Invertibility of switched linear systems

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Abstract
We address the invertibility problem for switched systems, which is the problem of recovering the switching signal and the input uniquely given an output and an initial state. In the context of hybrid systems, this corresponds to recovering the discrete state and the input from partial measurements of the continuous state. In solving the invertibility problem, we introduce the concept of singular pairs for two systems. We give a necessary and sufficient condition for a switched system to be invertible, which says that the individual subsystems should be invertible and there should be no singular pairs. When the individual subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs. Detailed examples are included.

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1. Introduction
Switched systems—systems that comprise a family of dynamical subsystems together with a switching signal determining the active system at a current time—arise in many situations. Switching behaviors can come from controller design, such as in switching supervisory control (Morse, 1996) or gain scheduling control. Switching can also be inherent by nature, such as when a physical plant has the capability of undergoing several operational modes (e.g., an aircraft during different thrust modes (Lygeros et al., 1999), a walking robot during leg impact and leg swing modes (Westervelt et al., 2003), different formations of a group of vehicles, (Olfati-Saber and Murray, 2004). Also, switched systems may be viewed as higher-level abstractions of hybrid systems (see, e.g., Henzinger and Sastry, 1998 for hybrid system definitions), obtained by neglecting details of the discrete behavior and instead considering switching signals from a suitable class. Switched systems have attracted growing research interest recently and several important results for switched systems have been achieved, including various stability results (Agrachev and Liberzon, 2001; Branicky, 1998; Hespanha and Morse, 1999; Mancilla-Aguilar and Garcia, 2000), controllability and observability results (Sun et al., 2002; Xie and Wang, 2004; Vidal et al., 2002), and input-to-state properties (Hespanha, 2003; Xie et al., 2001; Vu et al., 2007) (see, e.g., Liberzon, 2003, for other references on switched systems and switching control).

Here we address the invertibility problem for switched systems, which concerns with the following question: What is the condition on the subsystems of a switched system so that, given an initial state \( x_0 \) and the corresponding output \( y \) generated with some switching signal \( \sigma \) and input \( u \), we can recover the switching signal \( \sigma \) and the input \( u \) uniquely? The aforementioned problem is in the same vein with the classic invertibility problem for non-switched linear systems, where one wishes to recover the input uniquely knowing the initial state and the output. The invertibility problem for non-switched linear systems has been studied extensively, first by Brockett and Mesarovic (1965), then with other algebraic criteria by Silverman (1969) and Sain and Massey (1969); also, a geometric criterion is given

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by Morse and Wonham (1971) (see also the discussions and the references in Moylan, 1977). However, the invertibility problem for switched systems has not been investigated and it is the subject of this paper. As in the classic setting, we start with an output and an initial state, but here, the underlying process constitutes multiple models and we have an extra ingredient to recover apart from the input, namely, the switching signal.

We now briefly discuss how the invertibility problem for switched systems fits into the current literature. On the one hand, non-switched linear systems can be seen as switched systems with constant switching signals. In this regard, the invertibility problem for switched systems is an extension of the non-switched counterpart in the sense that we have to recover the switching signal in addition to the input, based on the output and the initial state. On the other hand, switched systems can be viewed as higher-level abstractions of hybrid systems. Recovering the switching signal for switched systems is equivalent to mode identification for hybrid systems. For autonomous hybrid systems, this mode identification task is a part of the observability problem for hybrid systems (Vidal et al., 2002; Vidal et al., 2003). Mode detection for non-autonomous hybrid systems using known inputs and outputs has been studied, for example, in De Santis et al. (2003), Babaali and Pappas (2005) (see also, Huang et al., 2004). Here, the difference is that we do not know the input and we wish to do both mode detection and input recovery at the same time using the outputs of switched systems (see also, Sundaram and Hadjicostis, 2006, for input recovery of switched systems with known switching signals).

1.1. A motivating example

Before going into detail in the next section, we present a motivating example to illustrate an interesting aspect of the invertibility problem for switched systems that does not have a counterpart in non-switched systems (details are in Example 2 in Section 5). Consider a switched system consisting of the two subsystems

\[
\begin{align*}
\Gamma_1 : & \ \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\
y & = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \\
\Gamma_2 : & \ \dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \\
y & = \begin{bmatrix} 1 & 2 \end{bmatrix} x.
\end{align*}
\]

Both \(\Gamma_1\) and \(\Gamma_2\) are invertible, which means that for every output \(y\) in the output space of \(\Gamma_1\) and every initial state \(x_0\), there exists a unique input \(u\) such that the system \(\Gamma_1\) with the initial state \(x_0\) and the input \(u\) generates the output \(y\), and similarly for \(\Gamma_2\) (procedures for checking invertibility of non-switched linear systems are well-known as discussed in the introduction).

One is given

\[
y(t) = \begin{cases} 
2e^{2t} - 3e^t & \text{if } t \in [0, t^*), \\
c_1e^t + c_3e^{2t} & \text{if } t \in [t^*, T),
\end{cases}
\]

where \(t^* = \ln 3, T = \frac{6}{5}, c_1 = 15 + 18\ln(\frac{2}{5}), c_2 = \frac{1}{4} - 4\ln(\frac{2}{5})\), and the initial state \(x_0 = (-1, 0)^T\), and is asked to find the switching signal \(\sigma\) and the input \(u\) that generate \(y\). Because the output \(y\) has a discontinuity at time \(t = t^*\) and is smooth everywhere else, one may guess that \(t^*\) must be a switching time and is the only one (assuming that there is no state jump at switching times and hence, \(x\) is always continuous; see Section 2.1). Following this reasoning, since there are two subsystems, there are only two possible switching signals:

\[
\begin{align*}
\sigma_1(t) & = \begin{cases} 
1, & t \in [0, t^*), \\
2, & t \in [t^*, T),
\end{cases} \\
\sigma_2(t) & = \begin{cases} 
1, & t \in [0, t^*), \\
2, & t \in [t^*, T).
\end{cases}
\end{align*}
\]

For a fixed switching signal, by invertibility of \(\Gamma_1\) and \(\Gamma_2\), given the output and the initial state, the input can be reconstructed uniquely. One can then try both the switching signals \(\sigma_1\) and \(\sigma_2\) above to see which one gives an input that generates the output \(y\). As it turns out, none of the switching signals \(\sigma_1\) and \(\sigma_2\) would give an input that generates \(y\). Nonetheless, \(y\) is generated, uniquely in this case, by the following switching signal and input:

\[
\begin{align*}
\sigma(t) & = \begin{cases} 
2, & t \in [0, t_1), \\
1, & t \in [t_1, t^*), \\
2, & t \in [t^*, T),
\end{cases} \\
u(t) & = \begin{cases} 
0, & t \in [0, t_1), \\
6e^{2t} - 6e^t, & t \in [t_1, t^*), \\
0, & t \in [t^*, T),
\end{cases}
\end{align*}
\]

where \(t_1 = \ln 2\) (details are in Example 2 in Section 5). In Section 4, we will show how a switch at a later time \(t = t^*\) can be used to recover a switch at an earlier time \(t = t_1\) even if the output is smooth at \(t = t_1\).

1.2. Paper organization

In Section 2, we cover background on switched systems and invertibility of non-switched linear systems, including the structure algorithm; in Section 3, we define the invertibility problem for switched systems and provide a necessary and sufficient condition for invertibility, using the concept of singular pairs; in Section 4, we present an algorithm for finding inputs and switching signals that produce a given output with a given initial state; examples are in Section 5; in Section 6, we highlight the main results of the paper and discuss future work.

2. Notations and background

2.1. Notations and definitions

Denote by \(C^n\) the set of \(n\) times continuously differentiable functions on a domain \(\mathcal{D}\); when the domain is not relevant, we write \(C^n\). Denote by \(\mathcal{F}^n_{\mathcal{D}}\) the set of piecewise right-continuous functions on a domain \(\mathcal{D}\); when the domain is not relevant, we write \(\mathcal{F}^n\). Denote by \(\mathcal{F}\) the subset of \(\mathcal{F}^n\) whose elements are \(C^\infty\) between two consecutive discontinuities. For \(u : \mathcal{D} \to \mathbb{R}^n\), denote by \(u_{\mathcal{D}}\) the restriction of \(u\) onto \(\mathcal{D} \subseteq \mathcal{D}\).

A switched linear system is written as

\[
\Gamma_\sigma : \begin{align*}
\dot{x} & = A_\sigma x + B_\sigma u, \\
y & = C_\sigma x + D_\sigma u,
\end{align*}
\]

where \(\sigma : [0, \infty) \to \mathcal{P}\) is a piecewise constant right-continuous switching signal that indicates the active subsystem.
The invertibility problem for linear systems (from here onward, signal set conditions on the subsystem dynamics so that the map $H$ is one-to-one (injective), where $H$ is an input set and $\mathcal{Y}$ is the corresponding output set. There are two major approaches to the problem: one is the algebraic approach where conditions are obtained in terms of matrix rank equalities; the other is the geometric approach that is based on the invariant properties of subspaces.

An algebraic approach relies on the observation that differentiating $y$ reveals extra information about $u$ via $C\dot{x} = CAx + CBu$. If one keeps differentiating the output, one obtains more information about $u$ from $y$, $\dot{y}$, $\ddot{y}$, ... Along this approach, most well-known are the rank condition for invertibility (Sain and Massey, 1969) and the structure algorithm (Silverman, 1969), where the latter differs from the former in that it only differentiates parts of the output and not the entire output.

A geometric approach is different from an algebraic one in that it does not involve output differentiations. Instead, the invertibility property is realized from the fact that (i) $H_{0}(i)$ is invertible if and only if the kernel of $H_{0}$ is trivial, and (ii) the kernel of $H_{0}$ is the same as the set of inputs that yield the set $\mathcal{Y}$ of states that are reachable from the origin while keeping the output zero. Left invertibility is equivalent to $\mathcal{Y} \cap \ker D = \{0\}$ and $\ker B \cap \ker D = \{0\}$ (see, e.g., Trentelman et al., 2001, Ch. 8.5; see also (Morse, 1971; Morse and Wonham, 1971)).

**Remark 1.** For non-switched linear systems, the following are equivalent: (i) There is a unique $u$ such that $y = H_{0}(u)$ for one particular pair $(x_{0}, y)$ where $y$ has a domain $\mathcal{D} \subseteq [0, \infty)$; (ii) There is a unique $u$ such that $y = H_{0}(u)$ for every $y$ in the range of $\Gamma_{x_{0}}$ for all possible domains and for all $x_{0} \in \mathbb{R}^{n}$. As a consequence, the output set $\mathcal{Y}$ can be taken as the set of all output functions that are generated by continuous inputs on arbitrary time domain. As we will see later in Section 3.1, the map $H_{0}$ for switched systems (as defined in Section 2.1) does not have this property and a careful consideration of the output set $\mathcal{Y}$ is needed.

### 2.2. Invertibility of non-switched linear systems

Consider a linear system:

$$
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &=Cx + Du.
\end{align*}
$$

The invertibility problem for linear systems (from here onward, when we say invertibility we mean left invertibility) concerns with finding conditions on (2) so that for a given initial state $x_{0}$, the input–output map $H_{x_{0}} : \mathcal{U} \to \mathcal{Y}$ is one-to-one (injective), where $\mathcal{U}$ is an input set and $\mathcal{Y}$ is the corresponding output set.
To simplify the presentation, we paraphrase the range theorem in Lemma 1 below so as to be easier to understand at the expense of having to define extra notations. Define

- \( N := (N_{i1}^T \cdots N_{iB_1}^T)^T \), \( L := L_{\beta_1} \),
- \( \mathcal{T} \) be the set of functions \( f : \mathcal{D} \rightarrow \mathbb{R}^\ell \) for all \( \mathcal{D} \subseteq [0, \infty) \)
which satisfy (i) and (iii) of Theorem 1.

**Lemma 1.** For a linear system \( \Gamma \), using the structure algorithm on the system matrices, construct the set \( \mathcal{T} \), the differential operator \( N : \mathcal{T} \rightarrow \mathcal{C} \), and the matrix \( L \). There exists \( u \in C^0 \) such that \( y = y_{\Gamma x_0}(u) \) if and only if \( y \in \mathcal{T} \) and \( N_y|_{t_0} = Lx_0 \) where \( t_0 \) is the initial time of \( y \).

Roughly speaking, the set \( \mathcal{T} \) characterizes functions that can be generated by the system from all initial positions (in some sense, the condition (iii) capture the relationship among the output components regardless of the input). The condition \( N_y|_{t_0} = Lx_0 \) guarantees that a particular \( y \) can be generated starting from the particular initial state \( x_0 \) at time \( t_0 \). We evaluate \( N_y \) at \( t_0 \), which means \( \lim_{t \rightarrow t_0} N_y|_t \), to reflect that \( y \) does not need to be defined for \( t < t_0 \). This is especially useful later when we consider switched systems where inputs and outputs can be piecewise right-continuous.

**Remark 2.** If we allow piecewise continuous inputs, the invertibility condition for \( \Gamma \) is still the same \( \{q_2 = m\} \) but \( \ell \) in (4) can be piecewise continuous functions. The inverse, the generalized inverse, and Theorem 1 are applicable in between every two consecutive discontinuities of the input (on every output segment \( y_{[t, \ell]} \) in which \( N_y|_{t_0} \) exists and is continuous).

**Remark 3.** When \( q_2 = \ell \), \( M_1 = M_2 = 0 \) and the condition (iii) becomes trivial. Also, in this case, \( \alpha = \beta = \delta \). The set \( \mathcal{T} \) is equivalent to the set of functions \( f \) for which \( N_y \) is defined and continuous. In particular, any \( C^\alpha \) function will be in \( \mathcal{T} \). For an invertible system, \( q_2 = \ell \) if the input and output dimensions are the same.

The differential operator \( N \) is used in Lemma 1 to deal with a general output \( y \) that may not be differentiable but \( Ny \) exists (an example is that \( y \) is not differentiable so \( y \) does not exist but \( Ny = \frac{d}{dt} (y - \bar{y}) \) exists and is continuous). Let’s take a closer look at \( N \). We have

\[
M_0 y_0 = \begin{bmatrix} 0 & 1 \end{bmatrix} y_0 = \begin{bmatrix} \bar{y}_0 & 0 \end{bmatrix} =: K_{0,0} y + \frac{d}{dt} K_{0,1} y .
\]

(5)

Let \( M_i y_i := K_{i,0} y + \frac{d}{dt} (K_{i,1} y + \cdots + \frac{d}{dt} \cdots + \frac{d}{dt} K_{i,i+1} y) \). Then in view of \( M_i y_i + M_{i+1} y_{i+1} = \begin{bmatrix} \bar{s}_{i+1} & 0 \end{bmatrix} M_i y_i + \frac{d}{dt} \begin{bmatrix} \bar{s}_{i+1} & 0 \end{bmatrix} M_{i+1} y_{i+1} \), we have \( K_{i,j} \) defined recursively as

\[
K_{i,j+1} = \begin{bmatrix} \bar{s}_{i+1} & 0 \end{bmatrix} K_{i,j} + \frac{d}{dt} \begin{bmatrix} \bar{s}_{i+1} & 0 \end{bmatrix} K_{i,j+1} \qquad (6)
\]

for \( 0 \leq j \leq \ell, i \geq 0 \), where \( K_{i-1} = 0 \) by convention and \( K_{0,0}, K_{0,1} \) are as in (5). In view of the definition of \( K_{i,j} \) and
the fact \( \tilde{N}_k y = \tilde{S}_k M_{i-1} y_{i-1} \), we have

\[
N_y := N_{0y} + \frac{d}{dt} (N_1 y + \cdots + \frac{d}{dt} N_{\beta-1} y),
\]

where \( N_i := \begin{bmatrix} \tilde{S}_k k_{i-1} \\ \tilde{S}_{\beta-1} k_{\beta-2,i} \end{bmatrix} \), \( 0 \leq i \leq \beta - 1 \), \( K_{1,0} = I \) and \( K_{j,k} = 0 \) \( \forall k \geq j + 2, \forall f \) by convention. We then have the following lemma.

**Lemma 2.** Consider the linear system (2). Let \( \beta \) be the number and \( N \) be the differential operator as described in the structure algorithm. For any \( \kappa \geq \beta \), for every \( y \) such that \( N_y \) exists and is continuous, we have \( N_y = N_{0y} + \frac{d}{dt} (N_1 y + \cdots + \frac{d}{dt} (N_{\kappa-1} y + \frac{d}{dt} N_{\kappa-1} y)) \), where \( N_i, 0 \leq i \leq \beta - 1 \), are as in (7) and \( N_i = 0, i > \beta \).

In general, \( N_y \) is calculated by a chain of differentiations and additions as in the lemma. However, whenever \( y \in C^{\beta-1} \), calculating \( N_y \) can be simplified to the matrix multiplication \( [N_0 \ldots N_{\beta-1}] (y, y^{(\beta-1)})^T \).

### 3. Invertibility of switched systems

**The Invertibility Problem.** Consider a (switching signal \times input)-output map \( H_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y} \) for the switched system \( 1 \). Find a set \( \mathcal{U} \) and a condition on the subsystems, independent of \( x_0 \), such that the map \( H_{x_0} \) is one-to-one.

#### 3.1. Invertibility and singular pairs

We say that \( H_{x_0} \) is **invertible** at \( y \) if \( H_{x_0}(\sigma_1, u_1) = H_{x_0}(\sigma_2, u_2) = y \Rightarrow \sigma_1 = \sigma_2, u_1 = u_2 \) (similarly for non-switched systems, \( H_{x_0} : \mathcal{U} \to \mathcal{Y} \) is invertible at \( y \) if \( H_{x_0}(u_1) = H_{x_0}(u_2) = y \Rightarrow u_1 = u_2 \)). We say that \( H_{x_0} \) is **invertible on \( \mathcal{Y} \)** if it is invertible at \( y \), \( \forall y \in \mathcal{Y} \). We say that the switched system is **invertible on \( \mathcal{Y} \)** if \( H_{x_0} \) is invertible on \( \mathcal{Y} \) for all \( x_0 \).

There is a major difference between the maps \( H_{x_0} \) for non-switched systems and for switched systems. The former is a linear map on vector spaces (i.e. the input functions). The latter is a nonlinear map on the domain \( \mathcal{S} \times \mathcal{U} \), of which \( \mathcal{S} \) is not a vector space. For switched systems, uniqueness of \( (\sigma, u) \) for one pair \((x_0, y)\) does not imply uniqueness for other pairs, and thus, a switched system may be invertible on one output set \( \mathcal{Y} \) but not invertible on another set \( \mathcal{Y} \) (which is not the case for non-switched systems; see Remark 1). This situation prompts a more delicate definition of the output set \( \mathcal{Y} \) for switched systems, instead of letting \( \mathcal{Y} \) be the set generated by all possible combinations of piecewise continuous inputs and switching signals. For the invertibility problem, we look for a suitable set \( \mathcal{Y} \) and a condition on the subsystems so that unique recovery of \((\sigma, u)\) is guaranteed for all \( y \in \mathcal{Y} \) and all \( x_0 \in \mathbb{R}^n \).

It is obvious that \( H_0(\sigma, 0) = 0 \) \( \forall \sigma \) regardless of the subsystem dynamics and therefore, the map \( H_0 \) is not one-to-one if the function \( 0 \in \mathcal{Y} \). Note that the available information is the same for both non-switched systems and switched systems, namely, the pair \((x_0, y)\), but the domain in the switched system case has been enlarged to \( \mathcal{S} \times \mathcal{U} \), compared to \( \mathcal{Y} \) in the non-switched system case. For non-switched systems, under a certain condition on the system dynamics (i.e. when the system is invertible), the information \( (x_0, y) = (0, 0) \) is sufficient to determine \( u \) uniquely while for switched systems, that information is insufficient to determine \((\sigma, u)\) uniquely, regardless of what the subsystems are. This illustrates the issue of why we cannot take the output set \( \mathcal{Y} \) to be all the possible outputs. We call those pairs \((x_0, y)\) for which \( H_{x_0} \) is not invertible at \( y \) **singular pairs**.

**Remark 4.** The singular pair notion relates to the scenario where there is a switch in the underlying dynamical system yet the output is still smooth at the switching time. A similar scenario but with a different objective can be found in the context of bumpless switching (Arehart and Wolovich, 1996), in which the objective is to design the subsystems so that the output of the switched system is continuous. In the invertibility problem considered here, the subsystems are fixed and the objective is to recover the switching signal and the input.

#### 3.2. A solution of the invertibility problem

We now return to the invertibility problem. Let \( \mathcal{Y}^{\text{all}} \) be the set of outputs of the system (1) generated by all possible piecewise continuous inputs and switching signals from all possible initial states for all possible durations (the set \( \mathcal{Y}^{\text{all}} \) can be seen as all the possible concatenations of all elements of \( \mathcal{Y}_p, \forall p \in \mathcal{S} \)). Let \( \mathcal{Y} \subset \mathcal{Y}^{\text{all}} \) be the largest subset of \( \mathcal{Y}^{\text{all}} \) such that if \( y \in \mathcal{Y} \)
Theorem 2. Consider the switched system (1) and the output set $\mathcal{Y}$. The switched system is invertible on $\mathcal{Y}$ if and only if all the subsystems are invertible and the subsystem dynamics are such that for all $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y} \cap C^0$, the pairs $(x_0, y)$ are not singular pairs of $\Gamma_p, \Gamma_q$ for all $p \neq q, p, q \in \mathcal{P}$.

Proof (Sufficiency). Suppose that $H_{x_0}(\sigma_1, u_1) = H_{x_0}(\sigma_2, u_2) = y \in \mathcal{Y}$. Let $t_1 := \min\{t > 0 : u_1 \neq u_2 \text{ or } \sigma_1 \neq \sigma_2 \}$ is discontinuous at $t_1$. Let $p = \sigma_1(0) = \sigma_2(0)$. From Lemma 1, we have $N_p y(\cdot) = L_p x_0$ and $N_q y(\cdot) = L_q x_0$, and $y(0) \in \mathcal{Y}_p \cap \mathcal{Y}_q$. By definition, $(x_0, y(0))$ is a singular pair of $\Gamma_p, \Gamma_q$ if $p \neq q$. Also by definition, $y(0) \not\in \mathcal{Y} \cap C^0$. Since there is no singular pair for $\Gamma_p, \Gamma_q$, we must have $p = q$, i.e., $\sigma(t) = \sigma(t) = p \in [0, t_1)$. Since $\Gamma_p$ is invertible, $u(0, t_1) = u_0(t_1) \neq u(0, t_1) = \Gamma_{p,x}(y(0))$ is uniquely recovered on $[0, t_1)$ (recall from Section 2.1 that $\Gamma_{p,x}(y(0))$ is the output of the inverse of $\Gamma_p$ starting with input $y(0)$). Let $x_1 = x(t_1^-)$. By continuity of the trajectory, we have $x_1 = x(t_1^-) = x(t_1)$. If $t_1 = \infty$, then we have $\sigma(t) = \sigma(t)$ and $u(t) = u(t)$ for all $t \in [0, \infty)$. Otherwise, let $t_2 := \min\{t > t_1 : u(t) \neq u(t) \text{ or } \sigma_1 = \sigma_2 \text{ is discontinuous at } t_2\}$. Repeating the arguments in the previous paragraph, we must have $\sigma(t) = \sigma(t) = q \in [t_1, t_2)$ for some $q \in \mathcal{P}$, and $u(0, t_2) = u(0, t_2) = \Gamma_{p,x}(y(0))$ is uniquely recovered on $[t_1, t_2)$.

If $t_2 = \infty$, then we have $\sigma(t) = \sigma(t) = u(t) = u(t)$ for all $t \in [0, \infty)$; otherwise, repeat the procedure with $y(t_2)$. Since there cannot be infinitely many discontinuities in a finite interval (in other words, if there are infinitely many discontinuities, the interval must be $[0, \infty)$), we conclude that $\sigma(t) = \sigma(t)$ and $u(t) = u(t)$ for all $t \in [0, \infty)$. Necessity: Suppose that $\Gamma_p$ is not invertible for some $p \in \mathcal{P}$. Pick some $x_0$ and $y \in \mathcal{Y} \cap \mathcal{Y}_q$; this is always possible by the definition of the set $\mathcal{Y}$. Since $y \in \mathcal{Y}_q$, there exists $u$ such that $y = H_{x_0}(u)$. Since $\Gamma_p$ is not invertible, there exists $\tilde{u} \neq u$ such that $\Gamma_{p,x}(\tilde{u}) = y$ (see Remark 1). Then $H_{x_0}(\tilde{u}) = H_{x_0}(u) = y$. That means the map $H_p$ is not invertible at $y$ and thus, the switched system is not invertible on $\mathcal{Y}_q$, a contradiction. Therefore, $\Gamma_p$ must be invertible for all $p \in \mathcal{P}$.

Suppose that there are some $x_0 \in \mathbb{R}^n$, $y \in \mathcal{Y} \cap C^0$ and $p \neq q, p, q \in \mathcal{P}$ such that $(x_0, y)$ is a singular pair of $\Gamma_p, \Gamma_q$. This means $\forall u_1, u_2 \in C^0$ (not necessarily different) such that $\Gamma_{p,x}(u_1) = \Gamma_{q,x}(u_2) = y$ and therefore, $H_{x_0}(\tilde{u}) = H_{x_0}(\tilde{u}) = y$. Since $\tilde{u} \neq \tilde{u}$, the foregoing equality implies that $H_{x_0}$ is not invertible at $y$, and thus, the switched system is not invertible on $\mathcal{Y}$, a contradiction. Therefore, for all $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y} \cap C^0$, $(x_0, y)$ are not singular pairs of $\Gamma_p, \Gamma_q$ for all $p \neq q, p, q \in \mathcal{P}$. □

While checking for singularity for given $x_0$ and $y$ is feasible using (8), in general, checking for singularity for all $x_0$ and all $y \in \mathcal{Y}$ (and hence, checking invertibility) is not an easy task. We now develop a rank condition for checking invertibility of switched systems, which is more computationally friendly. In the case when the subsystem input and output dimensions are equal, the rank condition is also necessary. We first have the following lemma. For an index $p$, let $W_p := [N_{p,0} N_{p,1} \ldots N_{p,n-1}]$ where $N_{p,0}, \ldots, N_{p,n-1}$ are the matrices as in Lemma 2 for the subsystem with index $p$. Define $W_{pq} := [W_p^T W_q^T]^T$.

Lemma 3. Consider the switched system (1) and the output set $\mathcal{Y}$. Consider the following two statements:

(S1) The subsystem dynamics are such that for all $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y} \cap C^0$, the pairs $(x_0, y)$ are not singular pairs of $\Gamma_p, \Gamma_q$ for all $p \neq q, p, q \in \mathcal{P}$.

(S2) The subsystem dynamics are such that

$$\text{Rank}[W_{pq} L_{pq}] = \text{Rank}[W_{pq} + \text{Rank} L_{pq}]$$

for all $p \neq q, p, q \in \mathcal{P}$ such that $\mathcal{Y}_p \cap \mathcal{Y}_q \neq \{0\}$.

Then S2 always implies S1. If the subsystems are invertible and the input and output dimensions are the same ($m_p = \ell \forall p \in \mathcal{P}$), then S1 also implies S2.

Proof. (S2) $\Rightarrow$ (S1): Suppose that $\exists x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y} \cap C^0$ such that $(x_0, y)$ is a singular pair of $\Gamma_p, \Gamma_q$ for some $p \neq q$. Let $N_{p,q,k} := [N_{p,k} N_{q,k}]^T$. From (8) and Lemma 2, we have $N_{p,q} y = N_{p,q,0} y + \frac{\partial}{\partial t} (N_{p,q,1} y + \cdots + \frac{\partial}{\partial t} N_{p,q,n-1} y)$ in view of the definition of $N_{p,q}, N_{q,q}, 0 \leq i \leq n-1$. Since $N_{p,q,n-1} x_0 \in \text{Range} N_{p,q,n-1}$, we also have $\frac{\partial}{\partial t} N_{p,q,n-1} y \in \text{Range} N_{p,q,n-1}$. It follows that $N_{p,q} y \in \text{Range} W_{pq}$. The rank condition (9) implies that $\text{Range} W_{pq} \cap \text{Range} L_{pq} = \{0\}$. Therefore, from (8), we must have $N_{p,q} y = 0$ if $p \neq q$. But this equality contradicts the fact that $y \in \mathcal{Y}$, and hence, there are no $x_0, y$ that can form a singular pair for some $\Gamma_p, \Gamma_q, p \neq q$.

(S1) + invertible subsystems + $(m_p = \ell \forall p \in \mathcal{P})$ $\Rightarrow$ (S2): Suppose that the rank condition is violated for some $p \neq q, p, q \in \mathcal{P}$. That implies that $\text{Range} W_{pq} \cap \text{Range} L_{pq} \neq \{0\}$ and hence, there exist $\xi$ and $x_0$ such that $W_{pq} \xi = L_{pq} x_0 \neq 0$. If the subsystems are invertible and the input and output dimensions are the same, then $C^0$ functions are always in $\mathcal{Y}_p, \mathcal{Y}_q$ (see Remark 3). There always exists a $C^0$ function $y$ on an interval $[0, \epsilon)$ for some $\epsilon > 0$ such that $(y, \ldots, y^{(n-1)})^T |_{t=0} = \xi$ and $(y, \ldots, y^{(n-1)})^T |_{t} \not\in \text{Ker} W_{pq} \forall t \in [0, \epsilon)$. Since $y \in C^n$, $N_{p,q} y = W_{pq} y + \cdots + y^{(n-1)} y = W_{pq} y$ and $N_{p,q} y = W_{pq} (y, \ldots, y^{(n-1)})^T |_{t} \neq 0 \forall t \in [0, \epsilon)$, i.e., $y \in \mathcal{Y}_p \cap \mathcal{Y}_q \cap \mathcal{Y}$. 


It follows that $x_0$, $y$ forms a singular pair for $\Gamma_p, \Gamma_q$, a contradiction. □

From Theorem 2 and Lemma 3, we have the following result.

**Theorem 3.** Consider the switched system (1) and the output set $\mathcal{X}$. The switched system is invertible on $\mathcal{X}$ if all the subsystems are invertible and the rank condition (9) holds. If the input and output dimensions of all the subsystems are the same, then invertibility of all the subsystems together with the rank condition (9) is also necessary for invertibility of the switched system.

**Remark 5.** If a subsystem has more inputs than outputs, then it cannot be (left) invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case.

When a switched system is invertible i.e. it satisfies the conditions of Theorem 2, a switched inverse system can be constructed as follows. Define the index inversion function $\Sigma^{-1}$: $\mathbb{R}^n \times \mathcal{X} \rightarrow \mathcal{P}$ as

$$\Sigma^{-1} : (x_0, y) \mapsto p : y \in \mathcal{Y}_p \quad \text{and} \quad N_p y|_{x_0} = L_p x_0, \quad (10)$$

where $t_0$ is the initial time of $y$. The function $\Sigma^{-1}$ is well-defined since there is no singular pair, and so, the index $p$ at every time $t_0$ is uniquely determined from the output $y$ and state $x$. In the invertibility problem, it is assumed that $y \in \mathcal{X}$ is an output so the existence of $p$ in (10) is guaranteed. Then a switched inverse system is

$$\Gamma_a^{-1} : \begin{cases} \sigma(t) = \Sigma^{-1}(\zeta(t), y(t+t_0)), \\ \bar{z} = (A - BD^{-1} C )\sigma(t)z + (B D^{-1} N_x)\sigma(t)y, \\ u = -(D^{-1} \bar{C} )\sigma(t)z + (D^{-1} N_x)\sigma(t)y 
\end{cases}$$

(11)

with $\zeta(0) = x_0$ where $\varepsilon > 0$ is sufficiently small. The notation $\sigma(t)$ denotes the object in the parentheses calculated for the subsystem with index $\sigma(t)$. The initial condition $z(0) = x_0$ helps determine the initial active subsystem, $\sigma(0) = \Sigma^{-1}(x_0, y(0), 0)$ at $t = 0$, from which time onwards, the switching signal and the input as well as the state are determined uniquely and simultaneously via (11). In (11), we use $\sigma(t)$ in the right-hand side of $\bar{z}$ and $z$ in the formula of $\sigma(t)$ for notational convenience. Indeed, if $t$ is a switching time, $\Sigma^{-1}$ helps recover $\sigma$ at the time $t$ using small enough $\varepsilon$ such that $t + \varepsilon$ is less than the next switching time. If $t$ is not a switching time, $\sigma$ is constant between $t$ and the next switching time and is equal to $\sigma$ at the last switching time.

**Remark 6.** In (11), since we use a full order inverse for each subsystem, the state $z$ is exactly the same as the state $x$ of the switched system, and hence, we can use $z$ in the index inversion function $\Sigma^{-1}$. If we use a reduced-order inverse for each subsystem (see, e.g., Silverman, 1969), we still get $u$ but then need to plug this $u$ into the switched system to get $x$ to use in $\Sigma^{-1}$.

**Remark 7.** Let $\tilde{\beta} := \max_{p \in \mathcal{P}} \{\beta_p\}$, where $\beta_p$, $p \in \mathcal{P}$, are the $\beta$ as in Theorem 1 for the subsystems. In Lemma 3, instead of $W_p$, we can work with $\overline{W}_p$ where $W_p y\tilde{\beta} := [N_p, 0 \ldots N_p, \tilde{\beta}^{-1}](y, \ldots, y(\tilde{\beta}^{-1})^T$. In general, $W_p$ have fewer columns than $\overline{W}_p$, which make checking the rank condition for systems with large dimensions simpler.

**Remark 8.** Our results can be extended to include the case where output dimensions are different. If subsystem output dimensions are different, one needs to use “hybrid functions” to describe the output set $\mathcal{X}$ of the switched system (which are now not functions but concatenations of functions of different dimensions). Other than that, Definition 1 is unchanged (since it implies that the output dimensions of the two systems must be the same in order for $(x_0, y)$ to be a singular pair) and the statements of Theorems 2 and 3 remain the same.

**Remark 9.** The results in this section can also be extended to include the case when there are state jumps at switching times. Denote by $f_{p,q} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the jump map (also called reset map) i.e. if $\tau$ is a switching time, $x(t) = x(t^+) = f_{\sigma(t^+), \sigma(t)}(x(t^-))$. Note that the case of identity jump maps $f_{p,q}(x) = x \forall p, q \in \mathcal{P}, \forall x \in \mathbb{R}^n$ is the case considered in this paper. For non-identity jump maps, the concept of singular pairs changes to “$(x_0, y)$ is a singular pair of $\Gamma_p, \Gamma_q$ if $\exists u_1, u_2$ such that $\Gamma^O_{p,x_0}(u_1) = \Gamma^O_{q,f_{p,q}(x_0)}(u_2) = y$ or $\Gamma^O_{p,f_{p,q}(x_0)}(u_1) = \Gamma^O_{q,x_0}(u_2) = y$. The Eq. (8) becomes:

$$N_{pq} y|_{x_0} \left[ \begin{array}{c} L_p x_0 \\ L_q f_{p,q}(x_0) \\ \end{array} \right] \quad \text{or} \quad N_{pq} y|_{x_0} \left[ \begin{array}{c} L_p f_{p,p}(x_0) \\ L_q x_0 \\ \end{array} \right]$$

There will also be a distinction between identifying the initial switching mode and subsequent switching modes. Another generalization is to include switching mechanisms, such as switching surfaces. Denote by $S_{p,q}$ the switching surface for system $p$ changing to system $q$ such that $x(t) = f_{p,q}(x(t^-))$ if $x(t^-) \in S_{p,q}$ and $\sigma(t^+) = p$. Then we only need to check for singularity for $x_0 \in S_{p,q}$ and $x_0 \in S_{q,p}$ instead of $x_0 \in \mathbb{R}^n$ for $\Gamma_p, \Gamma_q$.

**4. Output generation**

In the previous section, we considered the invertibility question of whether one can recover $(\sigma, u)$ uniquely for all $y$ in some output set $\mathcal{X}$. In this section, we address a different but closely related problem which concerns with finding $(\sigma, u)$ (there may be more than one) such that $H_{\sigma_0}(\sigma, u) = y$ for given $y$ and $x_0$.

For the switched system (1), denote by $H_{\sigma_0}^{-1}(y)$ the preimage of an output $y$ under $H_{\sigma_0}$,

$$H_{\sigma_0}^{-1}(y) := \{(\sigma, u) : H_{\sigma_0}(\sigma, u) = y\}. \quad (12)$$

By convention, $H_{\sigma_0}^{-1}(y) = \emptyset$ if $y$ is not in the image set of $H_{\sigma_0}$. In general, $H_{\sigma_0}^{-1}(y)$ is a set for a given $y$ (when $H_{\sigma_0}^{-1}(y)$ is a singleton, the map $H_{\sigma_0}$ is invertible at $y$). We want to find conditions and an algorithm to generate $H_{\sigma_0}^{-1}(y)$ when $H_{\sigma_0}^{-1}(y)$ is a finite set.
We need the individual subsystems to be invertible because if this is not the case, then the set $H_{x_0}^{-1}(y)$ will be infinite by virtue of the following lemma.

**Lemma 4.** Consider the non-switched linear system $G$ in (2) with piecewise continuous input. Suppose that $G$ is not invertible. Let $[a, b]$ be an arbitrary interval. For every $u \in \mathcal{P}_{[a, b]}$ and $x_0 \in \mathbb{R}^n$, there are infinitely many different $v \in \mathcal{P}_{[a, b]}$ such that $\Gamma_{x_0}^u(v) = \Gamma_{x_0}^u(u)$ and $\Gamma_{x_0}^u(v)|_{b} = \Gamma_{x_0}^u(u)|_{b}$.

**Proof.** If $G$ is not invertible, then $K \neq 0$ in the generalized inverse (4). Then the controllable subspace $\mathcal{C}$ of $(A - BD_{x}^2\mathcal{C}, BK)$ is non-trivial. Pick any $\xi \in \mathcal{C}$, $\xi \neq 0$ and $T_1, T_2 > 0$ such that $T_1 + T_2 = b - a =: T$. Since $0, \xi \in \mathcal{C}$, there is a nonzero $w_1 \in q_{0, T_1}$ such that $\Gamma_{x_0}^0(0, w_1)|_{T_1} = \xi$ and $w_2 \in q_{0, T_2}$ such that $\Gamma_{x_0}^0(0, w_2)|_{T_2} = 0$. By the time-invariant property, we then have $\Gamma_{x_0}^0(0, w)|_{T} = 0$ where $w = w_1 \oplus w_2$. Let $\hat{u} = \Gamma_{x_0}^0(0, w)$; then $\Gamma_{x_0}(\hat{u})|_{T} = 0$ and $\Gamma_{x_0}^0(\hat{u}) \equiv 0$. Also by the time-invariant property, if $u(t) = \hat{u}(t) - a$, $u \in \mathcal{P}_{[a, b]}$, then $\Gamma_{x_0}(u)|_{T} = 0$ and $\Gamma_{x_0}^0(u) \equiv 0$. Let $v = u + \hat{u}$. Linearity of $\Gamma$ implies $\Gamma_{x_0}(v) = \Gamma_{x_0}(u) + \Gamma_{x_0}(\hat{u})$ and $\Gamma_{x_0}^0(v) = \Gamma_{x_0}(u) + \Gamma_{x_0}^0(\hat{u})$, and hence, $\Gamma_{x_0}(v)|_b = \Gamma_{x_0}(u)|_b$ and $\Gamma_{x_0}^0(v)|_b = \Gamma_{x_0}^0(u)|_b$. Since there are infinitely many $\xi \in \mathcal{C}$, we have infinitely many such $v$. □

**Corollary 1.** Consider a function $y = H_{x_0}(\sigma, u)$ for some $\sigma, u$. Let $2 \subseteq \Sigma$ be the set of values of $\sigma$. If there exists $q \in 2$ such that $I_q$ is not invertible, then there exist infinitely many $u$ such that $y = H_{x_0}(\sigma, u)$.

The previous discussion motivates us to introduce the following assumption.

**Assumption 1.** The individual subsystems $G_p$ are invertible for all $p \in \mathcal{P}$.

However, we have no other assumption on the subsystem dynamics and the switched system may not be invertible as the subsystems may not satisfy the invertibility condition in the previous section. Since we look for an algorithm to find $H_{x_0}^{-1}(y)$, we only consider the functions $y$ of finite intervals (and hence, there is a finite number of switches) to avoid infinite loop reasoning when there are infinitely many switchings.

As we shall see, it is possible to use a switch at a later time to recover a “hidden switch” earlier (e.g. a switch at which the output is smooth). We now present a switching inversion algorithm for switched systems that returns $H_{x_0}^{-1}(y)$ for a function $y \in \mathcal{P}$ where $\mathcal{P}$ is a finite interval, when $H_{x_0}^{-1}(y)$ is a finite set. The parameters to the algorithm are $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{P}$, and the return is $H_{x_0}^{-1}(y)$ as in (12). Define the **index-matching map** $\Sigma^{-1} : \mathbb{R}^n \times \mathcal{P} \rightarrow 2^\mathcal{P}$ as

$$\Sigma^{-1}(x_0, y) := \{ p : y \in \mathcal{P}_p \text{ and } N_p y|_{t_0} = L_p x_0 \},$$

where $t_0$ is the initial time of $y$. If $\Sigma^{-1}(x_0, y)$ is empty, no subsystem is able to generate that $y$ starting from $x_0$.

**Begin of Function $H_{x_0}^{-1}(y)$**

Let the domain of $y$ be $[t_0, T]$.

Let $\mathcal{F} := \{ p \in \mathcal{P} : y|_{[t_0, t_0+\varepsilon]} \notin \mathcal{P}_p \text{ for some } \varepsilon > 0 \}$.

Let $t^* := \min \{ t \in [t_0, T) : y|_{[t, t+\varepsilon]} \notin \mathcal{P}_p \text{ for some } p \in \mathcal{P}, \varepsilon > 0 \}$ or $t^* = T$ otherwise.

Let $\mathcal{A}^* := \Sigma^{-1}(x_0, y|_{[t_0, t_0+\varepsilon]})$ for sufficiently small $\varepsilon$.

If $\mathcal{A}^* \neq \emptyset$,

Let $\mathcal{A} := \emptyset$.

For each $p \in \mathcal{A}^*$,

Let $u := \Gamma_{p, x_0}^{-1}(y|_{[t_0, t_0+\varepsilon]})$.

Let $\mathcal{F} := \{ t \in (t_0, t^*) : (x(t), y|_{[t, t+\varepsilon]}) \text{ is a singular pair of } G_p, I_q \text{ for some } q \neq p \}$.

If $\mathcal{F}$ is a finite set,

- For each $\tau \in \mathcal{F}$, let $\xi := \Gamma_p(u)(\tau)$.
  - $\mathcal{A} := \mathcal{A} \cup \{ (\sigma_p, u|_{[t_0, \tau]} \oplus H_{\xi}^{-1}(y|_{[\tau, T]}) \}$

End For each

Else if $\mathcal{F} = \emptyset$ and $t^* < T$, let $\xi := \Gamma_p(u)|_{t^*}$.

Else if $\mathcal{F} = \emptyset$ and $t^* = T$,

Else $\mathcal{F} = \emptyset$.

End If

End For each

Else $\mathcal{A} := \emptyset$.

End If

Return $H_{x_0}^{-1}(y) := \mathcal{A}$.

**End of Function**

The algorithm is based on the following relationship:

$$H_{x_0}^{-1}(y|_{[t_0, \tau]}) = \{ (\sigma, u) \oplus H_{x_0}^{-1}(y|_{[t_0, \tau], t, \tau}) \text{ such that } (\sigma, u) \in H_{x_0}^{-1}(y|_{[t_0, \tau]}), \forall t \in [t_0, \tau) \},$$

which follows from the semigroup property of trajectories of dynamical systems (which include switched systems). Now, if $t$ in (14) is the first switching time after $t_0$, then we can find $H_{x_0}^{-1}(y|_{[t_0, t]})$ by singling out which subsystems are capable of generating $y|_{[t_0, t]}$ using the index-matching map (13). The problem comes down to determining the first switching time $t$ (and then the procedure is repeated for the function $y|_{[t, T]}$).

In light of the discussion in the previous paragraph, it is noted that the switching inversion algorithm is a recursive procedure calling itself with different parameters within the main algorithm (e.g., $H_{x_0}^{-1}(y)$ uses the returns of $H_{\xi}^{-1}(y|_{[\tau, T]})$). There are three stopping conditions: it terminates either when $\mathcal{P} = \emptyset$, in which case there is no subsystem that can generate $y$ at time $t$ starting from $x_0$, or when $\mathcal{F}$ is not a finite set, in which case we cannot proceed because of infinitely many possible switching times, or when $\mathcal{F}$ is an empty set and $t^* = T$, in which case the switching signal is a constant switching signal.

If the algorithm returns an empty set, it means that there is no $\sigma$ and $u$ that can generate $y$, or there is an infinite number of possible switching times (it is possible to further distinguish between these two cases by using an extra variable in the algo-
We find the switched system generated by the following two subsystems:

\[ \frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x. \]

\[ \frac{dx}{dt} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 2 \end{bmatrix} x. \]

Using the structure algorithm, we can check that the two systems are invertible. The operators \( N_1, N_2 \) are matrix operators and \( N_1 = W_1 = [1], N_2 = W_2 = [1, 2]. \) In this example, the input and output dimensions are the same.

5. Examples

Example 1. Consider the switched system generated by the following two subsystems:

\[ \Gamma_1 : \begin{cases} \frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y = [0, 1] x, \end{cases} \quad \Gamma_2 : \begin{cases} \frac{dx}{dt} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \\ y = [0, 2] x. \end{cases} \]

As we can see, there is a switching at \( t_1 < t^* \) whilst the output is smooth at \( t_1. \) Using the switching inversion algorithm, we can detect the switching at \( t_1 \) and recover the switching signal and subsequently the input.

6. Conclusions

We formulated the invertibility problem for switched systems, which seeks a condition on the subsystems that guarantees unique recovery of the switching signal and the input from given output and initial state. We introduced the concept of singular pairs and presented a necessary and sufficient condition for invertibility of continuous-time switched linear systems, which says that the subsystems should be invertible and there should be no singular pairs with respect to the output set. For switched systems, not necessarily invertible but with invertible subsystems, we gave an algorithm for finding switching signals and inputs that generate a given output from a given initial state.

The invertibility problem for discrete-time switched systems remains an open problem. Other research topic is a geometric approach for continuous-time switched systems, which will complement the matrix-oriented approach presented here. Another direction is to extend the results to switched nonlinear systems, for which the extension of the structure algorithm for non-switched nonlinear systems (Singh, 1981) or the geometric approach (Benedetto et al., 1989) may be useful.

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References


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