathcal{P}$ is compact, and suitable continuity assumptions on $\left\{\overline{\alpha}_{1,p}, \overline{\alpha}_{2,p}, \overline{\gamma}_p\right\}_{p\in\mathcal{P}}$ and $\left\{\lambda_{\circ,p}\right\}_{p\in\mathcal{P}}$ with respect to $p$ hold, (7) and (8) follow. The set of possible ISS-Lyapunov functions is restricted by (9). This inequality does not hold, for example, if $V_p$ is quadratic for one value of $p$ and quartic for another. If $\mu = 1$, the relation (9) implies that $V = V_p,\ p \in \mathcal{P}$ is a common ISS-Lyapunov function for the family of systems (1). In this case, the switched system is ISS for *arbitrary switching* (also called *uniformly input-to-state stable* [14]).

## IV. Application to Switching Supervisory Control of Nonlinear Systems

We quickly review here the switching supervisory control framework; for details, see *e.g.*, [12, Chapter 6] and references therein. Suppose that an unknown process $\mathbb{P}$ belongs to a family of plants parameterized by a parameter $p \in \mathcal{P}$, for some known finite index set $\mathcal{P}$ of $m$ elements, and denote by $p^\star \in \mathcal{P}$ the true value of the unknown parameter:

$$\dot{x} = f(x, u, p^\star, d), \quad y = h(x),$$

where $x, y, u, d$ are the state, output, input and disturbance, respectively. A family of *candidate controllers*

$$\dot{x}_{\mathbb{C}} = g_q(x_{\mathbb{C}}, y, u), \quad u_q = r_q(x_{\mathbb{C}}, y), \qquad q \in \mathcal{P}, \tag{18}$$

are designed such that the controller indexed by $q$ stabilizes the plant with index $q$. Controller selection is carried out by a high-level *supervisor*, which comprises three subsystems:

(i) The first subsystem is a *multi-estimator*:

$$\dot{x}_{\mathbb{E}} = F(x_{\mathbb{E}}, y, u), \quad y_p = h_p(x_{\mathbb{E}}), \qquad p \in \mathcal{P}. \tag{19}$$

Let $e_p = y_p - y,\ p \in \mathcal{P}$ be the estimation errors. The multi-estimator has the following property.

**Assumption IV.1** *There exists a constant $c_0 > 0$ such that $|e_{p^\star}(t)| \leqslant c_0 \quad \forall\, t \geqslant 0$.*

There is a family of *injected systems*, where the injected system indexed by $q \in \mathcal{P}$ comprises the multi-estimator and the corresponding controller:

$$\dot{x}_{\mathbb{CE}} = \begin{bmatrix} g_q(x_{\mathbb{C}}, y, r_q(x_{\mathbb{C}}, y)) \\ F(x_{\mathbb{E}}, y, r_q(x_{\mathbb{C}}, y)) \end{bmatrix} =: f_q(x_{\mathbb{CE}}, e_q)$$

by virtue of $y = h_q(x_{\mathbb{E}}) - e_q\ \forall q \in \mathcal{P}$, where $x_{\mathbb{CE}} := [x_{\mathbb{C}}^{\mathrm{T}}\ x_{\mathbb{E}}^{\mathrm{T}}]^{\mathrm{T}}$ is the state of the injected system; $x_{\mathbb{CE}} \in \mathbb{R}^n$; $e_q \in \mathbb{R}^\ell$. The *switched injected system* is generated by the above family of injected systems and some switching signal $\sigma$ defined in (iii) below.[4] We assume that the hypotheses of Theorem III.1 are satisfied for this switched injected system (see also Remark 1).

(ii) The second subsystem is the *monitoring signal generator* generating the *monitoring signals* $\mu_p, p \in \mathcal{P}$ as

$$\dot{z}_p = -\lambda z_p + \overline{\gamma}(|e_p|),\ z_p(0) = 0,\ \mu_p(t) = \varepsilon + z_p(t), \tag{20}$$

for some $\varepsilon > 0$, $\lambda \in (0, \lambda_\circ)$, where $\lambda_\circ$, $\overline{\gamma}$ are as in (8).

(iii) The third subsystem is a *switching logic*. We use the *scale-independent hysteresis switching logic*, which produces the *switching signal* $\sigma$ as follows:

$$\sigma(t) := \begin{cases} \operatorname*{argmin}_{q\in\mathcal{P}} \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else,} \end{cases}$$

where $h > 0$ is a design parameter such

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu}{\lambda_\circ - \lambda}. \tag{21}$$

Note that the above hysteresis switching logic is scale-independent—the switching signal $\sigma$ is unaltered when we multiply all the monitoring signals by a positive scalar. Let $\bar{\mu}_p(t) := e^{\lambda t}\mu_p(t),\ t \geqslant 0,\ p \in \mathcal{P}$, be the scaled version of $\mu_p$. From (20), for each $p \in \mathcal{P}$, we have

$$\bar{\mu}_p(t) = \varepsilon \mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda s}\overline{\gamma}(|e_p(s)|)ds, \qquad t \geqslant 0. \tag{22}$$

It is evident from (22) that $\bar{\mu}_p$ is continuous and monotonically nondecreasing. The following lemma provides a characterization of the switching signal $\sigma$ (*cf.* [3, Theorem 1]); the proof is along the lines of [3] and is omitted.

**Lemma IV.2** *For arbitrary $t \geqslant t_0 \geqslant 0$, we have*

$$N_\sigma(t, t_0) \leqslant m + \frac{m}{\ln(1+h)} \ln\left(\frac{\bar{\mu}_q(t)}{\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\right), \tag{23}$$

$$\sum_{k=0}^{N_\sigma(t,t_0)} \left(\bar{\mu}_{\sigma(\tau_k)}(\tau_{k+1}) - \bar{\mu}_{\sigma(\tau_k)}(\tau_k)\right) \leqslant m\left((1+h)\bar{\mu}_q(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\right), \tag{24}$$

*for every index $q \in \mathcal{P}$ where $\tau_1, \tau_2, \ldots, \tau_{N_\sigma(t,t_0)}$ are the discontinuities of $\sigma$ on $(t_0, t)$ and $\tau_{N_\sigma(t,t_0)+1} := t$, $\tau_0 := t_0$.*

Letting $p = p^\star$ in (22), by Assumption IV.1 we obtain

$$\bar{\mu}_{p^\star}(t) \leqslant \kappa \mathrm{e}^{\lambda t}, \quad \kappa := \varepsilon + \overline{\gamma}(c_0)/\lambda. \tag{25}$$

Since $\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0) \geqslant \varepsilon \mathrm{e}^{\lambda t_0}\ \ \forall\, t_0 \geqslant 0$, (23) with $q = p^\star$ and (25) yield $N_\sigma(t, t_0) \leqslant N_\circ + \frac{t - t_0}{\tau_a}$, where $N_\circ := m + m\ln(\kappa/\varepsilon)/\ln(1+h)$, and $\tau_a := \ln(1+h)/(\lambda m)$. With $q = p$ in (24), using $\bar{\mu}_p(t)$ from (22), and (25), we arrive at

$$\int_0^t \mathrm{e}^{\lambda s}\overline{\gamma}(|e_{\sigma(s)}(s)|)ds + \varepsilon \mathrm{e}^{\lambda t} - \varepsilon = \sum_{k=0}^{N_\sigma(t,t_0)} \left(\bar{\mu}_{\sigma(\tau_k)}(\tau_{k+1}) - \bar{\mu}_{\sigma(\tau_k)}(\tau_k)\right) \leqslant m(1+h)\kappa \mathrm{e}^{\lambda t}. \tag{26}$$

[4] By switched injected system we mean that there are no jumps in $x_{\mathbb{CE}}$ at switching instants. When implementing (18), at each switching instant $\tau_i$, we can ensure that $x_{\mathbb{C}}(\tau_i^-) = x_{\mathbb{C}}(\tau_i)$, and thus $x_{\mathbb{C}}$ is continuous; $x_{\mathbb{E}}$ is continuous in view of (19).

We now have the following result on switching supervisory control of nonlinear plants in the presence of disturbances.

**Theorem IV.3** *Suppose that*

(i) *the state $x$ of the process $\mathbb{P}$ is bounded when the input $u$, output $y$ and disturbance $d$ are bounded,*
(ii) *the multi-estimator is designed such that Assumption IV.1 holds,*
(iii) *the candidate controllers are designed such that the hypotheses of Theorem III.1 hold for the switched injected system.*

*Then under the supervisor with scale-independent hysteresis switching logic, all continuous states of the closed-loop system are bounded for arbitrary initial conditions and bounded disturbances.*

*Proof:* From hypothesis (iii) and the condition on average dwell-time (21), it follows from Theorem III.1 that the state of switched injected system $x_{\mathbb{CE}}$ has the $e^{\lambda t}$-weighted iISS property (5) for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2$ and $\gamma$. This property together with (26) yield

$$|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(\alpha_2(|x_{\mathbb{CE}}(0)|) + cm(1+h)\kappa) =: c_2 \quad \forall t \geqslant 0. \tag{27}$$

We have $\forall\, q \in \mathcal{P},\ \forall\, t \geqslant 0,\ |y_q(t)| = |h_q(x_{\mathbb{E}}(t))| \leqslant \sup_{p\in\mathcal{P},|\xi|\leqslant c_2}\{|h_p(\xi)|\} =: c_3$. Since $y = y_{p^\star} - e_{p^\star}$ and $|e_{p^\star}(t)| \leqslant c_0\ \forall\, t \geqslant 0$ (by Assumption IV.1), it follows that $|y(t)| \leqslant c_3 + c_0 =: c_4\ \forall\, t \geqslant 0$. Also $e_q = y_q - y$, and therefore $|e_q(t)| \leqslant c_0 + 2c_3 =: c_5\ \forall\, q \in \mathcal{P},\ \forall\, t \geqslant 0$. Further, we have $\forall\, t \geqslant 0,\ |u(t)| \leqslant \sup_{q\in\mathcal{P},|\xi|\leqslant c_2,|\eta|\leqslant c_4}\{|r_q(\xi,\eta)|\} =: c_6$. Since $d$, $u$ and $y$ are bounded, the state $x$ remains bounded in view of hypothesis (i). Finally, every monitoring signal $\mu_q, q \in \mathcal{P}$, is bounded since $|e_q|$ is bounded for all $q \in \mathcal{P}$. ■

**Remark 2** Hypothesis (i) of Theorem IV.3 holds, for example, when the plant is input-output-to-state stable (IOSS) (see [20] for the definition). Hypothesis (ii) requires that at least one estimator provides a bounded estimation error in the presence of disturbances. This is more or less a standard assumption in multi-estimator design; a similar assumption was used in [8] for plants without disturbances. Hypothesis (iii) stipulates that the injected systems are ISS (which was also an assumption in [8]); the design of ISS injected systems is nontrivial, and is a topic of ongoing research (*cf.* [13]). All three hypotheses can be completely characterized via detectability and stabilizability of the plant for linear systems [15], but for nonlinear systems, there is no known criterion on the plant which guarantees that these requirements can be fulfilled. However, there are certain nonlinear systems for which these conditions hold (see Example 1 below).

**Remark 3** If the disturbance $d$ is vanishing and in Assumption IV.1 we replace the constant bound $c_0$ with a time-varying bound $c_0(t) \to 0$ as $t \to \infty$, and further, if the plant is IOSS, then we can have $|x(t)| \to 0$ as $t \to \infty$ if we use a non-negative decaying $\varepsilon(t)$ in the monitoring signal generator such that $\varepsilon(t) \to 0$ and $\overline{\gamma}(c_0(t))/\varepsilon(t) < \infty$ as $t \to \infty$ (which means $\varepsilon$ should decay more slowly than $\overline{\gamma}(c_0)$). If this is the case, $\kappa \to 0$ in (25) and the chatter bound $N_\circ < \infty$. Then the iISS property of the switched injected system together with (26) yields $|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(e^{-\lambda t}\alpha_2(|x_{\mathbb{CE}}(0)|) + cm(1+h)\kappa(t)) \to 0$ as $t \to \infty$; thus, $c_2$ in (27) becomes a time-varying $c_2(t) \to 0$ as $t \to \infty$. It then follows that $c_3(t), c_4(t), c_5(t), c_6(t) \to 0$ as $t \to \infty$. Since $|u(t)| \to 0$, $|y(t)| \to 0$ and the plant is IOSS, the state norm $|x(t)|$ goes to 0 as $t \to \infty$.

**Example 1** Consider a scalar nonlinear plant

$$\dot{y} = y^2 + p^\star u + d, \tag{28}$$

where $p^\star$ is an unknown constant belonging to a finite index set $\mathcal{P} := \{p_1, \dots, p_m\}$, and $d$ is a disturbance. Our objective is to keep the state bounded in the presence of a bounded disturbance. The unknown parameter enters as the input gain, which makes the problem challenging to solve in the framework of conventional adaptive control when the sign of $p^\star$ is unknown.

The multi-estimator and the candidate controllers are

$$\begin{aligned} \dot{y}_p &= -(y_p - y) - (y_p - y)^3 + pu + y^2, \\ u_p &= \frac{1}{p}(-y - y^2 - y^3), \end{aligned} \qquad p \in \mathcal{P}.$$

For the controller with index $q \in \mathcal{P}$, the injected system is

$$\dot{y}_p = -(y_p - y) - (y_p - y)^3 + \frac{p}{q}(-y - y^2 - y^3) + y^2,\ p \in \mathcal{P}. \tag{29}$$

Consider the candidate ISS-Lyapunov function

$$V_q(x_{\mathbb{CE}}) := a_1 y_q^4 + b_1 y_q^2 + \sum_{p\neq q, p\in\mathcal{P}} a_0 y_p^4 + b_0 y_p^2,\ q \in \mathcal{P},$$

where $x_{\mathbb{CE}} := [y_{p_1}, \dots, y_{p_m}]^{\mathrm{T}}$ is the state of the injected system, for some $a_1, b_1, a_0, b_0 > 0$ to be determined. One can pick $\mu := \max\left\{\frac{a_1}{a_0}, \frac{a_0}{a_1}, \frac{b_1}{b_0}, \frac{b_0}{b_1}\right\}$. The derivative of $V_q$ along the $q^{th}$ injected system is

$$\dot{V}_q = 4a_1 y_q^3 \dot{y}_q + 2b_1 y_q \dot{y}_q + \sum_{p\neq q, p\in\mathcal{P}} 4a_0 y_p^3 \dot{y}_p + 2b_0 y_p \dot{y}_p. \tag{30}$$

Substituting (29) into (30), after some expansions and simplifications, we arrive at

$$\begin{aligned} \dot{V}_q \leqslant\ & -a_1 y_q^6 - 4a_1 y_q^4 - 2b_1 y_q^2 + \\ & \sum_{\substack{p\neq q,\\ p\in\mathcal{P}}} -a_0 y_p^6 - 4a_0 y_p^4 - 2b_0 y_p^2 + (4a_0 y_p^3 + 2b_0 y_p)\kappa_{pq} g(y). \end{aligned} \tag{31}$$

where $\kappa_{pq} := (1 - p/q)$ and $g(y) := y + y^2 + y^3$. Define $\kappa_{\max} := \max\{|\kappa_{pq}| : p, q \in \mathcal{P}\}$. Using completions of the squares with $-a_0 y_p^6 - b_0 y_p^2 + (4a_0 y_p^3 + 2b_0 y_p^2)\kappa_{pq} g(y)$ and using the triangle inequality with $|g(y)|^2$ in (31), after some computations, we obtain

$$\begin{aligned} \dot{V}_q \leqslant\ & -V_q - (a_1 - 256(4a_0 + b_0)m\kappa_{\max}^2) y_q^6 \\ & - (b_1 - 16(4a_0 + b_0)m\kappa_{\max}^2) y_q^2 \\ & + (4a_0 + b_0)m\kappa_{\max}^2 (16e_q^2 + 256e_q^6). \end{aligned}$$

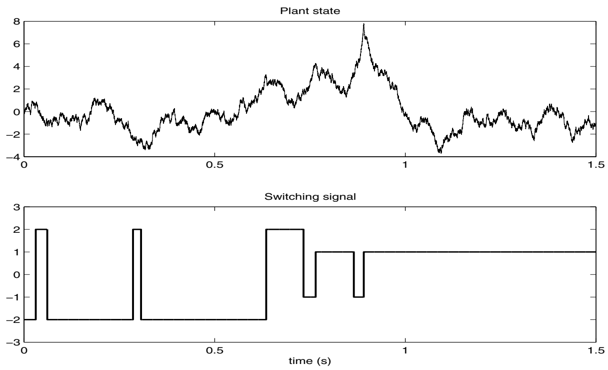


Fig. 1. Example 1

If $b_0$, $a_0$ chosen such that $(4a_0 + b_0)m\kappa_{\max}^2 \leqslant 1$, $a_1 \geqslant 256, b_1 \geqslant 16$, we then get $\dot{V}_q \leqslant -V_q + \overline{\gamma}(|e_q|)$, where $\overline{\gamma}(r) := 16r^2 + 256r^6$ is a class $\mathcal{K}_\infty$ function. The foregoing inequality shows that for each fixed controller with index $q$, the corresponding injected system is ISS with respect to the output error $e_q$. By Theorem IV.3, all the continuous states are bounded for arbitrary initial conditions and bounded disturbances under the supervisor with scale-independent hysteresis switching logic for a large enough $h$ satisfying (21).

For $\mathcal{P} = \{-2, -1, 1, 2\}$, $p^\star = 1$, numerical values are $m = 4$, $\kappa_{\max} = 3$, $a_0 = 6.5\times10^{-3}$, $b_0 = 0.5\times10^{-3}$, $a_1 = 256$, $b_1 = 16$, $\mu = 3.94\times10^4$. Choose $\varepsilon = 10^{-6}$, $\lambda = 2\times10^{-4}$. The hysteresis constant $h = 0.02$ satisfies the condition on average dwell-time (21). Simulation results in MATLAB® with disturbance uniformly distributed between $-10$ and 10, and $x_0 = 0.1$, $x_{\mathbb{E}}(0) = 0$ are plotted in Fig. 1. $\triangle$

*Boundedness under Weaker Hypotheses*

As noted in Remark 1, the existence of $\mu$ as in (9) for all $\xi$ restricts the set of possible ISS-Lyapunov functions. We now assume that we only have $\mu$ such that the inequality (9) holds in some annulus $\Omega := \{\xi \in \mathbb{R}^n : r_1 \leqslant |\xi| \leqslant r_2\}$, for some numbers $r_2 > r_1 \geqslant 0$.

Consider the switched injected system described in the previous section. Suppose $\mu \geqslant 1$ such that $V_p(\xi) \leqslant \mu V_q(\xi)$, $\forall\, r_1 \leqslant |\xi| \leqslant r_2\ \forall\, p, q \in \mathcal{P}$. We can set $x_{\mathbb{CE}}(0) = 0$. Let $\hat{t}_1 := \inf\{t \geqslant 0 : |x_{\mathbb{CE}}(t)| > r_1\}$. If $\hat{t}_1 = \infty$, then $|x_{\mathbb{CE}}(t)| \leqslant r_1\ \forall t \geqslant 0$. Otherwise, let $\hat{t}_2 := \inf\{t \geqslant \hat{t}_1 : |x_{\mathbb{CE}}(t)| > r_2\}$ and $\check{t}_1 := \inf\{t \geqslant \hat{t}_1 : |x_{\mathbb{CE}}(t)| < r_1\}$ and $\bar{t} := \min\{\hat{t}_2, \check{t}_1\}$. Since $r_1 \leqslant |x_{\mathbb{CE}}(t)| \leqslant r_2\ \forall t \in [\hat{t}_1, \bar{t})$, it follows from (27) that

$$|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(\alpha_2(r_1) + cm(1+h)\kappa) =: c_2 \quad \forall t \in [\hat{t}_1, \bar{t}) \tag{32}$$

with $c$ and $\kappa$ are as in (16), (25).

Let $\bar{x}_0$ and $\bar{d}$ be the bounds on the plant initial state and disturbance, respectively. Then the bound $c_0$ in Assumption IV.1 depends on $\bar{x}_0$ and $\bar{d}$ only. Suppose that $\bar{x}_0$ and $\bar{d}$ are sufficiently small such that

$$c_2 \leqslant r_2. \tag{33}$$

In view of (32) and (33), from the definition of $\hat{t}_2$, we must have $\hat{t}_2 = \infty$. If $\check{t}_1 = \infty$, then $\bar{t} = \infty$ and hence, $|x_{\mathbb{CE}}(t)| \leqslant c_2\ \forall\, t \geqslant 0$. If $\check{t}_1 < \infty$, then $\bar{t} = \check{t}_1$, and let $\hat{t}_3 := \inf\{t \geqslant \check{t}_1 : |x_{\mathbb{CE}}(t)| > r_1\}$. If $\hat{t}_3 = \infty$, then $|x_{\mathbb{CE}}(t)| \leqslant r_1 \leqslant c_2\ \forall t \geqslant \check{t}_1$, and hence, $|x_{\mathbb{CE}}(t)| \leqslant c_2\ \forall t \geqslant 0$; otherwise, repeat the current argument with $\hat{t}_3$ playing the role of $\hat{t}_1$. We can then conclude that $|x_{\mathbb{CE}}(t)| \leqslant c_2\ \forall t \geqslant 0$. From the boundedness of $x_{\mathbb{CE}}$, we can prove that all continuous states are bounded using similar arguments as in the proof of Theorem IV.3. We then have the following result.

**Theorem IV.4** *Suppose that*

(i) *the state $x$ of the process $\mathbb{P}$ is bounded when the input $u$, output $y$ and disturbance $d$ are bounded,*
(ii) *the multi-estimator is designed such that Assumption IV.1 holds,*
(iii) *the candidate controllers are designed such that hypotheses* (7), (8) *of Theorem III.1 hold for the switched injected system for some family of ISS-Lyapunov functions $\{V_p\}_{p\in\mathcal{P}}$,*
(iv) *there exist positive numbers $r_1, r_2, \mu$, such that $V_q(\xi) \leqslant \mu V_q(\xi)\ \forall\, r_1 \leqslant |\xi| \leqslant r_2$ and positive numbers $\bar{x}_0, \bar{d}$ such that $c_2 \leqslant r_2$ for some $\varepsilon > 0, h > 0, 0 < \lambda < \lambda_\circ$ where $c_2$ is as in* (32).

*Then under the supervisor with the scale-independent hysteresis switching logic, with hysteresis constant $h$, all continuous states of the closed-loop system are bounded for bounded disturbances $|d(t)| \leqslant \bar{d}$, $t \geqslant 0$ whenever the initial plant state $|x(0)| \leqslant \bar{x}_0$.*

**Example 2** Consider the scalar nonlinear plant in Example 1, and the following simpler multi-estimator and candidate controllers

$$\begin{aligned} \dot{y}_p &= -(y_p - y) + y^2 + pu, \\ u_p &= -\frac{1}{p}(y + y^2), \end{aligned} \qquad p \in \mathcal{P}.$$

The injected system with the controller indexed by $q$ is

$$\dot{y}_p = -(y_p - y) + y^2 + \frac{p}{q}(-y - y^2), \qquad p \in \mathcal{P}.$$

Using the candidate ISS-Lyapunov function $V_q := b_1 y_q^4 + b_2 y_q^2 + a\sum_{p\neq q,\, p\in\mathcal{P}} y_p^2$, it can be checked that for each fixed controller indexed by $q \in \mathcal{P}$, the injected system is ISS:

$$\begin{aligned} \dot{V}_q &= -4b_1 y_q^4 - 2b_2 y_q^2 + 2a\sum_{p\neq q, p\in\mathcal{P}} y_p(-y_p + \kappa_{pq} y + \kappa_{pq} y^2) \\ &\leqslant -\lambda_\circ V_q + \overline{\gamma}(|e_q|), \end{aligned}$$

with $y^2 := |y_q - e_q|^2 \leqslant 2(y_q^2 + e_q^2)$ and $y^4 \leqslant 8(y_q^4 + e_q^4)$ for some $0 < \lambda_\circ < 2$, $a_1, a_2 > 0$, such that $a_1 + a_2 = 2 - \lambda_\circ$, where $\kappa_{pq} := (1 - p/q)$, $\kappa_{\max} := \max\{|\kappa_{pq}| : p, q \in \mathcal{P}\}$, $b_3 := a(m-1)\kappa_{\max}^2$, $\overline{\gamma}(r) := b_3(2r^2/a_1 + 8r^4/a_2)$, and $b_1, b_2, a$ such that $b_3 < \min\{(4-\lambda_\circ)b_1 a_2/8, (2-\lambda_\circ)b_2 a_1/2\}$.

The ISS-Lyapunov function $V_q$ has the property (7), (8) for all $\xi \in \mathbb{R}^m$; however, there is no global $\mu$ as in (9) because $V_q$ is quartic in $y_q$ whilst $V_p, p \neq q$, are quadratic in $y_q$.

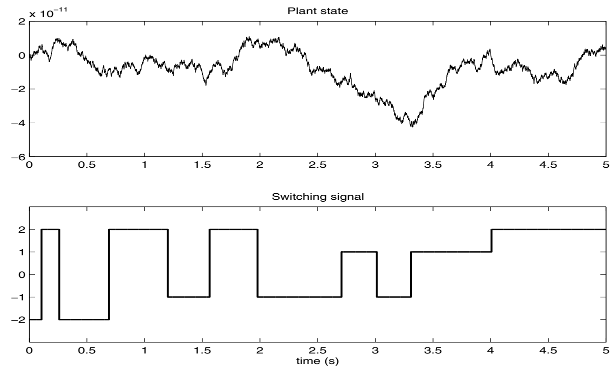


Fig. 2. Example 2

Nevertheless, we can obtain a stability result using Theorem IV.4.

We can choose $\overline{\alpha}_1(r) := \min\{b_2, a\}r^2 =: \eta_1 r^2$ and $\overline{\alpha}_2(r) := \max\{(b_1 r_2^2 + b_2), a\}r^2 =: \eta_2 r^2$. Then $\mu := \eta_2/\eta_1$. The error dynamics for $p = p^\star$ is $\dot{e}_{p^\star} = -e_{p^\star} - d$ and hence, the bound on $e_{p^\star}$ is $|e_{p^\star}(t)| \leqslant |e_{p^\star}(0)| + \bar{d} \leqslant \bar{x}_0 + \bar{d}$ since $|e_{p^\star}(0)| = |y_{p^\star}(0) - y(0)| \leqslant \bar{x}_0$ by virtue of $y_p(0) = 0 \;\forall\, p \in \mathcal{P}$. Now, $c_2 = \left(\mu^{1+N_\circ}(\eta_2 r_1^2 + m(1+h)\kappa)/\eta_1\right)^{1/2} < r_2$ if $r_1$ and $\bar{d}$ are small enough. Choosing the hysteresis constant $h$ to satisfy the average dwell-time condition, we conclude that all the continuous states $x, x_{\rm CE}$ are bounded.

Numerically, for $\mathcal{P} = \{-2, -1, 1, 2\}$, $p^\star = 1$, we have $m = 4$, $\kappa_{\max} = 3$. Let $r_2 = 0.1$, $r_1 = 10^{-8}$, $b_1 = 2.96 \times 10^{-9}, b_2 = 1.3 \times 10^{-10}$, $a = 8 \times 10^{-12}$, $a_1 = 1.75$, $a_2 = 0.15$, $\lambda_\circ = 0.1$. Then $\mu = 19.95$. Choose $h = 0.05$ and $\lambda = 0.0003$; the condition on average dwell-time is satisfied. Choose $\varepsilon = 3.2914 \times 10^{-21}$, then $N_\circ = 4.0819$. If $|x(0)| < 10^{-9}$ and $\bar{d} < 10^{-9}$, then all the states are bounded by $c_2 = 0.0836$ for all time. Simulation results are in Fig. 2. $\triangle$

On the one hand, when ISS-Lyapunov functions satisfying (9) are not available, Theorem IV.4 can provide a way to achieve local boundedness of the plant state. There are more choices of ISS-Lyapunov functions, which can lead to simpler controller and multi-estimator designs, but it may be difficult to find the positive numbers in hypothesis (iv) in Theorem IV.4. Also, the hysteresis constant $h$ cannot be chosen arbitrarily small since $\lambda$ cannot be arbitrarily small ($c_2$ increases when $\lambda$ decreases). On the other hand, if we can find ISS-Lyapunov functions satisfying (9), Theorem IV.3 provides a global boundedness result. It also provides the flexibility to choose a small hysteresis constant $h$, which can be made arbitrarily small by reducing $\lambda$ (see (21)), and a smaller $h$ possibly leads to a better performance.

## V. Conclusions

In this paper, we have shown that under switching signals with large enough average dwell-time, a switched system is ISS, $e^{\lambda t}$-weighted ISS, and $e^{\lambda t}$-weighted iISS, if the individual subsystems are ISS. We applied this result to show that the states of a nonlinear uncertain plant can be kept bounded for arbitrary initial conditions and bounded disturbances using switching supervisory control with the scale-independent hysteresis switching logic, provided that the injected systems are ISS with respect to the estimation errors and there is a global constant $\mu$ as in (9). We relaxed the requirement of a global $\mu$ and achieved local boundedness of the plant state in the presence of bounded disturbances. We illustrated our results on a plant where it may be difficult to apply traditional adaptive control tools. Future research is to study the scenario with measurement noises and unmodeled dynamics, as well as the case of a continuum $\mathcal{P}$.

## References


[1] C. De Persis, R. De Santis, and A. S. Morse. Switched nonlinear systems with state-dependent dwell-time switching logic. *Systems and Control Lett.*, 50:291–302, 2003.

[2] W. Feng and J. Zhang. Input-to-state stability of switched nonlinear systems. In *Proc. 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

[3] J. P. Hespanha, D. Liberzon, and A. S. Morse. Bounds on the number of switchings with scale-independent hysteresis: applications to supervisory control. In *Proc. 39th IEEE Conf. on Decision and Control*, volume 4, pages 3622–3627, 2000.

[4] J. P. Hespanha, D. Liberzon, and A. S. Morse. Supervision of integral-input-to-state stabilizing controllers. *Automatica*, 38:1327–1335, 2002.

[5] J. P. Hespanha, D. Liberzon, and A. S. Morse. Hysteresis-based switching algorithms for supervisory control of uncertain systems. *Automatica*, 39:263–272, 2003.

[6] J. P. Hespanha, D. Liberzon, and A. S. Morse. Overcoming the limitations of adaptive control by means of logic-based switching. *Systems and Control Lett.*, 49:49–65, 2003.

[7] J. P. Hespanha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. S. Brinsmead, and F. De Bruyne. Multiple model adaptive control, part 2: switching. *Int. J. Robust Nonlinear Control*, 11:479–496, 2001.

[8] J. P. Hespanha and A. S. Morse. Certainty equivalence implies detectability. *Systems and Control Lett.*, 36:1–13, 1999.

[9] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. 38th IEEE Conf. on Decision and Control*, pages 2655–2660, 1999.

[10] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 3rd edition edition, 2002.

[11] D. Liberzon. ISS and integral-ISS disturbance attenuation with bounded controls. In *Proc. 38th IEEE Conf. on Decision and Control*, pages 2501–2506, 1999.

[12] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[13] D. Liberzon, E. D. Sontag, and Y. Wang. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Systems and Control Lett.*, 4(2):111–127, 2002.

[14] J. L. Mancilla-Aguilar and R. A. Garcia. On the existence of common Lyapunov triples for ISS and iISS switched systems. In *Proc. 39th IEEE Conf. on Decision and Control*, pages 3507–3512, 2000.

[15] A. S. Morse. Supervisory control of families of linear set-point controllers-part I: exact matching. *IEEE Trans. Automat. Control*, 41:1413–1431, 1996.

[16] A. S. Morse. Supervisory control of families of linear set-point controllers-part II: robustness. *IEEE Trans. Automat. Control*, 42:1500–1515, 1997.

[17] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input to state stability. *Mathematics of Control, Signals and Systems*, 9:1–33, 1996.

[18] E. D. Sontag. Comments on integral variants of ISS. *Systems and Control Lett.*, 34:93–100, 1998.

[19] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Lett.*, 24:351–359, 1995.

[20] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems and Control Lett.*, 29:279–290, 1997.

[21] W. Xie, C. Wen, and Z. Li. Input-to-state stabilization of switched nonlinear systems. *IEEE Trans. Automat. Control*, 46:1111–1116, 2001.