# Stability Analysis and Stabilization of Randomly Switched Systems

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Abstract. This article is concerned with stability analysis and stabilization of randomly switched systems with control inputs. The switching signal is modeled as a jump stochastic process independent of the system state; it selects, at each instant of time, the active subsystem from among a family of deterministic systems. Three different types of switching signals are considered: the first is a jump stochastic process that satisfies a statistically slow switching condition; the second and the third are jump stochastic processes with independent identically distributed values at jump times together with exponential and uniform holding times, respectively. For each of the three cases we first establish sufficient conditions for stochastic stability of the switched system, when the subsystems do not possess control inputs, and are not all stable. Thereafter we design feedback controllers by employing our analysis results such that the switched control system is stable in closed loop, when subsystems are affine in control. Multiple Lyapunov functions and Sontag's universal formulae for feedback stabilization of nonlinear systems constitute the primary tools for analysis and control design.

## § 1. Introduction

Randomly switched systems generally consist of a finite family of subsystems and a random switching signal that specifies at each instant of time the active subsystem. The switching signal $\sigma$ is modeled as a continuous time stochastic process, which may be the state of a finite-state Markov chain, or a more general càdlàg jump stochastic process. Since the dynamics between two consecutive switching instants are governed by deterministic differential equations, these systems can be regarded as piecewise deterministic stochastic systems [7]. In this article our goal is twofold: one, to provide sufficient conditions for stochastic stability of randomly switched systems, and two, to provide a methodology for stabilizing controller synthesis when such systems possess control inputs.

A particular class of randomly switched systems has received widespread attention, namely, Markovian jump linear systems (MJLS). These systems may be realized as a family of linear subsystems, together with a switching signal generated by the state of a continuous-time Markov chain. Stability and stabilization of MJLS have been extensively investigated, specially under the assumption that the parameters of the Markov chain are completely known, see e.g. [3, 14, 8, 21] and the references therein. In particular, almost sure stabilization and mean stabilization of MJLS is discussed in [8], where the authors also establish precise equivalences between different stability notions for MJLS.

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Among the several stochastic stability notions, perhaps the most interesting is almost sure global asymptotic stability (GAS a.s.). We shall concentrate on this particular notion in this article; however, it is also possible to obtain stability in the mean and stability in probability with minimal extra work, which we indicate in Remark **??**. GAS a.s. of randomly switched systems was investigated in our earlier article [**6**]. There we assumed that each (nonlinear) subsystem was globally asymptotically stable, and $\sigma$ was a general jump stochastic process having an asymptotic bound on the probability mass function of the number of switches on each time interval $[0, t[$. Unless additional structure is imposed on the switching signal, switched systems with even one unstable subsystem cannot, in general, have the GAS a.s. property; see Remark 17. In the present article, we describe two possible scenarios where sufficient structure in the probabilistic properties of the switching signal make it possible to include unstable subsystems in the family. To be precise, in the first case the set of holding times of $\sigma$ is assumed to be a sequence of independent exponential variables of parameter $\lambda$, and in the second case the set of holding times is assumed to be a sequence of independent uniform random variables of parameter $T$. In addition, in both of the above cases we assume that values attained by $\sigma$ (at each switching instant) are independent and identically distributed, and are independent of the set of holding times. It follows naturally from our results that for the switched system to be GAS a.s., the unstable subsystems must have small probability of activation; see Remarks 19 and 20.

In [**6**] we also established a method of designing feedback controllers to achieve GAS a.s. of closed loop switched control systems, by employing the Artstein-Sontag universal formula [**23**]. The control took values in $\mathbb{R}$, and every subsystem was zero-input stable. In this article the controller design scheme allows the control to take values in general subsets of $\mathbb{R}^m$, (e.g., bounded sets, Minkowski balls, etc.,) and the subsystems are not necessarily zero-input stable. Our control design methodology works whenever each subsystem is affine in control, a suitable family of control-Lyapunov functions (one for each subsystem) is available, and a universal formula for feedback stabilization is available for the set of admissible inputs.

A myriad of techniques have been employed to study stability and stabilization of piecewise deterministic stochastic systems. HJB-based optimal control schemes for piecewise deterministic stochastic systems are well-studied, see e.g., [**7**] for a detailed account. Linear control systems admit analytically solvable linear quadratic optimal design methods, and such techniques have been effectively combined with the stochastic nature of structural variations in [**14**]; stabilization schemes based on Lyapunov exponents are studied in [**8**]. Game-theoretic techniques [**1**] in the presence of disturbance inputs, and spectral theory of Markov operators [**13**] have also been employed to study these systems. Stabilization schemes using robust control methods are investigated in [**24**]; see also the references cited in it. Stochastic hybrid systems, where the switching signal and its transition probabilities are state-dependent, are studied in [**5**] and [**12**], using an extended definition of the infinitesimal generator and optimal control strategies, respectively.

Our approach, in contrast to the above, parallels the one adopted in [**6**]. The stochastic switching signal is decoupled from the individual dynamical systems; instead of looking at the stochastic system as a whole, the properties of the random switching signal are decoupled from the deterministic properties of the switched system between consecutive switching instants. Consequently, we do not resort to infinitesimal generators for the stochastic process. The main analysis tool is the theory of multiple Lyapunov functions [**17**, Chapter 3], developed originally in the context of deterministic switched systems. The probabilistic properties of the switching signal, when suitably coupled with the dynamics of the Lyapunov functions, enable us to efficiently analyze the behavior of the overall switched system. Off-the-shelf universal formulae (see [**23, 18, 19, 20**]) and our analysis results provide the tools for our control design methodology.

The paper is arranged as follows. §2 contains the definitions of randomly switched systems and the stability notions that we study. The hypotheses on the switching signal and the associated analysis results are stated in §3. Controller synthesis results are stated and proved in §4. The proofs of all the results stated in §3 are collected in §5. We conclude the paper in §6 with a brief discussion of possible directions for further investigation.

## § 2. Preliminaries

Let the Euclidean norm be denoted by $\|\cdot\|$, the interval $[0,\infty[$ by $\mathbb{R}_{\geqslant 0}$, and the set of natural numbers $\{1,2,\ldots\}$ by $\mathbb{N}$. Recall that a continuous function $\alpha:\mathbb{R}_{\geqslant 0}\longrightarrow\mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if $\alpha$ is strictly increasing with $\alpha(0)=0$, of class $\mathcal{K}_\infty$ if in addition $\alpha(r)\to\infty$ as $r\to\infty$; we write $\alpha\in\mathcal{K}$ and $\alpha\in\mathcal{K}_\infty$ respectively. Let $\mathrm{L}_f h$ be the directional derivative of a continuously differentiable real-valued function $h$ defined on $\mathbb{R}^n$, along a vector field $f$ on $\mathbb{R}^n$. For $a,b\in\mathbb{R}$, we let $a\wedge b$ and $a\vee b$ stand for $\min\{a,b\}$ and $\max\{a,b\}$, respectively.

We define the family of systems affine in control:

$$\dot{x}=f_p(x),\qquad p\in\mathcal{P}, \tag{1}$$

where the state $x\in\mathbb{R}^n$, $\mathcal{P}$ is a finite index set of N elements: $\mathcal{P}=\{1,\ldots,\mathrm{N}\}$, the function $f_p:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is locally Lipschitz in $x$, $f_p(0)=0$, $p\in\mathcal{P}$. A *switched system* for the family (1) is generated by a *switching signal*—a piecewise constant function (continuous from the right by convention), $\sigma:\mathbb{R}_{\geqslant 0}\longrightarrow\mathcal{P}$, which specifies at every time $t$ the index $\sigma(t)\in\mathcal{P}$ of the active subsystem:

$$\dot{x}=f_\sigma(x),\qquad x(0)=x_0,\quad t\geqslant 0. \tag{2}$$

We assume that there are no jumps in the state $x$ at the switching instants, and let $x_0$ be given.

Let $\big(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geqslant 0},\mathsf{P}\big)$ be a complete filtered probability space [22], where $\Omega$ is the sample space, $\mathfrak{F}$ is a sigma-algebra on $\Omega$, $\mathsf{P}$ is a probability measure on the measurable space $(\Omega,\mathfrak{F})$, and $(\mathfrak{F}_t)_{t\geqslant 0}$ a right-continuous filtration with $\mathfrak{F}_0$ containing all the $\mathsf{P}$-measure 0 sets. Let $\sigma:=\big(\sigma(t)\big)_{t\geqslant 0}$ be a càdlàg stochastic process, (i.e., continuous from the right and possessing limits from the left,) taking values in $\mathcal{P}$, with $\sigma(0)$ completely known. Let the switching instants of $\sigma$ be denoted by $\tau_i$, $i\in\mathbb{N}$, and let $\tau_0:=0$ by convention. As a consequence of the hypotheses of our results, there is no explosion almost surely (see Lemma 32, Lemma 33 and Lemma 34); consequently the sequence $(\tau_i)_{i\in\mathbb{N}\cup\{0\}}$ is almost surely divergent. Finally, we assume that for every compact subset $K\subset\mathbb{R}_{\geqslant 0}\times\mathbb{R}^n$ there exists an integrable function $m_K$ satisfying $\sup_{p\in\mathcal{P}}\|f_p(x)\|\leqslant m_K(t)$ for all $(t,x)\in K$. Hence almost surely there exists a unique solution to (2) in the sense of Carathéodory [**9, 10**] over a nontrivial time interval containing 0; existence and uniqueness of a global solution will follow from the hypotheses of our results. We let $x(\cdot)$ denote this solution. For $x_0=0$, the solution to (2) is identically 0 for every $\sigma$; we shall ignore this trivial case in the sequel.

We are interested in the following two definitions of stability of (2).

3. Definition. The system (2) is said to be ***globally asymptotically stable almost surely*** (GAS a.s.) iff the following two properties are simultaneously verified:

(AS1) $\forall\,\varepsilon>0\;\exists\,\delta(\varepsilon)>0$ such that $\|x_0\|<\delta(\varepsilon)\implies\mathsf{P}\Big(\sup_{t\geqslant 0}\|x(t)\|<\varepsilon\Big)=1;$

(AS2) $\forall\,r,\varepsilon'>0\;\exists\,T(r,\varepsilon')\geqslant 0$ such that $\|x_0\|<r\implies\mathsf{P}\Big(\sup_{t\geqslant T(r,\varepsilon')}\|x(t)\|<\varepsilon'\Big)=1.$ ◇

4. Definition. The system (2) is said to be ***globally asymptotically stable in the mean*** (GAS-M) iff the following two properties are simultaneously verified:

(SM1) $\forall\,\varepsilon>0\quad\exists\,\widetilde{\delta}(\varepsilon)>0$ such that $\|x_0\|<\widetilde{\delta}(\varepsilon)\implies\sup_{t\geqslant 0}\mathsf{E}[\|x(t)\|]<\varepsilon;$

(SM2) $\forall\,r,\varepsilon'>0\quad\exists\,\widetilde{T}(r,\varepsilon')\geqslant 0$ such that $\|x_0\|<r\implies\sup_{t\geqslant T(r,\varepsilon')}\mathsf{E}[\|x(t)\|]<\varepsilon'.$ ◇

## § 3. Stability under Random Switching

In this section we establish sufficient conditions for GAS a.s. and GAS-M of the switched system (2). We treat three cases of different assumptions on $\sigma$, and corresponding to each assumption we present one theorem. The applicability and the differences among the theorems are discussed in the remarks that follow; the proofs may be found in §5. We mention that Theorem 8 was stated and proved in [6]; since it takes very little extra work, we provide some of the details once again for completeness.

Hereafter we shall denote the number of switches on the time interval $[t, t'[$ by $N_\sigma(t, t')$.

We make use of multiple Lyapunov functions (see [17, Chapter 3] for an extensive treatment of multiple Lyapunov functions in the deterministic case), one for each subsystem. The following assumption collects the properties we shall require from them.

5. Assumption. There exist a family of continuously differentiable real-valued functions $(V_p)_{p\in\mathcal{P}}$ on $\mathbb{R}^n$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, numbers $\mu > 1$ and $\lambda_p \in \Lambda \subseteq \mathbb{R}$, $p \in \mathcal{P}$, such that

(V1) $\alpha_1(\|x\|) \leqslant V_p(x) \leqslant \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, p \in \mathcal{P}$;

(V2) $\mathrm{L}_{f_p} V_p(x) \leqslant -\lambda_p V_p(x) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, p \in \mathcal{P}$;

(V3) $V_{p_1}(x) \leqslant \mu V_{p_2}(x) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, p_1, p_2 \in \mathcal{P}$. ◇

6. Remark. (V1) is a fairly standard hypothesis, ensuring $V_p$'s are each positive definite and radially unbounded. (V2) furnishes a quantitative estimate of the degree of stability or instability, depending on the sign of $\lambda_p$, of each subsystem of the family (1). The possible values that the $\lambda_p$'s are allowed to take is specified by the set $\Lambda$. (To wit, if there are unstable subsystems, we allow $\Lambda$ to contain negative real numbers so that the corresponding $\lambda_p$'s may be negative; if there are no unstable subsystems, $\Lambda$ is a subset of the positive real numbers.) The right-hand side of the inequality in (V2) being linear in $V_p$ is no loss of generality, see e.g., [16, Theorem 2.6.10] for details. (V3) certainly restricts the class of functions that the family $(V_p)_{p\in\mathcal{P}}$ can belong to; however this hypothesis is commonly employed in the deterministic case [17, Chapter 3]. Quadratic Lyapunov functions universally utilized in the case of linear subsystems satisfy this hypothesis. ◁

**§ 3.1. Global asymptotic stability almost surely.** We now present the results on GAS a.s. of (2) in the three different cases below.

First case. In this case $\sigma$ is a general càdlàg jump stochastic process, and merely an upper bound of its asymptotic probability distribution is known. The temporal probability distribution of $\sigma$ on $\mathcal{P}$ is completely unknown.

7. Assumption. The switching signal is characterized by: $\exists\, M \in \mathbb{N} \cup \{0\}$ and $\overline{\lambda}, \widetilde{\lambda} > 0$, such that $\forall\, k \geqslant M$ we have $\mathsf{P}\big(N_\sigma(0,t) = k\big) \leqslant \dfrac{(\overline{\lambda} t)^k}{k!} \mathrm{e}^{-\widetilde{\lambda} t}$. ◇

8. Theorem ([6]). *Consider the system (2). Suppose that*

(G1) *Assumption 5 holds with* $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$*;*

(G2) $\sigma$ *satisfies Assumption 7;*

(G3) $\mu < \left(\lambda_\circ + \widetilde{\lambda}\right)/\overline{\lambda}$.

*Then (2) is GAS a.s.*

9. Corollary ([6]). *Suppose the hypotheses of Theorem 8 hold true, and in addition suppose that one of the two following conditions hold:*

(G1′) $\alpha_1$ *is convex;*

(G2′) $\alpha_1$ *is continuously differentiable with a nonzero first derivative at* $0$.

*Then* (2) *is* GAS-M.

Second case. In this case Assumption 7 is replaced by Assumption 10 below; this imposes additional structure on the stochastic properties of $\sigma$.

10. Assumption. The switching signal $\sigma$ is characterized by:

(EH1) the sequence $(S_i)_{i\in\mathbb{N}}$, $S_i := \tau_i - \tau_{i-1}$, of holding times is an independent identically distributed sequence of exponential-$\lambda$ random variables;*

(EH2) $\exists\, q_p \in [0,1]$, $p \in \mathcal{P}$, such that $\forall\, i \in \mathbb{N}$, $\mathsf{P}\big(\sigma(\tau_i) = p \big| (\sigma(\tau_j))_{j=0}^{i-1}\big) = q_p$;

(EH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$. ♢

11. Theorem. *Consider the system* (2). *Suppose that*

(E1) *Assumption* 5 *holds with* $\Lambda = \mathbb{R}$;

(E2) $\sigma$ *satisfies Assumption* 10;

(E3) $\lambda_p + \lambda > 0 \quad \forall\, p \in \mathcal{P}$;

(E4) $\displaystyle\sum_{p\in\mathcal{P}} \frac{\mu q_p}{(1+\lambda_p/\lambda)} < 1.$

*Then* (2) *is* GAS *a.s.*

12. Corollary. *Suppose the hypotheses of Theorem* 11 *hold true, and in addition suppose that* <u>*one*</u> *of the two following conditions hold:*

(E1′) $\alpha_1$ *is convex;*

(E2′) $\alpha_1$ *is continuously differentiable with a nonzero first derivative at* 0.

*Then* (2) *is* GAS-M.

Third case. In this case Assumption 10 is replaced by Assumption 13 below; this imposes a different structure on the stochastic properties of $\sigma$ compared to the second case above.

13. Assumption. The switching signal $\sigma$ is characterized by:

(UH1) the sequence $(S_i)_{i\in\mathbb{N}}$, $S_i := \tau_i - \tau_{i-1}$, of holding times is an independent identically distributed sequence of uniform-$T$ random variables;†

(UH2) $\exists\, q_p \in [0,1]$, $p \in \mathcal{P}$, such that $\forall\, i \in \mathbb{N}$, $\mathsf{P}\big(\sigma(\tau_i) = p \big| (\sigma(\tau_j))_{j=0}^{i-1}\big) = q_p$;

(UH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$. ♢

14. Theorem. *Consider the system* (2). *Suppose that*

(U1) *Assumption* 5 *holds with* $\Lambda = \mathbb{R}$;

(U2) $\sigma$ *satisfies Assumption* 13;

(U3) $\displaystyle\sum_{p\in\mathcal{P}} \left(\frac{\mu q_p\left(1-\mathrm{e}^{-\lambda_p T}\right)}{\lambda_p T}\right) < 1.$

*Then* (2) *is* GAS *a.s.*

15. Corollary. *Suppose the hypotheses of Theorem* 14 *hold true, and in addition suppose that* <u>*one*</u> *of the two following conditions hold:*

(U1′) $\alpha_1$ *is convex;*

(U2′) $\alpha_1$ *is continuously differentiable with a nonzero first derivative at* 0.

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*Recall that a exponential-$\lambda$ random variable $X$ has the following probability distribution function: $\mathsf{P}\big(X \leqslant s\big) = 1 - \mathrm{e}^{-\lambda s}$ if $s \geqslant 0$, and 0 otherwise; see e.g. [2] for further details.

†Recall that a uniform-$T$ random variable $Y$ has the following probability distribution: $\mathsf{P}\big(Y \leqslant s\big) = s/T$ if $s \in [0,T]$, 0 if $s < 0$, and 1 if $s > T$; see e.g. [2] for further details.

*Then* (2) *is* GAS-M.

Remarks and discussion. We now examine in detail the three cases listed above.

16. REMARK. Intuitively, Assumption 7 requires that statistically the rate of switching is not too large in the long run. More specifically, the expected number of switches on the interval $[0,t[$ grows at most exponentially with $t$. Indeed, $\mathsf{E}[N_\sigma(0,t)] = \sum_{k=0}^{\infty} k\,\mathsf{P}\big(N_\sigma(0,t)=k\big)$, and this is upper bounded by $S+\sum_{k=M}^{\infty} k\,\mathsf{P}\big(N_\sigma(0,t)=k\big)$, where $S$ is a constant depending on $M$, which finally is in turn upper bounded by $S' + (\overline{\lambda}t)\mathrm{e}^{(\overline{\lambda}-\widetilde{\lambda})t}$, where $S'$ is a constant depending on $M$ and greater than $S$. Assumption 7 may therefore be regarded as a statistically slow switching condition. ◁

17. REMARK. On the one hand, note that Assumption 7 does not put any restrictions on the temporal probability distribution of $\sigma$ on $\mathcal{P}$. Consequently, if one subsystem in the family $(f_p)_{p\in\mathcal{P}}$ is unstable, and the switching signal obeys Assumption 7 but activates this subsystem for most of the time, the switched system may well become unstable. It follows that this assumption is not strong enough for almost sure global asymptotic stability of the switched system, unless we further stipulate that each subsystem is stable. On the other hand both Assumption 10 and Assumption 13 require the existence of a (stationary) probability distribution of $\sigma$ on $\mathcal{P}$ ((EH3) and (UH3), respectively), and are therefore better equipped to take into account instabilities of some subsystems. ◁

18. REMARK. Theorem 8 is quite intuitively appealing; it states that if each subsystem has sufficient stability margin, and $\sigma$ switches sufficiently slowly on an average, then the switched system is GAS a.s.. By (G1) there is a uniform stability margin (in terms of the Lyapunov functions) among the family of subsystems. (G3) links the deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 5, with the properties of the switching signal furnished by (G2). It is clear that the more stable the subsystems (larger the $\lambda_\circ$), the faster can be the switching signal (larger the $\overline{\lambda}$) that still renders (2) GAS a.s. This result is reminiscent of the well-known result [**17**, Theorem 3.2] on global asymptotic stability of deterministic switched systems under average dwell-time switching; see [**6**] for a detailed comparison. Moreover, this theorem applies to the case of $\sigma$ being the state of a continuous-time Markov chain with a given generator matrix; further details on this important case is given in [**6**]. ◁

19. REMARK. Let us examine the statement of Theorem 11 in some detail. Firstly, note that by (E1) not all subsystems are required to be stable, i.e., for some $p\in\mathcal{P}$, $\lambda_p$ can be negative; then (V2) provides a measure of the rate of instability of the corresponding subsystems. Secondly, note that condition (E3) is always satisfied if each $\lambda_p > 0$. However, if $\lambda_p < 0$ for some $p\in\mathcal{P}$, then (E3) furnishes a maximum instability margin of the corresponding subsystems that can still lead to GAS a.s. of (2). Intuitively, in the latter case, the process $N_\sigma(0,t)$ must switch fast enough ($\lambda > 0$ large enough) so that the unstable subsystems are not active for too long. Potentially this fast switching may have a destabilizing effect. Indeed, it may so happen that for a given $\mu$, a fixed set $(q_p)_{p\in\mathcal{P}}$, and a choice of functions $\big(V_p\big)_{p\in\mathcal{P}}$, (E3) and (E4) may be impossible to satisfy simultaneously, due to a very high degree of instability of even one subsystem for which the corresponding $q_p$ is also large. Then we need to search for a different family of functions $\big(V_p\big)_{p\in\mathcal{P}}$ for which the hypotheses hold. Thirdly, (E4) links the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 5, with the properties of the switching signal. From (E4) it is clear that larger degrees of instability of a subsystem (smaller $\lambda_p$) can be compensated by a smaller probability (smaller $q_p$) of the switching signal activating the corresponding subsystem. ◁

20. REMARK. Let us make some observations about the statement of Theorem 14. Once again, like Theorem 11, note that by (U1) not all subsystems are required to be stable, i.e., for

some $p \in \mathcal{P}$, $\lambda_p$ can be negative. (U3) links the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 5, with the properties of the switching signal. Also from (U3) it is clear that larger degrees of instability (larger $\lambda_p$) of a subsystem can be compensated by a smaller probability (smaller $q_p$) of the switching signal activating the corresponding subsystem. $\triangleleft$

21. Remark. It may appear that Theorem 11 requires a larger set of hypotheses compared to Theorem 14; however, this is only natural. Indeed, the switching signal in the latter case is constrained to switch at least once in $T$ units of time, whereas no such constraint is present on the switching signal in the former case. We observed in Remark 19 that it is necessary for the switching signal to switch fast enough if there are unstable subsystems in the family (1), which accounted for (E3). This fast enough switching is automatic if $\sigma$ satisfies Assumption 13, provided $T$ is related to the instability margin of the subsystems in a particular way. (U3) captures this relationship, for, observe that if $\lambda_p$ is negative and large in magnitude for some $p \in \mathcal{P}$, the ratio $\left(1 - \mathrm{e}^{-\lambda_p T}\right)/(\lambda_p T)$ is small provided $T$ is small, and a smaller ratio is better for GAS a.s. of (2); also for a given $T$, large and positive $\lambda_p$'s (i.e., subsystems with high margins of stability) make the aforesaid ratio small. $\triangleleft$

22. Remark. Let us recall that (2) is said to be *globally asymptotically stable in probability* (GAS-P) if the following two conditions hold simultaneously.

(P1) $\forall\, \eta, \varepsilon > 0\ \exists\, \delta(\eta, \varepsilon) > 0$ such that $\|x_0\| < \delta(\eta, \varepsilon) \implies \inf_{t \geqslant 0} \mathsf{P}\big(\|x(t)\| < \varepsilon\big) \geqslant 1 - \eta$;

(P2) $\forall\, \eta, r, \varepsilon' > 0\ \exists\, T(\eta, r, \varepsilon') \geqslant 0$ such that $\|x_0\| < r \implies \inf_{t \geqslant T(\eta, r, \varepsilon')} \mathsf{P}\big(\|x(t)\| < \varepsilon'\big) \geqslant 1 - \eta$.

The GAS-P property follows from the GAS-M property via a standard application of Chebyshev's inequality (see e.g., [**11**] for details), or from the GAS a.s. property because clearly (P1) is a weaker property than (AS1), and (P2) is a weaker property than (AS2). $\triangleleft$

## § 4. Stabilization under Random Switching

In this section we provide a methodology for designing controllers that ensure almost sure global asymptotic stability of control-affine randomly switched systems in closed loop.

Consider the affine in control switched system:

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{\sigma,i}(x) u_i, \qquad x(0) = x_0, \quad t \geqslant 0, \tag{23}$$

where $x \in \mathbb{R}^n$ is the state, $u_i,\ i = 1, \ldots, m$ are the control inputs, $f_p$ and $g_{p,i}$ are smooth vector fields on $\mathbb{R}^n$, with $f_p(0) = 0, g_{p,i}(0) = 0$, for each $p \in \mathcal{P}, i \in \{1, \ldots, m\}$. Let $\mathcal{U}$ be the set where the control $u := [u_1, \ldots, u_m]^{\mathrm{T}}$ takes its values. For the moment, we let $\mathcal{U}$ be a subset of $\mathbb{R}^m$; later we shall consider the case when $\mathcal{U}$ is a more general set, e.g. a Minkowski ball. With a feedback control function $\overline{u}_\sigma(x) = [u_{\sigma,1}(x), \ldots, u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed loop system stands as:

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{\sigma,i}(x) \overline{u}_{\sigma,i}(x), \quad x(0) = x_0, \ \ t \geqslant 0. \tag{24}$$

Our objective is to choose the control function $\overline{u}_\sigma$ so that (24) is GAS a.s. Let the switching signal $\sigma$ be a stochastic process as defined in §2, and let $x_0 \neq 0$.

We now describe the controller design methodology promised in §1.

A universal formula for stabilization of control-affine nonlinear systems was first constructed in [**23**], for the control taking values in $\mathcal{U} = \mathbb{R}^m$. The articles [**18**],[**19**], and [**20**] provide universal formulae for bounded controls, positive controls, and controls restricted to Minkowski balls, respectively. In view of the analysis results of §3 and the universal formulae provided in the aforementioned articles, it is possible to synthesize controllers $\overline{u}_\sigma$ for (23), such that the closed loop system (24) is GAS a.s. Recall that three different types of switching signals were considered in §3; the corresponding hypotheses on them appear in Assumptions 7, 10, and 13.

In general, we obtain one synthesis scheme for each type of $\mathcal{U}$ and $\sigma$; the following theorem provides a typical illustration of such a result. A complete recipe to obtain such results is provided in Remark 27.

25. THEOREM. *Consider the system (23), with $\mathcal{U} = \mathbb{R}^m$. Suppose that $\sigma$ satisfies Assumption 10, and there exists a family of continuously differentiable functions $\big(V_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}\big)_{p\in\mathcal{P}}$, such that*

(C1) *(V1) of Assumption 5 holds;*

(C2) *(V3) of Assumption 5 holds;*

(C3) $\exists\, \lambda_p \in \Lambda = \mathbb{R},\ p \in \mathcal{P}$, *such that* $\forall\, x \in \mathbb{R}^n \smallsetminus \{0\}$ *and* $\forall\, p \in \mathcal{P}$
$$\inf_{u\in\mathcal{U}} \left\{ \mathrm{L}_{f_p} V_p(x) + \lambda_p V_p(x) + \sum_{i=1}^{m} u_i \mathrm{L}_{g_{p,i}} V_p(x) \right\} < 0;$$

(C4) *((E3), (E4)) holds.*

*Then the feedback control*
$$\overline{u}_\sigma(x) = [k_{\sigma,1}(x), \ldots, k_{\sigma,m}(x)]^{\mathrm{T}},$$
*where*
$$k_{p,i}(x) := \begin{cases} -\mathrm{L}_{g_{p,i}} V_p(x) \cdot \varphi\Big(\overline{W}_p(x), \widetilde{W}_p(x)\Big) & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{26a}$$
$$\overline{W}_p(x) := \mathrm{L}_{f_p} V_p(x) + \lambda_p V_p(x), \tag{26b}$$
$$\widetilde{W}_p(x) := \sum_{i=1}^{m} \big(\mathrm{L}_{g_{p,i}} V_p(x)\big)^2, \tag{26c}$$
*and*
$$\varphi(a,b) := \begin{cases} \dfrac{a + \sqrt{a^2+b^2}}{b} & \text{if } b \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{26d}$$
*renders (24)* GAS *a.s.*

PROOF. The proof relies heavily on the construction of the universal formula in [**23**]. Fix $t \in \mathbb{R}_{\geqslant 0}$. If $x \neq 0$, applying the definition of $\varphi$, we get
$$\begin{aligned} &\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \sum_{i=1}^{m} k_{\sigma(t),i}(x) \mathrm{L}_{g_{\sigma(t),i}} V_{\sigma(t)}(x) \\ &= \mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) - \widetilde{W}_{\sigma(t)}(x) \cdot \varphi\Big(\overline{W}_{\sigma(t)}(x), \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^2\Big) \\ &= -\lambda_{\sigma(t)} V_{\sigma(t)}(x) - \sqrt{\big(\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x)\big)^2 + \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^2} \\ &< -\lambda_{\sigma(t)} V_{\sigma(t)}(x). \end{aligned}$$
Since $t$ is arbitrary, we conclude that the above inequality holds for all $t \in \mathbb{R}_{\geqslant 0}$. Note that by (C3), if for any $p \in \mathcal{P}$, $x \in \bigcap_{i=1}^{m} \ker\big(\mathrm{L}_{g_{p,i}} V_p\big)$, we automatically have $\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \lambda_{\sigma(t)} V_{\sigma(t)}(x) < 0$.

The above arguments, in conjunction with (C1) and (C2) enable us to conclude that the family $(V_p)_{p\in\mathcal{P}}$ satisfies Assumption 5 for the closed loop system (24) and $\Lambda = \mathbb{R}$. (C4) ensures that (E3) and (E4) hold, respectively, for (24). Since $\sigma$ satisfies Assumption 7, (E2) holds as well. Hence, it follows from Theorem 11 that (24) is GAS a.s. □

27. Remark. Theorem 25 can be modified to suit a different $\mathcal{U}$ and a different type of $\sigma$ using the following simple recipe. First, recall from the discussion preceding Theorem 25 that $\mathcal{U}$ may be any one among $\mathbb{R}^m$, the nonnegative orthant of $\mathbb{R}^m$, a bounded subset of $\mathbb{R}^m$, and a Minkowski ball in $\mathbb{R}^m$; $\sigma$ may satisfy any one of Assumptions 7, 10, and 13. Now suppose that a $\mathcal{U}$ and a $\sigma$ among the above possibilities is given to us. Then:

- (C1) and (C2) remain unchanged;
- the given $\mathcal{U}$ replaces the $\mathcal{U} = \mathbb{R}^m$ in Theorem 25;
- if the given $\sigma$ satisfies Assumption 7, then this assumption replaces Assumption 10, the pair ((E3), (E4)) appearing in hypothesis (C4) is replaced by (G3), and $\Lambda$ appearing in (C3) is replaced by the set $\{\lambda_\circ\}$;
- if the given $\sigma$ satisfies Assumption 13, then this assumption replaces Assumption 10, the pair ((E3), (E4)) appearing in hypothesis (C4) is replaced by (U3), and $\Lambda$ appearing in (C3) is replaced by the set $\mathbb{R}$;
- the universal formula corresponding to the given $\mathcal{U}$ replaces the one given in (26). $\lhd$

28. Remark. For linear systems it is possible to design controllers in a simpler fashion. For an illustration, let $\sigma$ satisfy Assumption 7. Consider the following linear version of (23):

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad x(0) = x_0, \quad t \geqslant 0, \tag{29}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^n \times \mathbb{R}^m$. Let us try to find a control $\overline{u}_\sigma(x) = K_{\sigma(t)}x$, where $K_p$ is a $(m \times n)$ matrix for each $p \in \mathcal{P}$, that achieves GAS a.s. of (29) in closed loop. For a square matrix $A$ of dimension $n$, with eigenvalues $\{\lambda_i\}_{i=1}^n$, let $\rho_1(A) := \min_i |\Re(\lambda_i)|$ and $\rho_2(A) := \max_i |\Re(\lambda_i)|$. Suppose that there exists a set of $(m \times n)$ matrices $(K_p)_{p\in\mathcal{P}}$ and a number $\lambda_\circ > 0$, such that the symmetric positive definite solution set $(M_p)_{p\in\mathcal{P}}$ to the linear matrix inequalities

$$\left(A_p + B_pK_p\right)^{\mathrm{T}} M_p + M_p \left(A_p + B_pK_p\right) \leqslant -\lambda_\circ M_p \tag{30}$$

satisfies the following estimate:

$$\mu := \frac{\max_{p\in\mathcal{P}} \rho_2(M_p)}{\min_{p\in\mathcal{P}} \rho_1(M_p)} < \frac{\lambda_\circ + \widetilde{\lambda}}{\overline{\lambda}}. \tag{31}$$

Standard and efficient computational tools for solving the linear matrix inequalities like (30) exist, see e.g., [4]; therefore finding the set $(K_p)_{p\in\mathcal{P}}$ is not difficult. It is clear that we have found a family of Lyapunov functions $\bigl(V_p(x) = x^{\mathrm{T}}M_px\bigr)_{p\in\mathcal{P}}$, for which (V1) and (V3) hold by the definitions of the $V_p$'s, and (V2) holds due to (30). Also, observe that $\Lambda = \{\lambda_\circ\}$, and (31) is nothing but (G3). It follows by Theorem 8 that the control function $\overline{u}_\sigma$ defined above renders (29) GAS a.s. in closed loop. $\lhd$

## § 5. Proofs

We present the proofs of the theorems in §3 in this section.

**§ 5.1. Auxiliary results.** This subsection consists of the statements and proofs of a number of technical lemmas which will be helpful in proving the theorems of §3, which are presented in the following subsection.

Recall that the random variable $N_\sigma(t, t')$ gives the number of switches of $\sigma$ on the interval $[t, t'[$, and $(\tau_i)_{i\in\mathbb{N}}$ is the set of switching instants. We define $N_\sigma(0, 0) = 0$. The extended real-valued random variable $\zeta := \sup_{\nu\in\mathbb{N}} \tau_\nu$ is the *explosion time* [22] of the process $(N_\sigma(0, t))_{t\in\mathbb{R}_{\geqslant 0}}$. If $\zeta = \infty$, then the process $(N_\sigma(0, t))_{t\in\mathbb{R}_{\geqslant 0}}$ is said to have *no explosions*; we shall also say that under this condition $\sigma$ has no explosions.

32. Lemma. *Suppose $\sigma$ satisfies Assumption 7. Then $N_\sigma(0, t) \longrightarrow \infty$ a.s. only if $t \longrightarrow \infty$; i.e., almost surely $\sigma$ has no explosion.*

PROOF. Suppose $\sigma$ satisfies Assumption 7. If $t' \in \mathbb{R}_{\geqslant 0}$, the event that there is an explosion at $t = t'$ is given by $\bigcap_{\nu\in\mathbb{N}} \big\{N_\sigma(0,t') \geqslant \nu\big\}$. But

$$\begin{aligned}\mathsf{P}\left(\bigcap_{\nu\in\mathbb{N}} \big\{N_\sigma(0,t') \geqslant \nu\big\}\right) &\leqslant \limsup_{\nu\to\infty} \mathsf{P}\left(\bigcup_{k=\nu}^{\infty} \big\{N_\sigma(0,t') = k\big\}\right)\\ &\leqslant \limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathsf{P}\big(N_\sigma(0,t') = k\big),\end{aligned}$$

and from the hypothesis of our assumption we get

$$\limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathsf{P}\big(N_\sigma(0,t') = k\big) \leqslant \limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathrm{e}^{-\widetilde{\lambda} t'} \frac{(\overline{\lambda} t')^k}{k!}.$$

Since $\sum_{k=\nu}^{\infty} (\overline{\lambda} t')^k / k!$ is the tail of $\mathrm{e}^{\overline{\lambda} t'}$, it vanishes as $\nu \longrightarrow \infty$. We conclude that since $t' \in \mathbb{R}_{\geqslant 0}$ is arbitrary, almost surely $\sigma$ has no explosion. □

33. LEMMA. *Suppose $\sigma$ satisfies Assumption 10. Then $N_\sigma(0,t) \longrightarrow \infty$ a.s. if and only if $t \longrightarrow \infty$.*

PROOF. To see sufficiency, consider the event

$$\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0} \text{ such that } \forall\, t \geqslant t' \;\; N_\sigma(0,t) = N_\sigma(0,t')\big\}.$$

But this event is equal to

$$\big\{\forall\, \nu \in \mathbb{N} \quad S_{N_\sigma(0,t')+1} > \nu\big\} = \bigcap_{\nu\in\mathbb{N}} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}.$$

In the light of (EH1), the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\left(\bigcap_{\nu\in\mathbb{N}} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}\right) &\leqslant \limsup_{\nu\to\infty} \mathsf{P}\left(\bigcup_{k=\nu}^{\infty} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}\right)\\ &\leqslant \limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathsf{P}\big(S_{N_\sigma(0,t')+1} > \nu\big)\\ &= \limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathrm{e}^{-\lambda k} = 0.\end{aligned}$$

Therefore, almost surely $N_\sigma(0,t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Conversely, to see necessity, consider the event of an explosion, i.e.,

$$\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0} \text{ such that } \forall\, \nu \in \mathbb{N} \;\; N_\sigma(0,t') \geqslant \nu\big\}.$$

But in view of (EH1) and (EH2) the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\left(\bigcap_{\nu\in\mathbb{N}} \big\{N_\sigma(0,t') \geqslant \nu\big\}\right) &\leqslant \mathsf{P}\Big(\exists\, M \in \mathbb{N} \text{ such that } \forall\, i \geqslant M \;\; S_i < 1\Big)\\ &\leqslant \prod_{i\geqslant M} \mathsf{P}\big(S_i < 1\big) = 0.\end{aligned}$$

Since $t'$ is arbitrary, it follows that almost surely $\sigma$ has no explosion. □

34. LEMMA. *Suppose $\sigma$ satisfies Assumption 13. Then $N_\sigma(0,t) \longrightarrow \infty$ a.s. if and only if $t \longrightarrow \infty$.*

PROOF. The proof mimics the proof of Lemma 33; for completeness we provide it below. To see sufficiency, consider the event

$$\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0} \text{ such that } \forall\, t \geqslant t' \;\; N_\sigma(0,t) = N_\sigma(0,t')\big\}.$$

But this event is equal to

$$\big\{\forall\, \nu \in \mathbb{N} \quad S_{N_\sigma(0,t')+1} > \nu\big\} = \bigcap_{\nu\in\mathbb{N}} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}.$$

In the light of (UH1), $\exists\, \nu \in \mathbb{N}$ such that $T < \nu$; therefore the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}\Bigg) &\leqslant \limsup_{\nu\to\infty} \mathsf{P}\Bigg(\bigcup_{k=\nu}^{\infty} \big\{S_{N_\sigma(0,t')+1} > \nu\big\}\Bigg)\\ &\leqslant \limsup_{\nu\to\infty} \sum_{k=\nu}^{\infty} \mathsf{P}\big(S_{N_\sigma(0,t')+1} > \nu\big) = 0\end{aligned}$$

Therefore, almost surely $N_\sigma(0,t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Conversely, to see necessity, consider the event of an explosion, i.e.,

$$\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0} \text{ such that } \forall\, \nu \in \mathbb{N} \;\; N_\sigma(0,t') \geqslant \nu\big\}.$$

But in view of (UH1) and (UH2) the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}} \big\{N_\sigma(0,t') \geqslant \nu\big\}\Bigg) &\leqslant \mathsf{P}\Big(\exists\, M \in \mathbb{N} \text{ such that } \forall\, i \geqslant M \;\; S_i < T/2\Big)\\ &\leqslant \prod_{i\geqslant M} \mathsf{P}\big(S_i < T/2\big) = \prod_{i\geqslant M} \left(\frac{1}{2}\right)^i = 0.\end{aligned}$$

Since $t'$ is arbitrary, it follows that almost surely $\sigma$ has no explosion. □

35. LEMMA. *Consider the system (2). Suppose that Assumption 5 holds, and that $\sigma$ satisfies one of Assumptions 7, or 10, or 13. Then for $\nu \in \mathbb{N}$, almost all $\omega \in \Omega$, and $t \in [\tau_\nu(\omega), \tau_{\nu+1}(\omega)[$, we have*

$$V_{\sigma(t,\omega)}(x(t,\omega)) \leqslant \mu^\nu V_{\sigma(0)}(x_0) \cdot \left(\prod_{i=1}^{\nu} \mathrm{e}^{-\lambda_{\sigma(\tau_{i-1},\omega)} S_i(\omega)}\right) \cdot \mathrm{e}^{-\lambda_{\sigma(t,\omega)}(t-\tau_\nu(\omega))}.$$

PROOF. Since $\sigma$ satisfies at least one of Assumptions 7, or 10, or 13, it follows from Lemma 32, or 33, or 34, respectively, that the sequence $(\tau_i)_{i\in\mathbb{N}}$ is almost surely monotonically increasing (and divergent). Therefore, for almost all $\omega \in \Omega$, at time $t \in [\tau_i(\omega), \tau_{i+1}(\omega)[$ we can write using (V2)

$$V_{\sigma(t,\omega)}(x(t,\omega)) \leqslant V_{\sigma(\tau_i,\omega)}(x(\tau_i,\omega))\mathrm{e}^{-\lambda_{\sigma(\tau_i,\omega)}(t-\tau_i(\omega))}.$$

In view of (V3), at $t = \tau_{i+1}(\omega)$ we have

$$V_{\sigma(\tau_{i+1},\omega)}(x(\tau_{i+1},\omega)) \leqslant \mu \cdot V_{\sigma(\tau_i,\omega)}(x(\tau_i,\omega)) \cdot \mathrm{e}^{-\lambda_{\sigma(\tau_i,\omega)} S_{i+1}(\omega)}.$$

Fix $\nu \in \mathbb{N}$. Iterating this inequality from $i = 0$ through $i = \nu - 1$, we get

$$V_{\sigma(\tau_\nu,\omega)}(x(\tau_\nu,\omega)) \leqslant \mu^\nu V_{\sigma(0)}(x_0) \prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i,\omega)} S_{i+1}(\omega)},$$

and for $t \in [\tau_\nu(\omega), \tau_{\nu+1}(\omega)[$,

$$V_{\sigma(\tau_\nu,\omega)}(x(t,\omega)) \leqslant \mu^\nu V_{\sigma(0)}(x_0) \cdot \left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i,\omega)} S_{i+1}(\omega)}\right) \cdot \mathrm{e}^{-\lambda_{\sigma(\tau_\nu,\omega)}(t-\tau_\nu(\omega))}.$$

Since $\sigma(t,\omega) = \sigma(\tau_\nu,\omega)$, the thesis follows. $\square$

36. LEMMA. *Consider the system* (2). *Suppose that Assumption* 5 *holds, and for every nonnegative monotonically increasing divergent sequence* $(s_i)_{i\in\mathbb{N}}$, $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$. *Then* $V_{\sigma(t)}(x(t)) \longrightarrow 0$ *as* $t \longrightarrow \infty$ *almost surely.*

PROOF. If the claim is false, there exists a set $\Omega' \subset \Omega$ with $\mathsf{P}\big(\Omega'\big) > 0$, such that $\forall\,\omega \in \Omega'$, $V_{\sigma(t,\omega)}(x(t,\omega)) \not\longrightarrow 0$; that is to say,

$$\forall\,\omega \in \Omega'\ \exists\,\eta > 0\ \forall\,t' \geqslant 0\ \exists\,t > t'\ \ V_{\sigma(t,\omega)}(x(t,\omega)) \geqslant \eta. \tag{37}$$

Fix $\omega \in \Omega'$. By (37), there exists $s'_1 > 0$ such that $V_{\sigma(s'_1,\omega)}(x(s'_1,\omega)) \geqslant \eta$. Similarly, by (37) there exists $s'_2 > s'_1 + 1$ such that $V_{\sigma(s'_2,\omega)}(x(s'_2,\omega)) \geqslant \eta$. Now suppose that $s'_j$ has been chosen. Then by (37) there exists $s'_{j+1} > s'_j + 1$ such that $V_{\sigma(s'_{j+1},\omega)}(x(s'_{j+1},\omega)) \geqslant \eta$. Continuing in this way, we construct a nonnegative, monotonically increasing, divergent sequence $(s'_i)_{i\in\mathbb{N}}$, such that $\forall\,i \in \mathbb{N}$, $V_{\sigma(s'_i,\omega)}(x(s'_i,\omega)) \geqslant \eta$. It is clear that

$$\Omega' = \bigcup_{\ell\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \left\{ V_{\sigma(s'_i)}(x(s'_i)) \geqslant \frac{1}{\ell} \right\}.$$

A trivial monotonicity argument shows

$$\mathsf{P}\left( \bigcap_{i\in\mathbb{N}} \big\{ V_{\sigma(s'_i)}(x(s'_i)) \geqslant \eta \big\} \right) \leqslant \limsup_{i\to\infty} \mathsf{P}\big( V_{\sigma(s'_i)}(x(s'_i)) \geqslant \eta \big)\,. \tag{38}$$

Employing Chebyshev's inequality,[‡] we have

$$\forall\,i \in \mathbb{N}\quad \mathsf{P}\big( V_{\sigma(s'_i)}(x(s'_i)) \geqslant \eta \big) \leqslant \mathsf{E}\big[ V_{\sigma(s'_i)}(x(s'_i)) \big] /\eta,$$

which leads to

$$\limsup_{i\to\infty} \mathsf{P}\big( V_{\sigma(s'_i)}(x(s'_i)) \geqslant \eta \big) \leqslant \limsup_{i\to\infty} \frac{\mathsf{E}\big[ V_{\sigma(s'_i)}(x(s'_i)) \big]}{\eta}. \tag{39}$$

In view of (39) and (38),

$$\mathsf{P}\left( \bigcap_{i\in\mathbb{N}} \big\{ V_{\sigma(s'_i)}(x(s'_i)) \geqslant \eta \big\} \right) \leqslant \limsup_{i\to\infty} \frac{\mathsf{E}\big[ V_{\sigma(s'_i)}(x(s'_i)) \big]}{\eta} = 0. \tag{40}$$

To compute the probability measure of the set $\Omega'$, corresponding to each $\ell \in \mathbb{N}$ we construct, if possible, a sequence $(s'_i)_{i\in\mathbb{N}}$ with $\eta = 1/\ell$, as outlined above. (Since by assumption $\mathsf{P}\big(\Omega'\big) > 0$, the set of such sequences is, in particular, nonempty.) Using (40) it follows that

$$\mathsf{P}\big(\Omega'\big) \leqslant \sum_{\ell\in\mathbb{N}} \mathsf{P}\left( \bigcap_{i\in\mathbb{N}} \left\{ V_{\sigma(s'_i)}(x(s'_i)) \geqslant \frac{1}{\ell} \right\} \right) = 0.$$

This contradicts our assumption that $\mathsf{P}\big(\Omega'\big) > 0$. Therefore, for almost all sample paths, $\lim_{t\to\infty} V_{\sigma(t)}(x(t)) = 0$. $\square$

41. LEMMA. *Suppose that hypothesis (G1) of Theorem* 8 *holds. Then* $\exists\,S \geqslant 0$ *such that the* moment generating function $\mathsf{E}\big[\mathrm{e}^{sN_\sigma(0,t)}\big]$ *of* $N_\sigma(0,t)$ *satisfies*

$$\mathsf{E}\Big[\mathrm{e}^{sN_\sigma(0,t)}\Big] \leqslant S + \mathrm{e}^{\big(\mathrm{e}^{s}\overline{\lambda} - \widetilde{\lambda}\big)t} \quad \forall\,s \geqslant 0. \tag{42}$$

[‡]Recall [2] Chebyshev's inequality: if $\varepsilon > 0$ and $Y$ is a nonnegative random variable on the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, then $\mathsf{P}\big(Y \geqslant \varepsilon\big) \leqslant \mathsf{E}[Y]/\varepsilon$.

PROOF. Using (G1), for $s \geqslant 0$,

$$\begin{aligned}\mathsf{E}\Big[\mathrm{e}^{sN_\sigma(0,t)}\Big] &= \sum_{k=0}^{\infty} \mathrm{e}^{sk}\mathsf{P}\big(N_\sigma(0,t)=k\big)\\ &\leqslant \sum_{k=0}^{M-1} \mathrm{e}^{sk}\mathsf{P}\big(N_\sigma(0,t)=k\big) + \sum_{k=M}^{\infty} \mathrm{e}^{sk}\frac{(\overline{\lambda}t)^k \mathrm{e}^{-\widetilde{\lambda}t}}{k!}\\ &\leqslant \sum_{k=0}^{M-1} \mathrm{e}^{sk} + \sum_{k=M}^{\infty} \mathrm{e}^{sk}\frac{(\overline{\lambda}t)^k \mathrm{e}^{-\widetilde{\lambda}t}}{k!} \leqslant S + \mathrm{e}^{(\mathrm{e}^s\overline{\lambda}-\widetilde{\lambda})t},\end{aligned}$$

where $S := \sum_{k=0}^{M-1} \mathrm{e}^{sk} \geqslant 0$. Clearly, $\mathsf{E}\big[\mathrm{e}^{sN_\sigma(0,t)}\big]$ is well defined for $t \geqslant 0$. □

43. LEMMA. *Consider the system (2). Suppose that the hypotheses of Theorem 8 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have* $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

PROOF. Fix $t \in \mathbb{R}_{\geqslant 0}$, and let $\nu := N_\sigma(0,t)$. It follows from Lemma 35 that

$$V_{\sigma(t)}(x(t)) \leqslant \mu^\nu \cdot V_{\sigma(0)}(x_0) \cdot \left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\right) \cdot \mathrm{e}^{-\lambda_{\sigma(t)}(t-\tau_\nu)} \quad \text{a.s.}$$

Taking expectations, and keeping in mind that by (G1) $\forall\, p \in \mathcal{P}\ \ \lambda_p = \lambda_\circ$, we obtain

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \mathsf{E}[\mu^\nu]\, \mathrm{e}^{-\lambda_\circ t} V_{\sigma(0)}(x_0).$$

From Lemma 41 it follows that (42) holds; the substitution $s = \ln \mu$ leads to

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant V_{\sigma(0)}(x_0)\mathrm{e}^{\lambda t} + SV_{\sigma(0)}(x_0)\mathrm{e}^{-\lambda_\circ t},$$

where, in view of (G3), $\lambda := \lambda_\circ - \Big(\mu\overline{\lambda} - \widetilde{\lambda}\Big) > 0$. Together with (V1) this yields

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \alpha_2(\|x_0\|)\left(\mathrm{e}^{-\lambda t} + S\mathrm{e}^{-\lambda_\circ t}\right). \tag{44}$$

For an arbitrary fixed, nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$, (44) shows that

$$\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0,$$

since $\lim_{t\to\infty}\left(\mathrm{e}^{-\lambda t} + S\mathrm{e}^{-\lambda_\circ t}\right) = 0$. □

45. LEMMA. *Consider the system (2). Suppose that the hypotheses of Theorem 11 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have* $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

PROOF. Fix $\nu \in \mathbb{N}$. From Lemma 35 we get, for $s \in [\tau_\nu, \tau_{\nu+1}[$,

$$V_{\sigma(s)}(x(s)) \leqslant \mu^\nu \cdot V_{\sigma(0)}(x_0)\left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\right)\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)} \quad \text{a.s.}$$

Taking expectations, and utilizing (V1) and (EH1)-(EH3) in successive steps,

$$\begin{aligned}\mathsf{E}\big[V_{\sigma(s)}(x(s))\big] &\leqslant \mathsf{E}\left[\mu^\nu V_{\sigma(0)}(x_0)\left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\right)\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right]\\ &\leqslant \alpha_2(\|x_0\|)\mathsf{E}\left[\mu^\nu \left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\right)\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right]\\ &= \alpha_2(\|x_0\|)\mu^\nu\left(\prod_{i=0}^{\nu-1} \mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\big]\right)\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\Big].\end{aligned} \tag{46}$$

The finiteness of $\mathcal{P}$ and (EH1)-(EH3) lead to

$$\mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\big] = \int_0^\infty \sum_{p\in\mathcal{P}} \mathrm{e}^{-\lambda_p v}\mathsf{P}\big(\sigma(\tau_i)=p\big)\,\lambda \mathrm{e}^{-\lambda v}\mathrm{d}v \\ = \sum_{p\in\mathcal{P}} \mathsf{P}\big(\sigma(\tau_i)=p\big)\,\lambda \int_0^\infty \mathrm{e}^{-(\lambda_p+\lambda)v}\mathrm{d}v.$$

In view of (E3) the preceding equation may be written as

$$\mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\big] = \sum_{p\in\mathcal{P}} \frac{\lambda q_p}{\lambda_p+\lambda}. \tag{47}$$

Since $\mathcal{P}$ is finite, $\overline{\lambda} := -\max_{p\in\mathcal{P}}\{-\lambda_p\}$ is well-defined; therefore

$$\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\Big] \leqslant \mathsf{E}\Big[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\Big].$$

If $\overline{\lambda} \geqslant 0$, then $\mathsf{E}\Big[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\Big] \leqslant 1$; if $\overline{\lambda} < 0$, then $\mathsf{E}\Big[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\Big] \leqslant \mathsf{E}\Big[\mathrm{e}^{-\overline{\lambda}(\tau_{\nu+1}-\tau_\nu)}\Big]$, and by (E3) and (EH1) this equals $\lambda/(\overline{\lambda}+\lambda)$. Hence

$$\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\Big] \leqslant 1 \vee \left(\frac{\lambda}{\overline{\lambda}+\lambda}\right). \tag{48}$$

Collecting the results of (46), (47) and (48), $\forall\, s \in [\tau_\nu, \tau_{\nu+1}[$ we obtain

$$\mathsf{E}\big[V_{\sigma(s)}(x(s))\big] \leqslant \alpha_2(\|x_0\|)\cdot\left(1 \vee \left(\frac{\lambda}{\overline{\lambda}+\lambda}\right)\right)\cdot\left(\sum_{p\in\mathcal{P}} \frac{\mu q_p}{(1+\lambda_p/\lambda)}\right)^{\nu}. \tag{49}$$

Fix a nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$. By Lemma 33, for every $i \in \mathbb{N}$ there exists $\nu(i) \in \mathbb{N}$ such that $s_i \in [\tau_{\nu(i)}, \tau_{\nu(i)+1}[$, and $\nu(i) \longrightarrow \infty$ as $i \longrightarrow \infty$ almost surely. Therefore, from (49) and (E4) we get

$$\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] \leqslant \alpha_2(\|x_0\|)\cdot\left(1 \vee \left(\frac{\lambda}{\overline{\lambda}+\lambda}\right)\right)\cdot\limsup_{i\to\infty}\left(\sum_{p\in\mathcal{P}} \frac{\mu q_p}{(1+\lambda_p/\lambda)}\right)^{\nu(i)} \\ = 0.$$

Since the sequence $(s_i)_{i\in\mathbb{N}}$ was arbitrary, we conclude that for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$, $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$. □

50. Lemma. *Consider the system (2). Suppose that the hypotheses of Theorem 14 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have* $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

Proof. Fix $\nu \in \mathbb{N}$. From Lemma 35 we get, for $s \in [\tau_\nu, \tau_{\nu+1}[$,

$$V_{\sigma(s)}(x(s)) \leqslant \mu^\nu \cdot V_{\sigma(0)}(x_0)\left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\right)\cdot \mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)} \quad \text{a.s.}$$

Taking expectations, and utilizing (V1) and (UH1)-(UH3) in successive steps,

$$\begin{aligned}\mathsf{E}\big[V_{\sigma(s)}(x(s))\big] &\leqslant \mathsf{E}\left[\mu^{\nu} V_{\sigma(0)}(x_0)\left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\right) \mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right]\\ &\leqslant \alpha_2(\|x_0\|)\mathsf{E}\left[\mu^{\nu}\left(\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\right) \mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right]\\ &= \alpha_2(\|x_0\|)\mu^{\nu}\left(\prod_{i=0}^{\nu-1} \mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\big]\right) \mathsf{E}\left[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right]. \qquad (51)\end{aligned}$$

The finiteness of $\mathcal{P}$ and (UH1)-(UH3) lead to

$$\begin{aligned}\mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\big] &= \int_0^T \sum_{p\in\mathcal{P}} \mathrm{e}^{-\lambda_p v} \mathsf{P}\big(\sigma(\tau_i)=p\big)\frac{1}{T}\mathrm{d}v\\ &= \sum_{p\in\mathcal{P}} \mathsf{P}\big(\sigma(\tau_i)=p\big)\frac{1}{T}\int_0^T \mathrm{e}^{-\lambda_p v}\mathrm{d}v.\end{aligned}$$

In view of (U3) the preceding equation may be written as

$$\mathsf{E}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\big] = \sum_{p\in\mathcal{P}} \frac{q_p\left(1-\mathrm{e}^{\lambda_p T}\right)}{\lambda_p T}. \qquad (52)$$

Since $\mathcal{P}$ is finite, $\overline{\lambda} := -\max_{p\in\mathcal{P}}\{-\lambda_p\}$ is well-defined; therefore

$$\mathsf{E}\left[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right] \leqslant \mathsf{E}\left[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\right].$$

If $\overline{\lambda}\geqslant 0$, then $\mathsf{E}\left[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\right]\leqslant 1$; if $\overline{\lambda}<0$, then $\mathsf{E}\left[\mathrm{e}^{-\overline{\lambda}(s-\tau_\nu)}\right]\leqslant \mathrm{e}^{-\overline{\lambda}T}$. Hence

$$\mathsf{E}\left[\mathrm{e}^{-\lambda_{\sigma(s)}(s-\tau_\nu)}\right] \leqslant 1\vee \mathrm{e}^{-\overline{\lambda}T}. \qquad (53)$$

Collecting the results of (51), (52) and (53), $\forall\, s\in[\tau_\nu,\tau_{\nu+1}[$ we obtain

$$\mathsf{E}\big[V_{\sigma(s)}(x(s))\big] \leqslant \alpha_2(\|x_0\|)\cdot\left(1\vee \mathrm{e}^{-\overline{\lambda}T}\right)\cdot\left(\sum_{p\in\mathcal{P}}\frac{\mu q_p\left(1-\mathrm{e}^{-\lambda_p T}\right)}{\lambda_p T}\right)^{\nu}. \qquad (54)$$

Fix a nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$. By Lemma 34, for every $i\in\mathbb{N}$ there exists $\nu(i)\in\mathbb{N}$ such that $s_i\in[\tau_{\nu(i)},\tau_{\nu(i)+1}[$, and $\nu(i)\longrightarrow\infty$ as $i\longrightarrow\infty$ almost surely. Therefore, from (54) and (U3) we get

$$\begin{aligned}\limsup_{i\to\infty}\mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] &\leqslant \alpha_2(\|x_0\|)\cdot\left(1\vee \mathrm{e}^{-\overline{\lambda}T}\right)\cdot\limsup_{i\to\infty}\left(\sum_{p\in\mathcal{P}}\frac{\mu q_p\left(1-\mathrm{e}^{-\lambda_p T}\right)}{\lambda_p T}\right)^{\nu(i)}\\ &= 0.\end{aligned}$$

Since the sequence $(s_i)_{i\in\mathbb{N}}$ was arbitrary, we conclude that for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$, $\limsup_{i\to\infty}\mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big]=0$. □

55. Lemma. *The system (2) has the following property: for every $\varepsilon>0$ there exists $L_\varepsilon>0$ such that $\|x(t)\|\leqslant\|x_0\|\,\mathrm{e}^{L_\varepsilon t}$ $\forall\, t\geqslant 0$ as long as $\|x(t)\|<\varepsilon$.*

Proof. Since the vector field of each individual subsystem of the family (1) is locally Lipschitz, and $\mathcal{P}$ is a finite set, there exists a constant $L_\varepsilon>0$ such that

$$\sup_{\substack{p\in\mathcal{P},\\ \|x\|\in[0,\varepsilon[}} \|f_p(x)\| \leqslant L_\varepsilon\|x\|. \qquad (56)$$

Also, $\forall\, x \in \mathbb{R}^n \smallsetminus \{0\}$, $\left|\frac{\mathrm{d}\|x\|^2}{\mathrm{d}t}\right| = \left\|2x^{\mathrm{T}}\frac{\mathrm{d}x}{\mathrm{d}t}\right\| \leqslant 2\|x\|\left\|\frac{\mathrm{d}x}{\mathrm{d}t}\right\|$, and $\left|\frac{\mathrm{d}\|x\|^2}{\mathrm{d}t}\right| = 2\|x\|\left\|\frac{\mathrm{d}\|x\|}{\mathrm{d}t}\right\|$, which leads to $\left|\frac{\mathrm{d}\|x\|}{\mathrm{d}t}\right| \leqslant \left\|\frac{\mathrm{d}x}{\mathrm{d}t}\right\|$. For the family (1), this and (56) lead to

$$\frac{\mathrm{d}\,\|x\|}{\mathrm{d}t} \leqslant L_\varepsilon \|x\| \qquad \forall\, x \in \left\{x \in \mathbb{R}^n \,\middle|\, \|x\| < \varepsilon\right\} \smallsetminus \{0\}. \tag{57}$$

An application of a standard differential inequality (see e.g. [16, Theorem 1.2.1]) indicates that every solution $x(\cdot)$ of (2) satisfies

$$\|x(t)\| \leqslant \|x_0\|\,\mathrm{e}^{L_\varepsilon t}$$

so long as $\|x(t)\| < \varepsilon$. This proves the claim. □

**§5.2. Proofs of the theorems in §3.** We are finally ready for the proofs of the theorems in §3.

Proof of Theorem 8. We need to establish the properties (AS1)-(AS2) of (2).

First we prove (AS2). Fix $r, \varepsilon' > 0$. Lemma 43 shows that the assertion of Lemma 36 holds. In view of (V1) and Lemma 36, we can now write

$$\lim_{t\to\infty} \alpha_1(\|x(t)\|) = 0 \quad \text{a.s.};$$

hence there exists $T(r, \varepsilon') \geqslant 0$ such that

$$\|x_0\| < r \implies \mathsf{P}\left(\sup_{t \geqslant T(r,\varepsilon')} \alpha_1(\|x(t)\|) < \alpha_1(\varepsilon')\right) = 1.$$

Since $r, \varepsilon'$ are arbitrary, we conclude that $\forall\, r, \varepsilon' > 0$ there exists $T(r, \varepsilon') \geqslant 0$ such that

$$\|x_0\| < r \implies \mathsf{P}\left(\sup_{t \geqslant T(r,\varepsilon')} \|x(t)\| < \varepsilon'\right) = 1.$$

The (AS2) property of (2) follows.

It remains to prove (AS1). Fix $\varepsilon > 0$. We know from the (AS2) property proved above that there exists a nonnegative real number $T(1, \varepsilon)$, so that

$$\|x_0\| < 1 \implies \mathsf{P}\left(\sup_{t \geqslant T(1,\varepsilon)} \|x(t)\| < \varepsilon\right) = 1.$$

Select $\delta(\varepsilon) = \varepsilon \mathrm{e}^{-L_\varepsilon T(1,\varepsilon)} \wedge 1$. By Lemma 55, $\|x_0\| < \delta(\varepsilon)$ implies

$$\|x(t)\| \leqslant \|x_0\|\,\mathrm{e}^{L_\varepsilon t} < \delta(\varepsilon)\mathrm{e}^{L_\varepsilon T(1,\varepsilon)} < \varepsilon \;\forall\, t \in [0, T(1, \varepsilon)].$$

Further, the (AS2) property guarantees that with the above choice of $\delta$ and $x_0$, we have $\mathsf{P}\big(\sup_{t \geqslant T(1,\varepsilon)} \|x(t)\| < \varepsilon\big) = 1$. Thus, $\|x_0\| < \delta(\varepsilon)$ implies $\mathsf{P}\big(\sup_{t\geqslant 0} \|x(t)\| < \varepsilon\big) = 1$. Since $\varepsilon$ is arbitrary, the (AS1) property of (2) follows.

We conclude that (2) is GAS a.s. □

Proof of Theorem 11. The proof repeats verbatim that of Theorem 8, with just Lemma 45 substituted in place of Lemma 43. □

Proof of Theorem 14. The proof repeats verbatim that of Theorem 8, with just Lemma 50 substituted in place of Lemma 43. □

Proof of Corollary 9. Suppose that (G1′) holds. Using Jensen's inequality[§] in (V1), in view of (44) we arrive at

$$\alpha_1(\mathsf{E}[\|x(t)\|]) \leqslant \mathsf{E}[\alpha_1(\|x(t)\|)] \leqslant \alpha_2(\|x_0\|)\left(S\mathrm{e}^{-\lambda_\circ t} + \mathrm{e}^{-\lambda t}\right).$$

[§]Recall [2] Jensen's inequality: if $Z$ is an integrable random variable on the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, then $\phi(\mathsf{E}[Z]) \leqslant \mathsf{E}[\phi(Z)]$.

If we let $\widetilde{\beta}(r,s) := \alpha_1^{-1}\left(\alpha_2(r)\left(S\mathrm{e}^{-\lambda_\circ s} + \mathrm{e}^{-\lambda s}\right)\right)$, then $\widetilde{\beta}$ is a class $\mathcal{KL}$ function.[¶] It follows that

$$\mathsf{E}[\|x(t)\|] \leqslant \widetilde{\beta}(\|x_0\|, t).$$

A standard argument (see [**15**, Lemma 4.5]) now shows that the (SM1) and (SM2) properties follow from the above inequality.

Suppose now that (G2′) holds instead of (G1′). Let the Taylor's series development of $\alpha_1$ around 0 be

$$\alpha_1(y) = c_1 y + R(y),$$

where $R$ is the remainder term, $R(y) = o(y)$,[‖] and $c_1 > 0$ by hypothesis.

First we prove (SM1). Fix $\varepsilon > 0$. By Theorem 8 and Definition 3, there exists $\delta(\varepsilon) > 0$ so that

$$\|x_0\| < \delta(\varepsilon) \implies \mathsf{P}\left(\sup_{t\geqslant 0}\|x(t)\| < \varepsilon\right) = 1; \tag{58}$$

i.e., (AS1) holds. Let $\Omega' \subset \Omega$ be the set of all $\omega$ such (AS1) fails; Theorem 8 guarantees that $\mathsf{P}(\Omega') = 0$. With $x_0$ selected such that $\|x_0\| < \delta(\varepsilon)$, it follows that

$$\mathsf{E}[\|x(t)\|] = \int_\Omega \|x(t,\omega)\|\,\mathsf{P}(\mathrm{d}\omega) = \int_{\Omega\smallsetminus\Omega'}\|x(t,\omega)\|\,\mathsf{P}(\mathrm{d}\omega) + \int_{\Omega'}\|x(t,\omega)\|\,\mathsf{P}(\mathrm{d}\omega) < \varepsilon.$$

The (SM1) property of (2) follows with $\widetilde{\delta} = \delta$.

Now we prove (SM2). Fix $r, \varepsilon' > 0$, and select $x_0$ with $\|x_0\| < r$. In view of (44) and (V1), there exists $T_1(r,\varepsilon') \geqslant 0$ such that

$$\mathsf{E}[\alpha_1(\|x(t)\|)] < \frac{c_1\varepsilon'}{2} \qquad \forall\, t \geqslant T_1(r,\varepsilon'). \tag{59}$$

Also, since $R(y) = o(y)$, there exists $\delta(\varepsilon') > 0$ such that $\forall\, y \in [0, \delta(\varepsilon')[$, we have $|R(y)| < c_1\varepsilon'/2$. In view of the assertion of Theorem 8, there exists $T_2(r,\varepsilon') \geqslant 0$ such that

$$\|x_0\| < r \implies \mathsf{P}\left(\sup_{t\geqslant T_2(r,\varepsilon')}\|x(t)\| < \delta(\varepsilon')\right) = 1; \tag{60}$$

i.e., (AS2) holds. Let $\Omega'' \subset \Omega$ be the set of all $\omega$ such that (AS2) fails; Theorem 8 guarantees that $\mathsf{P}(\Omega'') = 0$. From (59) it follows that $\forall\, t \geqslant T_1(r,\varepsilon') \vee T_2(r,\varepsilon')$,

$$\begin{aligned}
\frac{c_1\varepsilon'}{2} &> \mathsf{E}[\alpha_1(\|x(t)\|)]\\
&= \int_\Omega \big(c_1\|x(t,\omega)\| + R(\|x(t,\omega)\|)\big)\mathsf{P}(\mathrm{d}\omega)\\
&= c_1\int_\Omega \|x(t,\omega)\|\,\mathsf{P}(\mathrm{d}\omega) + \int_{\Omega''} R(\|x(t,\omega)\|)\mathsf{P}(\mathrm{d}\omega) + \int_{\Omega\smallsetminus\Omega''} R(\|x(t,\omega)\|)\mathsf{P}(\mathrm{d}\omega)\\
&\geqslant c_1\mathsf{E}[\|x(t)\|] - \int_{\Omega\smallsetminus\Omega''}|R(\|x(t,\omega)\|)|\,\mathsf{P}(\mathrm{d}\omega)\\
&\geqslant c_1\mathsf{E}[\|x(t)\|] - \frac{c_1\varepsilon'}{2},
\end{aligned}$$

and we arrive at $\mathsf{E}[\|x(t)\|] < \varepsilon'$. Since $r, \varepsilon'$ were arbitrary, with $\widetilde{T}(r,\varepsilon') := T_1(r,\varepsilon') \vee T_2(r,\varepsilon')$, the (SM2) property of (2) follows. □

[¶] A function $\beta : \mathbb{R}^2_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$, continuous in both arguments, is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is a function of class $\mathcal{K}$ for every fixed $t$, and $\beta(r,t) \to 0$ as $t \to \infty$ for every fixed $r$; we write $\beta \in \mathcal{KL}$.

[‖] Recall that $\lim_{h\to 0}\frac{o(h)}{h} = 0$, and $\lim_{h\to 0}\frac{O(h)}{h} \in\, ]-\infty, \infty[$.

PROOF OF COROLLARY 12. Suppose that (E1′) holds. Then for a fixed $\nu \in \mathbb{N} \cup \{0\}$, using Jensen's inequality in (49), $\forall\, s \in [\tau_\nu, \tau_{\nu+1}[$

$$\mathsf{E}[\|x(s)\|] \leqslant \alpha_1^{-1}\left(\alpha_2(\|x_0\|) \cdot \left(1 \vee \left(\frac{\lambda}{\overline{\lambda} + \lambda}\right)\right) \cdot \left(\sum_{p \in \mathcal{P}} \frac{\mu q_p}{(1 + \lambda_p/\lambda)}\right)^{\nu}\right). \tag{61}$$

Considering (E4), it follows that

$$\mathsf{E}[\|x(s)\|] \leqslant \alpha_1^{-1}\left(\alpha_2(\|x_0\|) \cdot \left(1 \vee \left(\frac{\lambda}{\overline{\lambda} + \lambda}\right)\right)\right).$$

The right hand side of the above inquality is independent of $\nu$; hence $\forall\, s \in \mathbb{R}_{\geqslant 0}$. It follows that for $\varepsilon > 0$, (SM1) holds with $\delta(\varepsilon) \in \left]0, \alpha_2^{-1}\big(\alpha_1(\varepsilon)/\big(1 \vee \big(\lambda/(\overline{\lambda} + \lambda)\big)\big)\big)\right[$. To see (SM2), it suffices to prove that $\lim_{t\to\infty} \mathsf{E}[\|x(t)\|] = 0$. But by Lemma 33 it follows that for every $t \in \mathbb{R}_{\geqslant 0}$ there exists $\nu(t) \in \mathbb{N} \cup \{0\}$ such that $t \in [\tau_{\nu(t)}, \tau_{\nu(t)+1}[$, and $\nu(t) \longrightarrow \infty$ almost surely as $t \longrightarrow \infty$; keeping in mind (E4), (61) now yields the desired limit.

Suppose now that (E2′) holds instead of (E1′). In this case the proof repeats verbatim that of Corollary 9 when (G2′) holds instead of (G1′), with just Theorem 11 replacing Theorem 8. □

PROOF OF COROLLARY 15. The proof repeats verbatim that of Corollary 12, with just (U1′), (U2′), (U3), and Theorem 14 in place of (E1′), (E2′), (E4), and Theorem 11, respectively. □

## § 6. Conclusion and Further Work

We established a methodology for almost sure global asymptotic stabilization of randomly switched systems. As mentioned in §1, a necessary condition for its applicability is that the controller for every subsystem can be so placed that the switching signal activates each closed loop subsystem. Or, if the controller is implemented as a central unit, then it has to have perfect information about $\sigma$ at each instant of time. This actually leads us to wonder whether it is possible to design one stabilizing controller for the switched control system, which gets imperfect or no information about $\sigma$.

In the deterministic context, the problem of simultaneous stabilization of multiple systems can be thought of as a possible approach to the case when the controller gets no information about $\sigma$. Indeed, if a single controller stabilizes each subsystem, then under a sufficiently slow switching hypotheses (e.g. Assumption 7 with small enough $\overline{\lambda}$), the closed loop switched system will be GAS a.s. But in general the problem of simultaneous stabilization is restrictive and difficult. However, if there exists a controller that stabilizes a subfamily of $(f_p)_{p \in \mathcal{P}}$ and at the same time does not destabilize the others subsystems too much, the theorems of §3 can be applied to the closed loop switched system. Such results will be reported elsewhere.

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