Basic Problems in Stability and Design of Switched Systems

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By a switched system, we mean a hybrid dynamical system consisting of a family of continuous-time subsystems and a rule that orchestrates the switching between them. This article surveys recent developments in three basic problems regarding stability and design of switched systems. These problems are: stability for arbitrary switching sequences, stability for certain useful classes of switching sequences, and construction of stabilizing switching sequences. We also provide motivation for studying these problems by discussing how they arise in connection with various questions of interest in control theory and applications.

Problem Statement and Motivation

Many systems encountered in practice exhibit switching between several subsystems that is dependent on various environmental factors. Some examples of such systems are discussed in [1]-[3]. Another source of motivation for studying switched systems comes from the rapidly developing area of switching control. Control techniques based on switching between different controllers have been applied extensively in recent years, particularly in the adaptive context, where they have been shown to achieve stability and improve transient response (see, among many references, [4]-[6]). The importance of such control methods also stems in part from the existence of systems that cannot be asymptotically stabilized by a single continuous feedback control law [7].

Switched systems have numerous applications in control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters, and many other fields. The book [8] contains reports on various developments in some of these areas. In the last few years, every major control conference has had several regular and invited sessions on switching systems and control. Moreover, workshops and symposia devoted specifically to these topics are regularly taking place. Almost every major technical control journal has had or is planning to have a special issue on switched and hybrid systems. These sources can be consulted for further references.

Mathematically, a *switched system* can be described by a differential equation of the form

$$\dot{x} = f_\sigma(x), \tag{1}$$

where $\{f_p : p \in \mathcal{P}\}$ is a family of sufficiently regular functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is parametrized by some index set $\mathcal{P}$, and $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant function of time, called a *switching signal*. In specific situations, the value of σ at a given time t might just depend on t or $x(t)$, or both, or may be generated using more sophisticated techniques such as hybrid feedback with memory in the loop. We assume that the state of (1) does not jump at the switching instants, i.e., the solution $x(\cdot)$ is everywhere continuous. Note that the case of infinitely fast switching (chattering), which calls for a concept of generalized solution, is not considered in this article. The set $\mathcal{P}$ is typically a compact (often finite) subset of a finite-dimensional linear vector space.

In the particular case where all the individual subsystems are linear, we obtain a *switched linear system*

$$\dot{x} = A_\sigma x. \tag{2}$$

This class of systems is the one most commonly treated in the literature. In this article, whenever possible, problems will be formulated and discussed in the more general context of the switched system (1).

The first basic problem that we will consider can be formulated as follows.

Problem A. *Find conditions that guarantee that the switched system (1) is asymptotically stable for any switching signal.*

One situation in which Problem A is of great importance is when a given plant is being controlled by means of switching among a family of stabilizing controllers, each of which is designed for a specific task. The prototypical architecture for such a multicontroller switched system is shown in Fig. 1. A high-level decision maker (supervisor) determines which controller is to be connected in closed loop with the plant at each instant of time. Stability of the switched system can usually be ensured by keeping each controller in the loop for a long enough time, to allow the transient effects to dissipate. However, modern computer-controlled systems are capable of very fast switching rates, which creates the need to be able to test stability of the switched system for arbitrarily fast switching signals.

We are assuming here that the individual subsystems have the origin as a common equilibrium point: $f_p(0) = 0$, $p \in \mathcal{P}$. Clearly, a necessary condition for (asymptotic) stability under arbitrary

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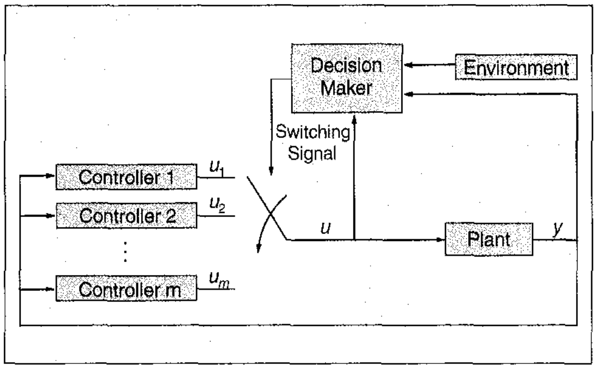

Fig. 1. Multicontroller architecture.

switching is that all of the individual subsystems are (asymptotically) stable. Indeed, if the pth system is unstable, the switched system will be unstable if we set $\sigma(t) \equiv p$. To see that stability of all the individual subsystems is not sufficient, consider two second-order asymptotically stable systems whose trajectories are sketched in the top row of Fig. 2. Depending on a particular switching signal, the trajectories of the switched system might look as shown in the bottom left corner (asymptotically stable) or as shown in the bottom right corner (unstable).

The above example shows that Problem A is not trivial in the sense that it is possible to get instability by switching between asymptotically stable systems. (However, there are certain limitations as to what types of instability are possible in this case. For example, it is easy to see that the trajectories of such a switched system cannot escape to infinity in finite time.) If this happens, one may ask whether the switched system will be asymptotically stable for certain useful classes of switching signals. This leads to the following problem.

Problem B. *Identify those classes of switching signals for which the switched system (1) is asymptotically stable.*

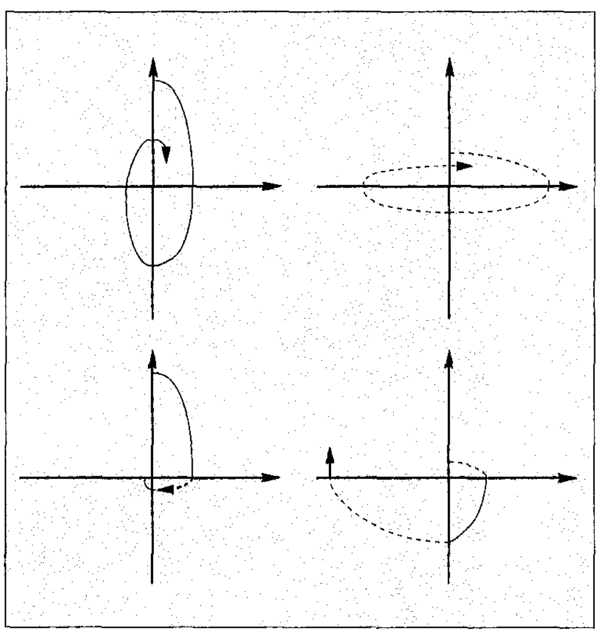

Fig. 2. Possible trajectories of a switched system.

Since it is often unreasonable to exclude constant switching signals of the form $\sigma(t) \equiv p$, Problem B will be considered under the assumption that all the individual subsystems are asymptotically stable. Basically, we will find that stability is ensured if the switching is sufficiently slow. We will specify several useful classes of slowly switching signals and show how to analyze stability of the resulting switched systems.

One reason for the increasing popularity of switching control design methods is that sometimes it is actually easier to find a switching controller performing a desired task than to find a continuous one. In fact, there are situations where continuous stabilizing controllers do not exist, which makes switching control techniques especially suitable (nonholonomic systems provide a good example [9]-[11]). In the context of the multicontroller system depicted in Fig. 1, it might happen that none of the individual controllers stabilize the plant, yet it is possible to find a switching signal that results in an asymptotically stable switched system (one such situation will be discussed in detail later in the article). We thus formulate the following problem.

Problem C. *Construct a switching signal that makes the switched system (1) asymptotically stable.*

Of course, if at least one of the individual subsystems is asymptotically stable, the above problem is trivial (just keep $\sigma(t) \equiv p$ where p is the index of this stable system). Therefore, in the context of Problem C, it will be understood that none of the individual subsystems are asymptotically stable.

The last problem is more of a design problem than a stability problem, but the previous discussion illustrates that all three problems are closely related. In what follows, we will give an exposition of recent results that address these problems. Open questions are pointed out throughout. We present many ideas and results on the intuitive level and refer the reader to the literature for technical details. As the above remarks demonstrate, a deeper understanding of the behavior of switched systems is crucial for obtaining efficient solutions to many important real-world problems. We therefore hope that this article will help bridge the gap between theory and practice by providing a detailed overview, accessible to a broad control engineering audience, of the theoretical advances.

Stability for Arbitrary Switching

It is easy to see that if the family of systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P} \tag{3}$$

has a common Lyapunov function (i.e., a positive definite radially unbounded smooth function V such that $\nabla V(x) f_p(x) < 0$ for all $x \neq 0$ and all $p \in \mathcal{P}$), then the switched system (1) is asymptotically stable for any switching signal σ. Hence, one possible approach to Problem A is to find conditions under which there exists a common Lyapunov function for the family (3).

In the next two subsections, we discuss various results on common Lyapunov functions and stability for arbitrary switching. The third subsection is devoted to converse Lyapunov theorems. We refer the reader to [12]-[14] for some related results not covered here. Another line of research that appears to be relevant is the work on connective stability—see [15] and the references therein. Our discussion throughout the article is restricted to state-space methods. For some frequency domain results, the reader may consult Hespanha [16, Chapter 3], where it is shown that if a linear process and a family of linear controllers are given by their transfer matrices, there always exist realizations such that the family of feedback connections of the process with the controllers possesses a quadratic common Lyapunov function.

Commutation Relations and Stability

Let us start by considering the family of linear systems

$$\dot{x} = A_p x, \quad p \in \mathcal{P}, \tag{4}$$

such that the matrices A_p are stable (i.e., with eigenvalues in the open left half of the complex plane) and the set $\{A_p : p \in \mathcal{P}\}$ is compact in $\mathbb{R}^{n \times n}$. If all the systems in this family share a quadratic common Lyapunov function, the switched linear system (2) is globally uniformly exponentially stable (the word "uniform" is used here to describe uniformity with respect to switching signals). This means that if there exist two symmetric positive definite matrices P and Q such that we have

$$A_p^T P + PA_p \leq -Q \quad \forall p \in \mathcal{P},$$

then there exist positive constants c and μ such that the solution of (2) for any initial state $x(0)$ and any switching signal σ satisfies

$$\|x(t)\| \leq ce^{-\mu t}\|x(0)\| \quad \forall t \geq 0. \tag{5}$$

One could also define the property of uniform (over the set of all switching signals) asymptotic stability, local or global. For linear systems all of the above properties are equivalent; see, e.g., [17] for more information on this.

In this subsection we present sufficient conditions for uniform exponential stability that involve the commutation relations among the matrices A_p, $p \in \mathcal{P}$. The simplest case is when these matrices commute pairwise (i.e., $A_pA_q = A_qA_p$ for all $p, q \in \mathcal{P}$) and $\mathcal{P}$ is a finite set, say, $\mathcal{P} = \{1,\ldots,m\}$. It is not difficult to check that in this case the system (2) is asymptotically stable for any switching signal σ. Taking for simplicity $\mathcal{P} = \{1, 2\}$, it is enough to write $e^{A_1 t_1} e^{A_2 \tau_1} e^{A_1 t_2} e^{A_2 \tau_2} \cdots e^{A_1 t_k} e^{A_2 \tau_k} x(0) = e^{A_1(t_1+\ldots+t_k)} e^{A_2(\tau_1+\ldots+\tau_k)} x(0) \to 0$ as $t_1+\ldots+t_k+\tau_1+\ldots+\tau_k \to \infty$ because A_1 and A_2 are both stable. An explicit construction of a quadratic common Lyapunov function for a finite commuting family of linear systems is given in [18].

Proposition 1 [18] Let $P_1, \ldots, P_m$ *be the unique symmetric positive definite matrices that satisfy the Lyapunov equations*

$$A_1^T P_1 + P_1 A_1 = -I,$$
$$A_i^T P_i + P_i A_i = -P_{i-1}, \quad i = 2,\ldots,m.$$

Then the function $V(x) := x^T P_m x$ *is a common Lyapunov function for the systems* $\dot{x} = A_i x$, $i = 1,\ldots,m$.

The matrix P_m is given by the formula

$$P_m = \int_0^\infty e^{A_m^T t_m} \ldots \left(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1 \right) \ldots e^{A_m t_m} dt_m.$$

Since the matrices A_i commute, for each $i \in \{1,\ldots,m\}$ we can rewrite this in the form

$$P_m = \int_0^\infty e^{A_i^T t_i} \, Q_i \, e^{A_i t_i} \, dt_i$$

with $Q_i > 0$, which makes the claim of Proposition 1 obvious.

Recent work reported in [19] directly generalizes the result and the proof technique of [18] to the switched nonlinear system (1). For each $p \in \mathcal{P}$, denote by $\varphi_p(t, z)$ the solution of the system $\dot{x} = f_p(x)$ starting at a point z when $t = 0$. Suppose that all these systems are exponentially stable, and that the corresponding vector fields commute pairwise (which can be expressed as $[f_p, f_q](x) := \frac{\partial f_q(x)}{\partial x} f_p(x) - \frac{\partial f_p(x)}{\partial x} f_q(x) = 0$ for all $p, q \in \mathcal{P}$). Take $\mathcal{P} = \{1,\ldots,m\}$. Then a common Lyapunov function for the family (3) locally in the neighborhood of the origin[1] can be constructed by the following iterative procedure (where T is a sufficiently large positive constant).

Proposition 2 [19] *Define the functions*

$$V_1(x) := \int_0^T |\varphi_1(s,x)|^2 ds,$$
$$V_i(x) := \int_0^T V_{i-1}(\varphi_i(s,x))ds, \quad i = 2,\ldots,m. \tag{6}$$

Then V_m *is a local common Lyapunov function for the systems* $\dot{x} = f_i(x)$, $i = 1,\ldots,m$.

Let us return to the linear case. A useful object that reveals the nature of the commutation relations among the matrices A_p, $p \in \mathcal{P}$ is the matrix Lie algebra $g := \{A_p : p \in \mathcal{P}\}_{LA}$ generated by them (with respect to the standard Lie bracket $[A_p, A_q] := A_pA_q - A_qA_p$). This is the linear space (over $\mathbb{R}$) spanned by iterated Lie brackets of these matrices. First, we recall some definitions (see, for example, [20]). If g_1 and g_2 are linear subspaces of a Lie algebra g, one writes $[g_1, g_2]$ for the linear space spanned by all the products $[a, b]$ with $a \in g_1$ and $b \in g_2$. Given a Lie algebra g, the sequence $g^{(k)}$ is defined inductively as follows: $g^{(1)} := g$, $g^{(k+1)} := [g^{(k)}, g^{(k)}] \subset g^{(k)}$. If $g^{(k)} = 0$ for k sufficiently large, g is called *solvable*. Similarly, one defines the sequence g^k by $g^1 := g$, $g^{k+1} := [g, g^k] \subset g^k$ and calls g *nilpotent* if $g^k = 0$ for k sufficiently large. For example, if g is a Lie algebra generated by two matrices A_1 and A_2, i.e., $g = \{A_1, A_2\}_{LA}$, we have:

$$g^{(1)} = g^1 = g = \mathrm{span}\{A_1, A_2, [A_1, A_2], [A_1, [A_1, A_2]], \ldots\},$$
$$g^{(2)} = g^2 = \mathrm{span}\{[A_1, A_2], [A_1, [A_1, A_2]], \ldots\},$$
$$g^{(3)} = \mathrm{span}\{[[A_1, A_2], [A_1, [A_1, A_2]]], \ldots\} \subset g^3$$
$$= \mathrm{span}\{[A_1, [A_1, A_2]], [A_2, [A_1, A_2]], \ldots\},$$

and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

The connection between asymptotic stability of a switched linear system and the properties of the corresponding Lie algebra was explicitly discussed for the first time by Gurvits in [21]. That paper is concerned with the discrete-time counterpart of (2), which takes the form

$$x(k+1) = A_{\sigma(k)} x(k), \tag{7}$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{P}$ and $A_p = e^{L_p}$, $p \in \mathcal{P}$ for some matrices L_p. Gurvits conjectured that if the Lie algebra $\{L_p : p \in \mathcal{P}\}_{LA}$ is nilpotent, the system

[1] If all the functions f_p are globally Lipschitz, one obtains a global common Lyapunov function.

(7) is asymptotically stable for any switching signal σ. He was able to prove this conjecture for the particular case where $\mathcal{P} = \{1,2\}$ and the third-order Lie brackets vanish: $[L_1,[L_1,L_2]] = [L_2,[L_1,L_2]] = 0$.

It was recently shown in [22] that if the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, the family (4) possesses a quadratic common Lyapunov function. One can derive the corresponding statement for the discrete-time case in similar fashion, thereby confirming and directly generalizing the above conjecture. The proof of the result given in [22] relied on the facts that matrices in a solvable Lie algebra can be simultaneously put in the upper-triangular form, and that a family of linear systems with stable upper-triangular matrices has a quadratic common Lyapunov function. This result incorporates the ones mentioned before (for a commuting family and a family generating a nilpotent Lie algebra) as special cases.

Theorem 3 [22] *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of stable matrices and the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable,[2] the switched linear system (2) is globally uniformly exponentially stable.*

Various conditions for simultaneous triangularizability are reviewed in [24], [25]. Note, however, that while it is a nontrivial matter to find a basis in which all matrices take the triangular form or even to decide whether such a basis exists, the Lie-algebraic condition given by Theorem 3 is formulated in terms of the original data and can always be checked in a finite number of steps if P is a finite set.

We now briefly discuss an implication of this result for switched nonlinear systems of the form (1). Consider, together with the family (3), the corresponding family of linearized systems

$$\dot{x} = F_p x, \quad p \in \mathcal{P},$$

where $F_p = \frac{\partial f_p}{\partial x}(0)$. Assume that the matrices F_p are stable, that $\mathcal{P}$ is a compact set, and that $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p. A straightforward application of Theorem 3 and Lyapunov's first method (see, for example, [26]) gives the following result.

Corollary 4 [22] *If the Lie algebra $\{F_p : p \in \mathcal{P}\}_{LA}$ is solvable, the system (1) is locally uniformly exponentially stable.*[3]

Note that while the condition of Corollary 4 involves the linearizations, the commuting condition of Proposition 2 is formulated in terms of the original nonlinear vector fields f_p, $p \in \mathcal{P}$. An important problem for future research is to investigate how the structure of the Lie algebra generated by these nonlinear vector fields is related to stability properties of the switched system (1). Taking higher-order terms into account, one may hope to obtain conditions that guarantee global stability of general switched nonlinear systems, or conditions that guarantee at least local stability when the above linearization test fails.

Finally, we comment on the issue of robustness. Both exponential stability and the existence of a quadratic common Lyapunov function are robust properties in the sense that they are not destroyed by sufficiently small perturbations of the systems' parameters. Regarding perturbations of upper-triangular matrices, one can obtain explicit bounds that have to be satisfied by the elements below the diagonal so that the quadratic common Lyapunov function for the unperturbed systems remains a common Lyapunov function for the perturbed ones [27]. Unfortunately, Lie-algebraic conditions, such as the one given by Theorem 3, do not have this robustness property.

Matrix Pencil Conditions

We now turn to some recently obtained sufficient, as well as necessary and sufficient, conditions for the existence of a quadratic common Lyapunov function for a pair of second-order asymptotically stable linear systems

$$\dot{x} = A_i x, \quad A_i \in \mathbb{R}^{2\times 2}, \ i = 1,2. \tag{8}$$

These conditions, presented in [24], [28], are given in terms of eigenvalue locations of convex linear combinations of the matrices A_1 and A_2.

Given two matrices A and B, the *matrix pencil* $\gamma_\alpha(A, B)$ is defined as the one-parameter family of matrices $\alpha A + (1-\alpha)B$, $\alpha \in [0,1]$. One obtains the following result.

Proposition 5 [28] *If A_1 and A_2 have real distinct eigenvalues and all the matrices in $\gamma_\alpha(A_1, A_2)$ have negative real eigenvalues, the pair of linear systems (8) has a quadratic common Lyapunov function.*

Shorten and Narendra [24] considered, together with the matrix pencil $\gamma_\alpha(A_1, A_2)$, the matrix pencil $\gamma_\alpha(A_1, A_2^{-1})$. They obtained the following *necessary and sufficient* condition for the existence of a quadratic common Lyapunov function.

Theorem 6 [24] *The pair of linear systems (8) has a quadratic common Lyapunov function if and only if all the matrices in $\gamma_\alpha(A_1, A_2)$ and $\gamma_\alpha(A_1, A_2^{-1})$* are stable.

The above results are limited to a pair of second-order linear systems. Observe that the conditions of Proposition 5 and Theorem 6 are in general robust in the sense specified at the end of the previous subsection. Indeed, the property that all eigenvalues of a matrix have negative real parts is preserved under sufficiently small perturbations. Moreover, if these eigenvalues are real, they will remain real under small perturbations, provided that they are distinct (because eigenvalues of a real matrix come in conjugate pairs).

Converse Lyapunov Theorems

In the preceding subsections we have relied on the fact that the existence of a common Lyapunov function implies asymptotic stability of the switched system, uniform over the set of all switching signals. The question arises whether the converse holds. A converse Lyapunov theorem for differential inclusions proved by Molchanov and Pyatnitskiy [17] gives a positive answer to this question. Their result can be adapted to the present setting as follows.

Theorem 7 [17] *If the switched linear system (2) is uniformly exponentially stable, the family of linear systems (4) has a strictly convex, homogeneous (of second order) common Lyapunov function of a quasi-quadratic form*

$$V(x) = x^T L(x) x,$$

where $L(x) = L^T(x) = L(\tau x)$ for all nonzero $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$.

The construction of such a Lyapunov function given in [17] (see also [1]) proceeds in the same spirit as the classical one that

[2]This condition has recently been improved by Agrachev [23]. The new result states, loosely speaking, that one still has uniform exponential stability if the above Lie algebra is a sum of two components, one solvable and the other consisting of matrices with purely imaginary eigenvalues.

[3]This is to say that the estimate (5) holds for all trajectories starting in a certain neighborhood of the origin.

is used to prove standard converse Lyapunov theorems (cf. [26, Theorem 4.5]), except that one must take the supremum of the usual expression, of the form (6), over all indices $p \in P$. It is also shown in [17] that one can find a common Lyapunov function that takes the piecewise quadratic form

$$V(x) = \max_{1 \le i \le k} \langle l_i, x \rangle^2,$$

where l_i, $i = 1, \ldots, k$ are constant vectors. A typical level set of such a function is shown in Fig. 3.

Interestingly, a *quadratic* common Lyapunov function does not always exist. Dayawansa and Martin [1] give an example of two second-order linear systems that do not share any quadratic common Lyapunov function, yet the switched system is uniformly exponentially stable. This result clearly shows that conditions leading to the existence of a quadratic common Lyapunov function (such as the condition of Theorem 3) cannot be necessary conditions for asymptotic stability. It is also shown in [1] that Theorem 7 can be generalized to a class of switched nonlinear systems as follows.

Theorem 8 [1] *If the switched system (1) is globally uniformly asymptotically stable and in addition locally uniformly exponentially stable, the family (3) has a common Lyapunov function.*

The converse Lyapunov theorem of Lin et al., is also relevant in this regard [29].

Stability for Slow Switching

We have seen above that a switched system might become unstable for certain switching signals, even if all the individual subsystems are asymptotically stable. Thus, if the goal is to achieve stability of the switched system, one often needs to restrict the class of admissible switching signals. This leads us to Problem B. As already mentioned, one way to address this problem is to make sure that the intervals between consecutive switching times are large enough. Such slow switching assumptions greatly simplify the stability analysis and are, in one form or another, ubiquitous in the switching control literature (see, for example, [30]-[32]).

Below we discuss multiple Lyapunov function tools that are useful in analyzing stability of slowly switched systems (and will also play a role later in the article). We then present stability results for such systems. Some of these results parallel the more familiar ones on stability of slowly time-varying systems (cf. [33] and references therein).

Multiple Lyapunov Functions

We discussed various situations in which asymptotic stability of a switched system for arbitrary switching signals can be established by means of showing that the family of individual subsystems possesses a common Lyapunov function. We also saw that the existence of a common Lyapunov function is necessary for asymptotic stability under arbitrary switching. However, if the class of switching signals is restricted, this converse result might not hold. In other words, the properties of admissible switching signals can sometimes be used to prove asymptotic stability of the switched system, even in the absence of a common Lyapunov function.

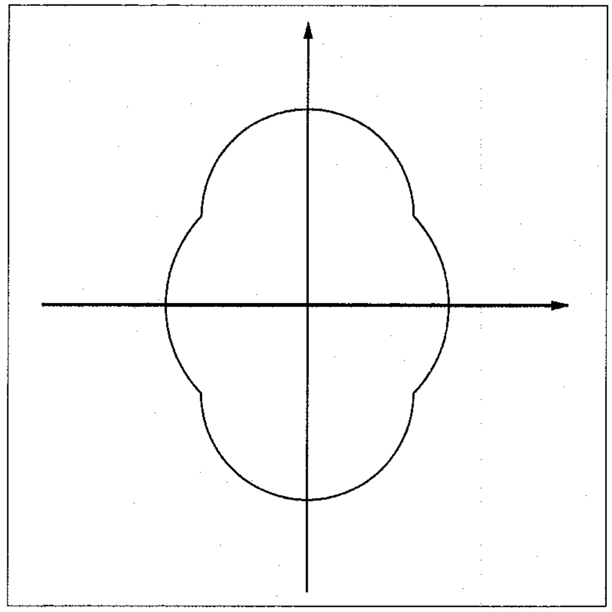

Fig. 3. A level set of a common Lyapunov function.

One tool for proving stability in such cases employs multiple Lyapunov functions. Take $\mathcal{P}$ to be a finite set. Fix a switching signal σ with switching times $t_0 < t_1 < \ldots$ and assume for concreteness that it is continuous from the right everywhere: $\sigma(t_i) = \lim_{t \to t_i^+} \sigma(t)$ for each i. Since the individual systems in the family (3) are assumed to be asymptotically stable, there is a family of Lyapunov functions $\{V_p : p \in \mathcal{P}\}$ such that the value of V_p decreases on each interval where the p-th subsystem is active. If for every p the value of V_p at the end of each such interval exceeds the value at the end of the next interval on which the p-th subsystem is active (see Fig. 4), the switched system can be shown to be asymptotically stable. The precise statement is as follows.

Lemma 9 [34] *Suppose that there exists a constant $\rho > 0$ with the property that for any two switching times t_i and t_j such that $i < j$ and $\sigma(t_i) = \sigma(t_j)$ we have*

$$V_{\sigma(t_j)}(x(t_{j+1})) - V_{\sigma(t_i)}(x(t_{i+1})) \le -\rho |x(t_{i+1})|^2.$$

Then the switched system (1) is globally asymptotically stable.

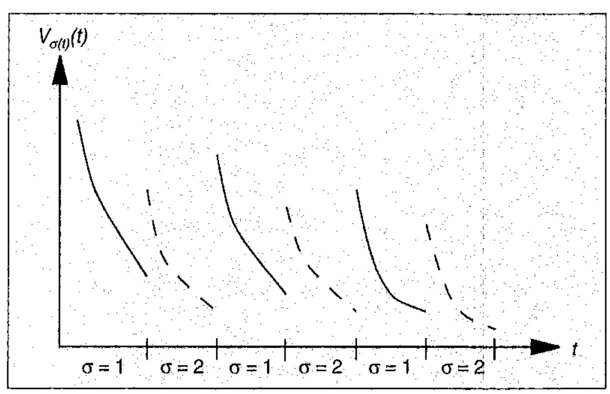

Fig. 4. Two Lyapunov functions (solid graphs correspond to V_1, dashed graphs corresponds to V_2).

To see why this is true, observe that due to the finiteness of $\mathcal{P}$ there exists an index $p \in \mathcal{P}$ that has associated with it an infinite sequence of switching times $t_{i_1}, t_{i_2}, \ldots$ such that $\sigma(t_{i_j}) = p$. The sequence $V_{\sigma(t_{i_1})}(x(t_{i_1+1}))$, $V_{\sigma(t_{i_2})}(x(t_{i_2+1}))$, ... is decreasing and positive, and therefore has a limit $L \geq 0$. We have

$$\begin{aligned} 0 = L - L &= \lim_{j \to +\infty} V_{\sigma(t_{i_{j+1}})}(x(t_{i_{j+1}+1})) - \lim_{j \to +\infty} V_{\sigma(t_{i_j})}(x(t_{i_j+1})) \\ &= \lim_{j \to +\infty} [V_{\sigma(t_{i_{j+1}})}(x(t_{i_{j+1}+1})) - V_{\sigma(t_{i_j})}(x(t_{i_j+1}))] \\ &\leq \lim_{j \to +\infty} [-\rho |x(t_{i_j+1})|^2] \leq 0, \end{aligned}$$

which implies that $x(t)$ converges to zero. As pointed out in [35], Lyapunov stability should and can be checked via a separate argument.

Some variations and generalizations of this result are discussed in [35]-[39]. See also the work reported in [3] for an application of similar ideas to the problem of controlling a mobile robot. A closely related problem of computing multiple Lyapunov functions numerically using linear matrix inequalities is addressed in [39] and [40].

Dwell Time

The simplest way to specify slow switching is to introduce a number $\tau > 0$ and restrict the class of admissible switching signals to signals with the property that the interval between any two consecutive switching times is no smaller than τ. This number τ is sometimes called the *dwell time* (because σ "dwells" on each of its values for at least τ units of time). It is a fairly well-known fact that when all the linear systems in the family (4) are asymptotically stable, the switched linear system (2) is globally exponentially stable if the dwell time τ is large enough. In fact, the required lower bound on τ can be explicitly calculated from the parameters of the individual subsystems. For details, see [4, Lemma 2].

What is perhaps less well known is that under suitable assumptions a sufficiently large dwell time also guarantees asymptotic stability of the switched system in the nonlinear case. Arguably the best way to prove most general results of this kind is by using multiple Lyapunov functions. We will not discuss the precise assumptions that are needed here (in fact, there is considerable work still to be done in that regard), but will present the general idea instead. Assume for simplicity that all the systems in the family (3) are globally exponentially stable. Then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p that for some positive constants a_p, b_p and c_p satisfies

$$a_p |x|^2 \leq V_p(x) \leq b_p |x|^2 \tag{9}$$

and

$$\nabla V_p(x) f_p(x) \leq -c_p |x|^2 \tag{10}$$

(see, for example, [26, Theorem 4.5]). Combining (9) and (10), we obtain

$$\nabla V_p(x) f_p(x) \leq -\lambda_p V_p(x), \quad p \in \mathcal{P},$$

where $\lambda_p = c_p / b_p$. This implies that

$$V_p(x(t_0 + \tau)) \leq e^{-\lambda_p \tau} V_p(x(t_0)),$$

provided that $\sigma(t) = p$ for almost all $t \in [t_0, t_0 + \tau]$. To simplify the next calculation, let us consider the case where $\mathcal{P} = \{1,2\}$ and σ takes on the value 1 on $[t_0, t_1)$ and 2 on $[t_1, t_2)$, where $t_{i+1} - t_i \geq \tau$, $i = 0,1$. From the above inequalities one has

$$V_2(t_1) \leq \frac{b_2}{a_1} V_1(t_1) \leq \frac{b_2}{a_1} e^{-\lambda_1 \tau} V_1(t_0),$$

and furthermore

$$V_1(t_2) \leq \frac{b_1}{a_2} V_2(t_2) \leq \frac{b_1}{a_2} e^{-\lambda_2 \tau} V_2(t_1) \leq \frac{b_1 b_2}{a_1 a_2} e^{-(\lambda_1 + \lambda_2)\tau} V_1(t_0).$$

We see that $V_1(t_2) < V_1(t_0)$ if τ is large enough. In fact, it is not hard to compute an explicit lower bound on τ that ensures that the hypotheses of Lemma 9 are satisfied, which means that the switched system is globally asymptotically stable.

We do not discuss possible extensions and refinements here because a more general result will be stated in the next subsection. Note, however, that the exponential stability assumption is not necessary; for example, the above reasoning would still be valid if the quadratic estimates in (9) and (10) were replaced by, say, quartic ones. In essence, all we used was the fact that

$$\mu := \sup \left\{ \frac{V_p(x)}{V_q(x)} : x \in \mathbb{R}^n,\ p, q \in \mathcal{P} \right\} < \infty. \tag{11}$$

If this inequality does not hold globally in the state space, only local asymptotic stability can be established.

Average Dwell Time

For each $T > 0$, let $N_\sigma(T)$ denote the number of discontinuities of a given switching signal σ on the interval $[0, T)$. Following Hespanha [41], we will say that σ has the *average dwell time* property if there exist two nonnegative numbers a and b such that for all $T > 0$ we have $N_\sigma(T) \leq a + bT$. This terminology is prompted by the observation that, if we discard the first a switchings, the average time between consecutive switchings is at least $1/b$. Dwell-time switching signals considered in the previous subsection satisfy this definition with $a = 0$ and $b = 1/\tau$. Loosely speaking, while the counterpart of a dwell-time switching signal for continuously time-varying systems is a tuning signal with bounded derivative, the counterpart of an average dwell-time switching signal is a *nondestabilizing* tuning signal in the sense of [42].

Consider the family of nonlinear systems (3), and assume that all the systems in this family are globally asymptotically stable. Then for each $p \in \mathcal{P}$ there exist positive definite, radially unbounded smooth functions V_p and D_p such that $\nabla V_p(x) f_p(x) \leq -D_p(x)$ for all x. As explained in [43], there is no loss of generality in taking $D_p(x) = \lambda_p V_p(x)$ for some $\lambda_p > 0$ (changing V_p if necessary). Since $\mathcal{P}$ is a compact set, we can also assume that the numbers λ_p are the same for all $p \in \mathcal{P}$, so that we have

$$\nabla V_p(x) f_p(x) \leq -\lambda V_p(x), \quad \lambda > 0. \tag{12}$$

The following result was recently proved by Hespanha [41] with the help of Lyapunov function techniques similar to those we alluded to in the previous subsection.

Theorem 10 [41] *If (11)-(12) hold, the switched system (1) is globally asymptotically stable*[4] *for any switching signal that has the average dwell time property with* $b < \lambda / \log\mu$.

The study of average dwell-time switching signals is motivated by the following considerations. Stability problems for switched systems arise naturally in the context of switching control. Switching control techniques employing a dwell time have been successfully applied to linear systems with imprecise measurements or modeling uncertainty (cf. [4], [30], [31], [44]). In the nonlinear setting, however, such methods are often unsuitable because of the possibility of finite escape time. Namely, if a "wrong" controller has to remain in the loop with an imprecisely modeled system for a specified amount of time, the solution to the system might escape to infinity before we switch to a different controller (of course, this will not happen if all the controllers are stabilizing, but when the system is not completely known, such an assumption is not realistic).

An alternative to dwell-time switching control of nonlinear systems is provided by the so-called *hysteresis* switching proposed in [45] and its scale-independent versions, which were recently introduced and analyzed in [16] and [46] and applied to control of uncertain nonlinear systems in [11] and [47]. When the uncertainty is purely parametric and there is no measurement noise, switching signals generated by scale-independent hysteresis have the property that the switching stops in finite time, whereas in the presence of noise under suitable assumptions they can be shown to have the average dwell time property (see [41]). Thus Theorem 10 opens the door to provably correct stabilization algorithms for uncertain nonlinear systems corrupted by noise, which is the subject of ongoing research.

Stabilizing Switching Signals

Since some switching signals lead to instability, it is natural to ask, given a family of systems, whether it is possible to find a switching signal that renders the switched system asymptotically stable. Such stabilizing switching signals may exist even in the extreme situation when all the individual subsystems are unstable. For example, consider two second-order systems whose trajectories are sketched in Fig. 5, left, and Fig. 5, center. If we switch in such a way that the first system is active in the second and fourth quadrants while the second one is active in the first and third quadrants, the switched system will be asymptotically stable (see Fig. 5, right). Note that this switching control strategy (as well as the ones to be discussed below) is closed-loop, i.e., the switching signal takes the feedback form $\sigma(x)$. A less commonly considered alternative is to employ open-loop switching signals (for example, periodic ones [48]-[50]). All switching control strategies discussed here are deterministic; for available results concerning stochastic switching, the reader may consult [51, Chapter 9], [52] and the references therein.

In this section we present various methods for constructing stabilizing switching signals in the case where none of the individual subsystems are asymptotically stable (Problem C). We also discuss how these ideas apply to the problem of stabilizing a linear system with finite-state hybrid output feedback. Although we only address stabilizability here, there are other interesting questions, such as attainability and optimal control via switching [53], [54], exact output tracking [55], [56], and switched observer design [57], [58].

Stable Convex Combinations

Assume that $\mathcal{P} = \{1,2\}$ and that we are switching between two linear systems

$$\dot{x} = A_1 x \tag{13}$$

and

$$\dot{x} = A_2 x \tag{14}$$

of arbitrary dimension n. As demonstrated by Wicks, Peleties, and DeCarlo in [2], [59], one assumption that leads to an elegant construction of a stabilizing switching signal is the following one.

Assumption 1. The matrix pencil $\gamma_\alpha(A_1, A_2)$ contains a stable matrix.

According to the definition of a matrix pencil given earlier, this means that for some $\alpha \in (0,1)$, the convex combination $A := \alpha A_1 + (1-\alpha)A_2$ is stable (the endpoints 0 and 1 are excluded because A_1 and A_2 are not stable). Thus there exist symmetric positive definite matrices P and Q such that we have

$$A^T P + PA = -Q.$$

This can be rewritten as

$$\alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) = -Q$$

or

$$\alpha x^T(A_1^T P + PA_1)x + (1-\alpha)x^T(A_2^T P + PA_2)x = -x^T Qx < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Since $0 < \alpha < 1$, it follows that for each nonzero $x \in \mathbb{R}^n$, at least one of the quantities $x^T(A_1^T P + PA_1)x$ and $x^T(A_2^T P + PA_2)x$ is negative. In other words, $R^n \setminus \{0\}$ is covered by the union of two open conic regions $\Omega_1 := \{x : x^T(A_1^T P + PA_1)x < 0\}$ and $\Omega_2 := \{x : x^T(A_2^T P + PA_2)x < 0\}$. The function $V(x) := x^T Px$ decreases along solutions of the system (13) in the region Ω_1 and decreases along solutions of the system (14) in the region Ω_2. Using this property, it is possible to construct a switching signal such that V decreases along solutions of the switched system, which implies asymptotic stability. The precise result is this.

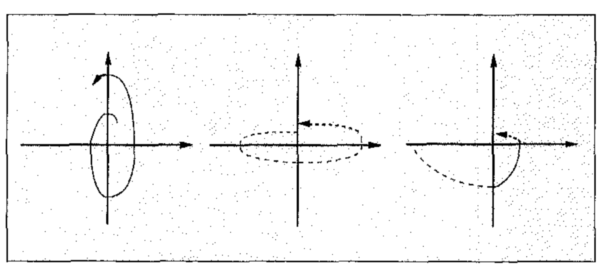

Fig. 5. A stabilizing switching signal.

[4]If *exponential* stability of the switched system is desired, certain specific growth bounds on the functions V_p must be imposed.

Theorem 11 [2], [59] *If Assumption 1 is satisfied, there exists a piecewise constant switching signal which makes the switched system quadratically stable.* ("Quadratic stability" means that there exists a positive ε such that $\dot{V} < -\varepsilon x^T x$.)

This stabilizing switching signal takes the state feedback form; i.e., the value of σ at any given time $t \geq 0$ depends on $x(t)$. An interesting observation due to Feron [60] is that Assumption 1 is not only sufficient but also necessary for quadratic stabilizability via switching.

Proposition 12 [60] *If there exists a quadratically stabilizing switching signal in the state feedback form, the matrices A_1 and A_2 satisfy Assumption 1.*

One can gain insight into the issue of quadratic stabilizability with the help of the following example. Take

$$A_1 := \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}$$

and

$$A_2 := \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}.$$

The trajectories of the systems (13) and (14) will then look, at least qualitatively, as depicted in Fig. 5, left and center, respectively. We explained earlier how to construct a stabilizing switching signal that yields the switched system with trajectories as shown in Fig. 5, right. This system is asymptotically stable; in fact, we see that the function $V(x_1, x_2) := x_1^2 + x_2^2$ decreases along solutions. However, it is easy to check that no convex combination of A_1 and A_2 is stable, and Proposition 12 tells us that the switched system cannot be quadratically stable. Indeed, on the coordinate axes (which form the set where the switching occurs) we have $\dot{V} = 0$.

When the number of individual subsystems is greater than 2, one can try to single out from the corresponding set of matrices a pair that has a stable convex combination (an algorithm for doing this is discussed in [59]). If that fails, it might be possible to find a stable convex combination of three or more matrices from the given set, and then the above method for constructing a stabilizing switching signal can still be implemented with minor modifications. Observe that the converse result of Proposition 12 is only known to hold for the case of two systems. We note that the problem of identifying stable convex combinations (of matrices with rational coefficients) is NP-hard [61]. A discussion of computational issues associated with some problems related to the one addressed in this section, as well as relevant bibliography, can be found in Chapters 11 and 14 of [62].

Unstable Convex Combinations

The previous example suggests that even when there exists no stable convex combination of A_1 and A_2, and thus quadratic stabilization is impossible, asymptotic stabilization may be quite easy to achieve (by using techniques that can actually be applied to general systems, not necessarily linear ones). An interesting source of motivation for pursuing this idea comes from the following problem. Suppose that we are given a linear time-invariant control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{15}$$

that is *stabilizable* and *detectable* (i.e., there exist matrices F and K such that the eigenvalues of $A + BF$ and $A + KC$ have negative real parts). Then, as is well known, there exists a continuous lin-

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ear dynamic output feedback law that asymptotically stabilizes the system (see, for example, [63, Section 6.4]). In practice, however, such a continuous dynamic feedback might not be implementable, and a suitable discrete version of the controller is often desired. Recent references [44], [64]-[67] discuss some issues related to control of continuous plants by various types of discontinuous feedback.

In particular, in [65] it is shown that the system (15) admits a stabilizing hybrid output feedback that uses a countable number of discrete states. A logical question to ask next is whether it is possible to stabilize (15) by using a hybrid output feedback with only a *finite* number of discrete states. Artstein [68] explicitly raised this question and discussed it in the context of a simple example (that paper can also be consulted for a formal definition of hybrid feedback). This problem seems to require a solution that is significantly different from the ones mentioned above, because a finite-state stabilizing hybrid feedback is unlikely to be obtained from a continuous one by means of any discretization process.

One approach to the problem of stabilizing the linear system (15) via finite-state hybrid output feedback is prompted by the following observation. Suppose that we are given a collection of gain matrices $K_1, \ldots, K_m$ of suitable dimensions. Setting $u = K_i y$ for some $i \in \{1,\ldots,m\}$, we obtain the system

$$\dot{x} = (A + BK_iC)x.$$

Thus the stabilization problem for the original system (15) will be solved if we can orchestrate the switching between the systems in the above form in such a way as to achieve asymptotic stability. Denoting $A + BK_iC$ by A_i for each $i \in \{1,\ldots,m\}$, we are led to the following question: Using the measurements of the output $y = Cx$, can we find a switching signal σ such that the switched system $\dot{x} = A_\sigma x$ is asymptotically stable? The value of σ at a given time t might just depend on t and/or $y(t)$, or a more general hybrid feedback may be used. We are assuming, of course, that none of the matrices A_i are stable, as the existence of a matrix K such that the eigenvalues of $A + BKC$ have negative real parts would render the problem trivial.

First of all, observe that the existence of a stable convex combination $A := \alpha A_i + (1-\alpha)A_j$ for some $i, j \in \{1,\ldots,m\}$ and $\alpha \in (0,1)$ would imply that the system (15) can be stabilized by the linear static output feedback $u = Ky$ with $K := \alpha K_i + (1-\alpha)K_j$, contrary to the assumption that we just made. In view of Proposition 12, this implies that a quadratically stabilizing switching signal does not exist. However, it might still be possible to construct an asymptotically stabilizing switching signal and even base a stability proof on a single Lyapunov function.

To illustrate this point, we discuss a modified version of the stabilizing switching strategy for the harmonic oscillator with position measurements described in [68]. Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$
$$y = x_1.$$

Although this system is both controllable and observable, it cannot be stabilized by (even discontinuous) static output feedback. On the other hand, it can be stabilized by hybrid output feedback; several ways to do this were presented in [68]. We will now sketch one possible stabilizing strategy. Letting $u = -y$ we obtain the system

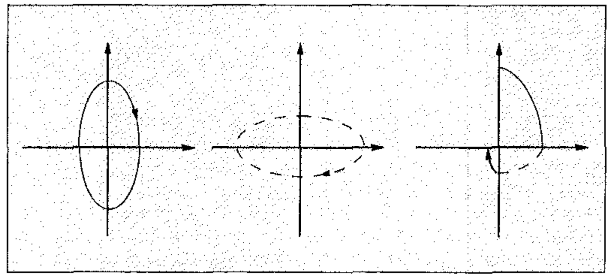

Fig. 6. Stabilizing the harmonic oscillator.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{16}$$

whose trajectories look as shown in Fig. 6, left, while letting $u = \frac{1}{2}y$ we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{17}$$

whose trajectories look as shown in Fig. 6, center.

Define $V(x) := x_1^2 + x_2^2$. This function decreases along the solutions of (16) when $x_1x_2 > 0$ and decreases along the solutions of (17) when $x_1x_2 < 0$. Therefore, if the system (16) is active in the first and third quadrants, while the system (17) is active in the second and fourth quadrants, we will have $\dot{V} < 0$ whenever $x_1x_2 \neq 0$; hence the switched system is asymptotically stable by LaSalle's principle. A possible trajectory of the switched system is sketched in Fig. 6, right. (This situation is similar to the one shown in Fig. 5, except that here the individual subsystems are critically stable.) It is important to notice that, since both systems being switched are linear time-invariant, the time between a crossing of the x_1-axis and the next crossing of the x_2-axis can be explicitly calculated and is independent of the trajectory. This means that the above switching strategy can be implemented via hybrid feedback based just on the measurements of the output; see [68] and [69] for details. The problem of stabilizing second-order switched linear systems was also studied in [70] and [71].

If one cannot carry out the stability analysis with the help of a single Lyapunov function that decreases along the trajectories of the switched system, in view of the results presented earlier it might still be possible to find a stabilizing switching signal and prove stability by using multiple Lyapunov-like functions. Although this line of thinking does not seem to lead to such a simple and constructive procedure as the one described in [2] and [59], some preliminary ideas have been explored in the literature. These are discussed next.

The method proposed in [34] is to associate with the system (13) a function $V_1(x) = x^T P_1 x$ that decreases along solutions in an appropriate region Ω_1. This is always possible unless A_1 is a nonnegative multiple of the identity matrix. Similarly, associate with the system (14) a function $V_2(x) = x^T P_2 x$ that decreases along solutions in an appropriate region Ω_2. If the union of the regions Ω_1 and Ω_2 covers $R^n \setminus \{0\}$, then one can try to orchestrate the switching in such a way that the conditions of Lemma 9 are satisfied. The paper [34] contains an example that illustrates how this stabilizing switching strategy works.

In a more recent paper [72], this investgation is continued with the goal of putting the above idea on more solid ground by means of formulating algebraic sufficient conditions for a switching strategy based on multiple Lyapunov functions to exist. Consider the situation when the following condition holds:

Condition 1. $x^T(P_1A_1 + A_1^TP_1)x < 0$ whenever $x^T(P_1 - P_2)x \geq 0$ and $x \neq 0$, and $x^T(P_2A_2 + A_2^TP_2)x < 0$ whenever $x^T(P_2 - P_1)x \geq 0$ and $x \neq 0$.

If this condition is satisfied, a stabilizing switching signal can be defined by $\sigma(t) := \arg\max\{V_i(x(t)) : i = 1,2\}$. Indeed, the function V_σ will then be continuous and will decrease along solutions of the switched system, which guarantees asymptotic stability. Similar techniques were used independently in [73] in a more general, nonlinear context. That paper shows an application to the interesting problem of stabilizing an inverted pendulum via a switching control strategy.

Condition 1 holds if the following condition is satisfied (by virtue of the S-procedure [74], the two conditions are equivalent, provided that there exist $x_1, x_2 \in R^n$ such that $x_1^T(P_1 - P_2)x_1 > 0$ and $x_2^T(P_2 - P_1)x_2 > 0$).

Condition 2. There exist $\beta_1, \beta_2 \geq 0$ such that $-P_1A_1 - A_1^TP_1 + \beta_1(P_2 - P_1) > 0$ and $-P_2A_2 - A_2^TP_2 + \beta_2(P_1 - P_2) > 0$.

Alternatively, if $\beta_1, \beta_2 \leq 0$, a stabilizing switching signal can be defined by $\sigma(t) := \arg\min\{V_i(x(t)) : i = 1,2\}$. In [72] Condition 2 is further reformulated in terms of eigenvalue locations of certain matrix operators. In [69] it is shown how the above inequalities can be adapted to the context of the finite-state hybrid output feedback stabilization problem discussed earlier. It would be interesting to compare these results with the characterization of stabilizability via switched *state* feedback obtained in [75], and also with the dynamic programming approach presented in [76] and [77].

Concluding Remarks

We have surveyed recent developments in three basic problems regarding switched dynamical systems: stability for arbitrary switching signals, stability for slow switching signals, and construction of stabilizing switching signals. We have aimed at providing an overview of general results and ideas involved. For technical details, the reader may consult the references listed below. These references also address many issues that are relevant to switched systems but fell outside the scope of this survey. Despite a number of interesting results presented here, it is safe to say that the subject is still largely unexplored. Various open questions, some of which we have mentioned in the article, remain to be investigated.

The three problems studied here are very general and address fundamental issues concerning stability and design of switched systems. As we have noted throughout the article, special cases of these problems arise frequently in various contexts associated with control design. In such situations, the specific structure of a problem at hand can sometimes be used to obtain satisfactory results, even in the absence of a general theory. Examples of results that use such additional structure include the so-called Switching Theorem, which plays a role in the supervisory control of uncertain linear systems [4] and conditions for existence of a common Lyapunov function that exploit positive realness [78, Chapter 33], [79], [80]. We believe that to make significant further progress, one must stay in close contact with particular applications that motivate the study of switched systems.

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References

[1] W.P. Dayawansa and C.F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 751-760, 1999.

[2] M.A. Wicks, P. Peleties, and R.A. DeCarlo, "Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems," *European J. Contr.*, vol. 4, pp. 140-147, 1998.

[3] M. Zefran and J.W. Burdick, "Design of switching controllers for systems with changing dynamics," *Proc. 37th Conf. Decision and Contr.*, pp. 2113-2118, 1998.

[4] A.S. Morse, "Supervisory control of families of linear set-point controllers, part 1: Exact matching," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1413-1431, 1996.

[5] S.R. Kulkarni and P.J. Ramadge, "Model and controller selection policies based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1594-1604, 1996.

[6] K.S. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 171-187, 1997.

[7] R.W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, R.W. Brockett, et al., eds., pp. 181-191, Boston: Birkhauser, 1983.

[8] A.S. Morse, ed., *Control Using Logic-Based Switching*, London: Springer-Verlag, 1997.

[9] F.M. Pait and B. Piccoli, "A hybrid controller for a nonholonomic system," *Proc. 30th Conf. Inform. Sci. and Syst.*, pp. 416-420, 1996.

[10] J.P. Hespanha and A.S. Morse, "Stabilization of nonholonomic integrators via logic-based switching," *Automatica*, vol. 35, pp. 385-393, 1999.

[11] J.P. Hespanha, D. Liberzon, and A.S. Morse, "Logic-based switching control of a nonholonomic system with parametric modeling uncertainty," *Systems Control Lett.*, special issue on hybrid systems, to appear.

[12] Y. Mori, T. Mori, and Y. Kuroe, "On a class of linear constant systems which have a common quadratic Lyapunov function," *Proc. 37th Conf. Decision and Control*, pp. 2808-2809, 1998.

[13] T. Ooba and Y. Funahashi, "Two conditions concerning common quadratic Lyapunov functions for linear systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 719-721, 1997.

[14] T. Ooba and Y. Funahashi, "On a common quadratic Lyapunov function for widely distant systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1697-1699, 1997.

[15] D.D. Šiljak and D.M. Stipanović, "Polytopic connective stability," *Proc. IFAC Symposium on Large Scale Systems*, pp. 26-32, 1998.

[16] J.P. Hespanha, "Logic-Based Switching Algorithms in Control," Ph.D. Thesis, Dept. of Electrical Engineering, Yale University, 1998.

[17] A.P. Molchanov and Y.S. Pyatnitskiy, "Criteria of absolute stability of differential and difference inclusions encountered in control theory," *Systems Contr. Lett.*, vol. 13, pp. 59-64, 1989.

[18] K.S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting *A*-matrices," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2469-2471, 1994.

[19] H. Shim, D.J. Noh, and J.H. Seo, "Common Lyapunov function for exponentially stable nonlinear systems," presented at the *4th SIAM Conf. Control and its Applications*, Jacksonville, FL, 1998.

[20] H. Samelson, *Notes on Lie Algebras*, Van Nostrand Reinhold, New York, 1969.

[21] L. Gurvits, "Stability of discrete linear inclusion," *Lin. Alg. Appl.*, vol. 231, pp. 47-85, 1995.

[22] D. Liberzon, J.P. Hespanha, and A.S. Morse, "Stability of switched systems: a Lie-algebraic condition," *Systems Contr. Lett.*, vol. 37, no. 3, pp. 117-122, 1999.

[23] A.A. Agrachev and D. Liberzon, "Lie-algebraic conditions for exponential stability of switched systems," *Proc. 38th Conf. Decision and Control*, 1999, to appear.

[24] R.N. Shorten and K.S. Narendra, "On the existence of a common Lyapunov function for linear stable switching systems," *Proc. 10th Yale Workshop on Adaptive and Learning Systems*, pp. 130-140, 1998.

[25] R.N. Shorten and K.S. Narendra, "On the stability and existence of common Lyapunov functions for stable linear switching systems," *Proc. 37th Conf. Decision and Control*, pp. 3723-3724, 1998.

[26] H.K. Khalil, *Nonlinear Systems*, Macmillan Co., New York, 1992.

[27] Y. Mori, T. Mori, and Y. Kuroe, "A solution to the common Lyapunov function problem for continuous-time systems," *Proc. 36th Conf. Decision and Control*, pp. 3530-3531, 1997.

[28] R.N. Shorten and K.S. Narendra, "A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems," *Proc. 36th Conf. Decision and Control*, pp. 3521-3522, 1997.

[29] Y. Lin, E. D. Sontag, Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control Optim.*, vol. 34, pp. 124-160, 1996.

[30] J. Hocherman-Frommer, S.R. Kulkarni, and P.J. Ramadge, "Controller switching based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 596-607, 1998.

[31] F.M. Pait and F. Kassab, Jr., "Parallel algorithms for adaptive control: Robust stability," in *Control Using Logic-Based Switching*, A.S. Morse, ed., Springer-Verlag, London, 1997.

[32] C.J. Tomlin and S.S. Sastry, "Switching through singularities," *Proc. 36th Conf. Decision and Control*, pp. 1-6, 1997.

[33] V. Solo, "On the stability of slowly time-varying linear systems," *Math. Contr. Signals Systems*, vol. 7, pp. 331-350, 1994.

[34] P. Peleties and R.A. DeCarlo, "Asymptotic stability of *m*-switched systems using Lyapunov-like functions," *Proc. 1991 American Control Conf.*, pp. 1679-1684, 1991.

[35] M.S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 475-482, 1998.

[36] M.S. Branicky, "Stability of switched and hybrid systems," *Proc. 33rd Conf. Decision and Control*, pp. 3498-3503, 1994.

[37] M.S. Branicky, "Stability of hybrid systems: State of the art," *Proc. 36th Conf. Decision and Control*, pp. 120-125, 1997.

[38] L. Hou, A.N. Michel, and H. Ye, "Stability analysis of switched systems," *Proc. 35th Conf. Decision and Control*, pp. 1208-1212, 1996.

[39] S. Pettersson and B. Lennartson, "Stability and robustness for hybrid systems," *Proc. 35th Conf. Decision and Control*, pp. 1202-1207, 1996.

[40] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 555-559, 1998.

[41] J. P. Hespanha, "Stability of switched systems with average dwell time," *Proc. 38th Conf. Decision and Control*, 1999, to appear.

[42] A.S. Morse, "Towards a unified theory of parameter adaptive control: Tunability," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1002-1012, 1990.

[43] L. Praly and Y. Wang, "Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability," *Math. Contr. Signals Systems*, vol. 9, pp. 1-33, 1996.

[44] R.W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, to appear.

[45] A.S. Morse, D.Q. Mayne, and G.C. Goodwin, "Applications of hysteresis switching in parameter adaptive control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1343-1354, 1992.

[46] J.P. Hespanha and A.S. Morse, "Scale-independent hysteresis switching," in *Hybrid Systems: Computation and Control*, F.W. Vaandrager and J.H. van Schuppen, eds., *Lecture Notes in Computer Science*, vol. 1569, Berlin: Springer-Verlag, pp. 117-122, 1999.

[47] J.P. Hespanha and A.S. Morse, "Certainty equivalence implies detectability," *Systems Contr. Lett.*, vol. 36, pp. 1-13, 1999.

[48] M. Araki and T. Hagiwara, "Pole assignment by multirate sampled-data output feedback," *Int. J. Contr.*, vol. 44, 1986, pp. 1661-1673.

[49] D. Aeyels and J.L. Willems, "Pole assignment for linear time-invariant systems by periodic memoryless output feedback," *Automatica*, vol. 28, pp. 1159-1168, 1992.

[50] B.A. Francis and T.T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 820-832, 1988.

[51] D.D. Siljak, *Decentralized Control of Complex Systems*, Boston: Academic Press, 1991.

[52] C.-C. Tsai and A.H. Haddad, "Averaging, aggregation and optimal control of singularly perturbed stochastic hybrid systems," *Int. J. Contr.*, vol. 68, pp. 31-50, 1997.

[53] K.A. Loparo, J.T. Aslanis, and O. Hajek, "Analysis of switching linear systems in the plane, part 1: Local behavior of trajectories and local cycle geometry," *J. Optim. Theory Appl.*, vol. 52, pp. 365-394, 1987.

[54] K.A. Loparo, J.T. Aslanis, and O. Hajek, "Analysis of switching linear systems in the plane, part 2: Global behavior of trajectories, controllability and attainability," *J. Optim. Theory Appl.*, vol. 52, pp. 395-427, 1987.

[55] M. Jirstrand, *Constructive Methods for Inequality Constraints in Control*, Ph.D. Thesis, Linköping University, 1998.

[56] D. Nesic, E. Skafidas, I.M.Y. Mareels, and R.J. Evans, "Analysis of minimum phase properties for non-affine nonlinear systems," *Proc. 36th Conf. Decision and Control*, pp. 606-611, 1997.

[57] Y. Liu, "Switching observer design for uncertain nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1699-1703, 1997.

[58] V. Polotski, "Time-varying state feedback and output injection: Duality and distinctions," *Proc. 35th Conf. Decision and Control*, pp. 2159-2160, 1996.

[59] M.A. Wicks, P. Peleties, and R.A. DeCarlo, "Construction of piecewise Lyapunov functions for stabilizing switched systems," *Proc. 33rd Conf. Decision and Control*, pp. 3492-3497, 1994.

[60] E. Feron, "Quadratic stabilizability of switched systems via state and output feedback," technical report CICS-P-468, MIT, 1996, submitted to *SIAM J. Contr. Optimization*.

[61] V. Blondel and J.N. Tsitsiklis, "NP-hardness of some linear control design problems," *SIAM J. Contr. Optimization*, vol. 35, pp. 2118-2127, 1997.

[62] V.D. Blondel, E.D. Sontag, M. Vidyasagar, and J.C. Willems, eds., *Open Problems in Mathematical Systems and Control Theory*, London: Springer-Verlag, 1999.

[63] W.A. Wolovich, *Linear Multivariable Systems*, New York: Springer-Verlag, 1974.

[64] B. Hu and A.N. Michel, "Stability analysis of digital feedback control systems with time-varying sampling periods," *Automatica*, to appear.

[65] E. Litsyn, Y.V. Nepomnyashchikh, and A. Ponosov, "Stabilization of linear differential systems via hybrid feedback controls," submitted to *SIAM J. Contr. Optimization*.

[66] N.H. McClamroch, C. Rui, I. Kolmanovsky, and M. Reyhanoglu, "Hybrid closed loop systems: A nonlinear control perspective," *Proc. 36th Conf. Decision and Control*, pp. 114-119, 1997.

[67] E.D. Sontag, "Stability and stabilization: Discontinuities and the effect of disturbances," in *Proc. NATO Advanced Study Institute "Nonlinear Analysis, Differential Equations, and Control,"* Kluwer, pp. 551-598, 1999.

[68] Z. Artstein, "Examples of stabilization with hybrid feedback," in *Hybrid Systems III: Verification and Control*, R. Alur, et al., eds., *Lecture Notes in Computer Science*, Berlin: Springer-Verlag, vol. 1066, pp. 173-185, 1996.

[69] D. Liberzon, "Stabilizing a linear system with finite-state hybrid output feedback," *Proc. 7th IEEE Mediterranean Conf. Control and Automation*, Haifa, Israel, 1999, pp. 176-183.

[70] X. Xu and P.J. Antsaklis, "Design of stabilizing control laws for second-order switched systems," *Proc. 14th IFAC World Congress*, 1999, to appear.

[71] B. Hu, X. Xu, P.J. Antsaklis, and A.N. Michel, "Robust stabilizing control law for a class of second-order switched systems," *Systems Contr. Lett.*, to appear.

[72] M.A. Wicks and R.A. DeCarlo, "Solution of coupled Lyapunov equations for the stabilization of multimodal linear systems," *Proc. 1997 American Control Conf.*, pp. 1709-1713, 1997.

[73] J. Malmborg, B. Bernhardsson, and K.J. Åström, "A stabilizing switching scheme for multi controller systems," *Proc. 13th IFAC World Congress*, vol. F, pp. 229-234, 1996.

[74] S. Boyd, L.E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, 1994.

[75] A. Yu. Pogromsky, M. Jirstrand, and P. Spångéus, "On stability and passivity of a class of hybrid systems," *Proc. 37th Conf. Decision and Control*, pp. 3705-3710, 1998.

[76] A.V. Savkin, I.R. Petersen, E. Skafidas, and R.J. Evans, "Hybrid dynamical systems: Robust control synthesis problems," *Systems Contr. Lett.*, vol. 29, pp. 81-90, 1996.

[77] A.V. Savkin, E. Skafidas, and R.J. Evans, "Robust output feedback stabilizability via controller switching," *Automatica*, vol. 35, pp. 69-74, 1999.

[78] R.W. Brockett, *Finite Dimensional Linear Systems*, John Wiley, New York, 1970.

[79] S. Boyd and Q. Young, "Structured and simultaneous Lyapunov functions for system stability problems," *Int. J. Contr.*, vol. 49, pp. 2215-2240, 1989.

[80] H. Kokame and T. Mori, "On quadratic Hurwitz stability of segment polynomials," *Proc. 37th Conf. Decision and Control*, pp. 2806-2808, 1998.