Energy control of a pendulum with quantized feedback

Ruslan Seifullaev\textsuperscript{a,b}, Alexander Fradkov\textsuperscript{a,b}, Daniel Liberzon\textsuperscript{c}

\textsuperscript{a}Saint Petersburg State University, Russia
\textsuperscript{b}Institute of Problems of Mechanical Engineering, Saint Petersburg, Russia
\textsuperscript{c}Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, IL, U.S.A.

Abstract

The problem of controlling a nonlinear system to an invariant manifold using quantized state feedback is considered by the example of controlling the pendulum’s energy. A feedback control law based on the speed gradient algorithm is chosen. The main result consisting in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds is presented.

Key words: nonlinear control, quantized signals.

1 Introduction

Control theory has initially been developed under idealistic assumptions regarding information transmission in a feedback loop. More recently, however, researchers have been increasingly interested in the question of how much information is really needed to perform a desired control task, or conversely, what control objectives can be achieved with a given amount of information. Such considerations arise from applications where scarce communication resources, sensor limitations, or security concerns play a role, and are also motivated by theoretical interest in understanding the interplay between information and control.

Among the various phenomena responsible for a limited amount of information available in a feedback loop, quantization is one of the most basic and widely investigated. By a quantizer we mean a function that maps a continuous real-valued system signal into a piecewise constant one taking a finite set of values, thereby encoding this signal using a finite alphabet. Notable early studies of the effect of quantization on the behavior of control systems include \cite{11,4,13,5}, and a brief overview of the recent literature can be found in \cite{16}.

One approach to analysis of quantized control systems, taken in \cite{12} and elsewhere, involves modeling quantization effects as additive errors. If the controller possesses suitable robustness with respect to such errors, then the system performance can be shown to degrade gracefully due to quantization. In the context of stabilizing an equilibrium, instead of global asymptotic stabilization one typically obtains two nested invariant regions such that all trajectories starting in the larger one converge to the smaller one, a fact usually established by Lyapunov arguments. While robustness to additive errors is automatic for linear systems and linear feedback controllers, for general nonlinear systems the robustness requirements can be quite restrictive and finding a controller meeting such requirements can be challenging \cite{12}.

Pendulum dynamics is a popular and important benchmark system in control theory. The problem of stabilizing the upright equilibrium, as well as the problem of controlling the pendulum’s energy to a desired level, have been widely studied and call for innovative solutions. In particular, it is known (see, e.g., \cite{17}) that the upright equilibrium cannot be globally asymptotically stabilized by continuous feedback. See \cite{3,17,20,2,15} and the references therein for some interesting contributions to pendulum control. More generally, the problem of energy control for Hamiltonian systems was first considered in \cite{7}. In \cite{18,19} extended conditions for control of invariant sets were proposed with application to energy control of the pendulum.
In this paper we consider the problem of controlling the pendulum’s energy to a desired level using quantized state feedback. As the nominal feedback law, we choose one based on the speed gradient method from [6] (which stabilizes any energy level without quantization). As a candidate Lyapunov function, we choose the squared difference between the current and the desired energy levels (which decreases for the closed-loop system without quantization). We show that in the presence of sufficiently small state quantization errors, even though the Lyapunov function may not always decrease, the time periods on which it may increase and the amount by which it may increase are suitably bounded and decreasing behavior still dominates. Using these properties, we are able to establish that if the initial energy level is not too far from the desired one, then it will remain not too far from it and will eventually become close to it. While this result may appear intuitively not surprising, our main contribution lies in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds.

The rest of the paper is structured as follows. In Section 2 the general problem of the pendulum’s energy control using quantized state feedback is described. Our main result is presented in Section 3. Section 4 is devoted to a numerical example demonstrating the performance predicted by the main theorem.

2 Problem formulation

Consider the pendulum equations

\[ \ddot{\varphi}(t) = -\frac{g}{l} \sin \varphi(t) + \frac{1}{ml^2} u(t), \]  

where \( \varphi \) is a deviation angle (\( \varphi = 0 \) at the lower position), \( u \) is a controlling torque, \( g \) is a gravity acceleration, \( m \) and \( l \) are the mass and the length of the pendulum respectively.

Assume that \( H(\varphi, \dot{\varphi}) \) is the full energy of the pendulum, i.e.

\[ H(\varphi, \dot{\varphi}) = \frac{1}{2} ml^2 \dot{\varphi}^2 + mgl(1 - \cos \varphi). \]

Consider the problem of energy level stabilization of system (1). Let \( z = [\varphi, \dot{\varphi}]^T, z \in \mathbb{R}^2 \). Let \( h \) \( (h < 2mgl) \) be a positive number. Consider a set

\[ X_h = \{ z : 0 < H(z) \leq h \}. \]

Let \( H_* (0 < H_* < h) \) be desired energy level and the goal function be as follows

\[ V(z) = \frac{1}{2} (H(z) - H_*)^2. \]  

It is required to design a feedback law

\[ u = U(z), \]

providing the achievement of the control goal

\[ \lim_{t \to \infty} V(z(t, z_0)) = 0, \]  

where the initial energy level \( H(z_0) \) satisfies the following assumption:

\[ z_0 \in X_h, \]

i.e. \( z_0 \) belongs to energy layer between 0 and \( h \).

The algorithm design is based on the speed gradient method [6,9,10]. According to the speed gradient method it is required to calculate the function \( \omega(z, u) = V(z) \), i.e. \( \omega(z, u) \) is the speed of variation of the quantity \( V \) along the trajectories of system (1)

\[ \omega(z, u) = (H(z) - H_*) B^T z u, \]

where \( B = [0, 1]^T \). Let us find \( u \)-derivative of \( \omega(z, u) \) and write down the control algorithm in the finite form

\[ u = U(z) = -\gamma \frac{\partial \omega}{\partial u} = -\gamma (H(z) - H_*) B^T z, \]  

where \( \gamma > 0 \).

The idea of algorithm (5) can be explained as follows [8]. To achieve control goal (3), it is advisable to vary \( u \) such that \( V \) decreases. But because \( V \) does not depend on \( u \), it is difficult to find the direction of such decrease. Instead, one can decrease \( \dot{V} \) by ensuring that \( \dot{V} < 0 \), which is the condition that \( V \) decreases. The function \( \dot{V}(z) = \omega(z, u) \) explicitly depends on \( u \), which makes it possible to design algorithm (5).

The following theorem, characterizing the performance of control algorithm (5), can be directly concluded from Theorem 3.1 and Remark 3.1 in [9].

**Theorem 1** If the initial energy layer between the levels \( H(z_0) \) and \( H_* \) does not contain an equilibrium of the unforced system, then the goal level \( H_* \) will be achieved in the controlled system (1), (5) for any \( \gamma > 0 \) from all initial conditions.

The fulfillment of the condition in Theorem 1 follows from (4).

Let the set \( Z = \{ z_i : z_i \in X_h, i \in \mathbb{N} \} \cup z_{sat} \) be a finite subset of \( X_h \cup z_{sat} \), where \( z_{sat} \in \mathbb{R}^2 \). Consider quantizer \( q(z) : \mathbb{R}^2 \to Z \) proposed in [12]. Assume that \( Z_i = \{ z \in \mathbb{R}^2 : q(z) = z_i \} \) are quantizer regions (Fig.1), such that \( \bigcup Z_i = X_h \). Hence, \( q(z) = z_i \) for all \( z \in Z_i \),
Law (6) can be rewritten as follows:

\[
\text{Hence, } \phi(z) \text{ state }
\]

Suppose that only quantized measurements \( q(z) \) of the state \( z \) are available. Then the state feedback law (5) is non-implementable. Hence, instead of continuous control (5) consider quantized feedback control law (5):

\[
u = U(q(z)) = -\gamma (H(q(z)) - H_\ast) B^T q(z). \tag{6}
\]

Therefore, it is required to analyze the conditions of achievement of control goal (3) using quantized state feedback control (6). Note that assumption (4) is essential in the case of control algorithm (5) but can be omitted with using modifications of (5). In [17] it is shown that the global attractivity of the upright equilibrium can be achieved by a modification of the speed gradient energy method based on the idea of variable structure systems (VSS). However, such a modified algorithm is not suitable for application to the case of quantized measurements.

3 Main result

Let \( e(z) = q(z) - z = [e_1(z), e_2(z)]^T \) be a quantizer error vector. Assume that quantizer is chosen such that

\[|e_1(z)| \leq \Delta_1, \quad |e_2(z)| \leq \Delta_2 \quad \text{for all } z \in X_h. \tag{7}\]

Hence, \(|e(z)| \leq \sqrt{\Delta_1^2 + \Delta_2^2} = \Delta \quad \text{for all } z \in X_h.\]

Law (6) can be rewritten as follows:

\[U(q(z)) = U(z) + e_u(z), \]

where

\[
e_u(z) = -\gamma \left( \frac{1}{2} ml^2(e_2(z) + \dot{\phi})^2 + mgl (1 - \cos(\phi + e_1(z))) - H_\ast \right) e_2(z)
\]

\[
+ \left( ml^2 \left( \frac{1}{2} e_2(z) \right) e_2(z)
\]

\[
+ mgl(\cos \phi - \cos(\phi + e_1(z))) \right) \dot{\phi}. \]

Note that for all \( z \in X_h \)

\[
mgl (1 - \cos(\phi + e_1(z))) = H(\phi + e_1(z), 0) \leq h, \quad \tag{8}
\]

and

\[
|\cos \phi - \cos(\phi + e_1(z))| = 2 \left| \sin \frac{e_1(z)}{2} \sin \frac{e_1(z)}{2} \right|
\]

\[
\leq 2 \left| \sin \frac{e_1(z)}{2} \right| \leq 2 \sin \frac{\delta}{2}, \quad \tag{9}
\]

where \( \delta = \min \{\pi, \Delta \}. \)

From (7), (8), (9) and \( \dot{\delta} \geq 0 \) the set

\[
\frac{\delta}{2}
\]

\[
\Delta_e = \frac{1}{2} ml^2 \Delta_1 \Delta_2 + \frac{3l \sqrt{2mh}}{2} \Delta_2^2
\]

\[
+(4h - H_\ast) \Delta_2 + 2 \sqrt{2mh} \sin \frac{\delta}{2}
\]

Denote for any \( b > a \geq 0 \) the set

\[
V_{[a,b]} = \{ z \in X_h : a \leq V(z) \leq b \},
\]

and for any \( d > c \geq 0 \) the set

\[
H_{[c,d]}^{-1} = \{ z \in X_h : c \leq H(z) \leq d \}.
\]

It is easy to see that

\[
V_{[a,b]}^{-1} = H_{[c,d]}^{-1} \cup H_{[c,d]}^{-1} [H_\ast - \sqrt{2h}, H_\ast, - \sqrt{2h}].
\]

Let \( h_\ast \) be a positive constant, satisfying the condition \( h_\ast < \min \{H_\ast, h - H_\ast\}. \) Consider the following functions

\["
of scalar variable $y$:

\[ f_1(y) = H_\ast - h_\ast - \frac{3ml^2 \Delta_2^2}{y^2}, \]

\[ f_2(y) = \frac{g}{I} \sqrt{1 - \left(1 - \frac{f_0(y)}{mgI} \right)^2} - \frac{\gamma \Delta_\ast}{ml^2} \left( \frac{h_\ast \sqrt{6}}{y} + 1 \right), \]

where $f_0(y) = \min \{2mgI - H_\ast - h_\ast, f_1(y)\}$,

\[ f_3(y) = 4h_2^2 - y^2 - \frac{16\sqrt{6} \gamma \Delta_3^3}{f_2(y) y}, \]

\[ f_4(y) = \frac{1}{12} y^2 \arccos \left(1 - \frac{f_1(y)}{f_2(y)} \right) f_2(y) - \Delta_2^2. \]

The main result is the following theorem.

**Theorem 2** If the following system of inequalities

\[ 0 < y < 2h_\ast, \quad f_1(y) > 0, \quad f_2(y) > 0, \quad f_3(y) > 0, \quad f_4(y) > 0 \] (11)

is feasible with respect to $y$, then for any solution $y = h_1$ of (11) and for any given initial condition $z(0) \in H_{[-\tilde{x}_2, H_\ast + \tilde{x}_2]}^1$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H_{[-\tilde{x}, H_\ast + \tilde{x}]}^1$ for all $t \geq 0$ and there exists $T > 0$ such that $z(t) \in H_{[-\tilde{x}, H_\ast + \tilde{x}]}^1$ for all $t \geq T$, where

\[ \tilde{x}_1 = \sqrt{\frac{h_2^2 + 2\sqrt{6} \gamma \Delta_3^3}{h_1 f_2(h_1)}}, \quad \tilde{x}_2 = \sqrt{h_2^2 + \sqrt{6} \gamma \Delta_3^3}. \]

**Corollary 1** For any $\tilde{x}_1, \tilde{x}_2$ satisfying $\tilde{x}_1 < \tilde{x}_2 < h_\ast$ there exist sufficiently small $\Delta_1, \Delta_2$ such that for any given initial condition $z(0) \in H_{[-\tilde{x}_2, H_\ast + \tilde{x}_2]}^1$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H_{[-\tilde{x}, H_\ast + \tilde{x}]}^1$ for all $t \geq 0$ and there exists $T > 0$ such that $z(t) \in H_{[-\tilde{x}, H_\ast + \tilde{x}]}^1$ for all $t \geq T$. \[ \Box \]

**Proof of Corollary 1.** Let $h_1 = 2\tilde{x}_1$. If $\Delta_1 \to 0$ and $\Delta_2 \to 0$ then $\Delta_\ast \to 0$. Hence, $\tilde{x}_2 \to h_\ast$.

Since the negative terms in $f_1(h_1), f_2(h_1)$ and $f_4(h_1)$ vanish and $f_3(h_1) \to 4(h_2^2 + \tilde{x}_2^2) > 0$ if $\Delta_\ast \to 0$ and $\tilde{x}_2 \to h_\ast$, it is easy to see that $y = 2\tilde{x}_1$ is a solution of system (11).

Finally, from $\tilde{x}_2 \to h_\ast$ one obtains $H_{[-\tilde{x}_2, H_\ast + \tilde{x}_2]}^1 \subset H_{[-\tilde{x}, H_\ast + \tilde{x}]}^1$. Therefore, Corollary 1 directly follows from Theorem 2. \[ \Box \]

The proof of Theorem 2 is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in [14].

**Lemma 1** Let there exist a constant $\alpha > 0$, such that $\bar{V}(z) \leq \alpha \mathbf{I}^T \{0\}$, \[ \forall z \in X_h, t \geq 0, \] where $V(z)$ is the time derivative of $V(z)$ along the trajectories of closed-loop system (1), (6). Let there exist positive constants $A, v_0, v_3 \ (0 < v_0 < v_3 < \infty)$ and continuous functions $\beta: \bar{V}_{[v_0, v_3]} \to \mathbb{R}^+ \{0\}$, such that for arbitrary trajectory $z(\cdot)$ of system (1), (6) if

\[ z(t) \in V_{[v_0, v_3]}^{-1} \setminus \Pi \quad \forall t \in [t_1, t_2] \text{ for some } 0 < t_1 < t_2, \]

then

\[ a) \ t_2 - t_1 \leq A, \text{ and } \]
\[ b) \text{ there exists } t_3 > t_2, \text{ such that } V(z(t_3)) - V(z(t_1)) \leq -\epsilon(z(t_1)), \forall t \in (t_2, t_3) \text{ from } z(t) \in V_{[v_0, v_3]}^{-1} \rightarrow \text{ it follows } z(t) \in \Pi. \]

Suppose $\alpha A < \frac{1}{2}(v_3 - v_0)$. Denote $v_1 = v_0 + \alpha A, v_2 = v_3 - \alpha A$. If $z(0) \in V_{[v_0, v_3]}^{-1}$, then $z(t) \in V_{[v_0, v_3]}^{-1}$ for all $t \geq 0$ and there exists $T > 0$ such that $z(t) \in V_{[v_0, v_3]}^{-1}$ for all $t \geq T$. \[ \Box \]

**Proof of Theorem 2.** Let $v_0 = \frac{1}{8}h_1^2, v_3 = \frac{1}{2}h_2^2$. Hence, from $0 < h_1 < 2h_\ast$ it follows that $0 < v_0 < v_3$.

By direct calculations

\[ \bar{V}(z) = (H(z) - H_\ast) \hat{\phi} (e_u(z) - \gamma (H(z) - H_\ast)) \hat{\phi}. \]

- Estimate $\bar{V}$.

Since

\[ \left(\frac{\sqrt{7}}{2}(H(z) - H_\ast) \frac{\hat{\phi}}{\gamma} - \frac{1}{2\gamma} e_u(z) \right)^2 = \frac{\gamma}{2} (H(z) - H_\ast)^2 \hat{\phi}^2 - (H(z) - H_\ast) \hat{\phi} e_u(z) + \frac{1}{2\gamma^2} e_u^2(z) \geq 0, \]

\[ \bar{V}(z) \leq -\gamma (H(z) - H_\ast)^2 \hat{\phi}^2 + \frac{\gamma}{2} (H(z) - H_\ast)^2 \hat{\phi}^2 + \frac{1}{2\gamma^2} e_u^2(z) \]

\[ \leq \frac{\gamma \Delta_2^2}{2}. \] (12)

Let $\alpha = \frac{\gamma \Delta_2^2}{2}$. Therefore, $\bar{V} \leq \alpha$ for all $z \in X_h, t \geq 0$.

- Let the function $\beta(z) = \text{const} = \frac{\alpha}{2}$. Prove that for arbitrary trajectory $z(t)$ if $z(t) \in V_{[v_0, v_3]}^{-1}$ and $\bar{V} \geq -\frac{\alpha}{2}$
for all $t \in [t_1, t_2]$, then there exists $A > 0$ such that $t_2 - t_1 \leq A$.

From $V(z) \in [v_0, v_3]$ it follows that $|H(z) - H_*| \in \left[\frac{1}{2} h_1, h_*\right]$.

Since $\frac{\gamma}{2} (H(z) - H_*)^2 \psi^2 + \frac{\gamma}{2} \Delta_e^2 \geq \dot{V}(z) \geq -\frac{\gamma \Delta_e^2}{4}$,

$$(H(z) - H_*)^2 \psi^2 \leq \frac{3 \Delta e^2}{2}. $$

Hence,

$$|\dot{\psi}| \leq \frac{\Delta_e \sqrt{6}}{h_1}. \tag{13}$$

Estimate $|\ddot{\psi}|$.

$$|\ddot{\psi}| = \left| \frac{g}{t} \sin \varphi - \frac{1}{mt^2} (U(z) + e_u(z)) \right| \geq \frac{g}{t} |\sin \varphi| - \frac{1}{mt^2} |U(z) + e_u(z)|. \tag{14}$$

Since

$$|U(z) + e_u(z)| \leq |U(z)| + |e_u(z)| \leq \gamma |H(z) - H_*| |\dot{\psi}| + \gamma \Delta_e \leq \gamma h_* \frac{\Delta_e \sqrt{6}}{h_1} + \gamma \Delta_e = \gamma \Delta_e \left( \frac{h_* \sqrt{6}}{h_1} + 1 \right),$$

$$|\ddot{\psi}| \geq \frac{g}{t} |\sin \varphi| - \frac{2 \Delta_e}{mt^2} \left( \frac{h_* \sqrt{6}}{h_1} + 1 \right).$$

Now let us estimate $|\sin \varphi|$. From (13) and $|H(z) - H_*| \in \left[\frac{1}{2} h_1, h_*\right]$ one obtains (see Fig. 2)

$$|\sin \varphi| \geq |\sin \varphi_{\text{min}}|,$$

where $\varphi_{\text{min}}$ is such that

$$P(\varphi_{\text{min}}) = \min \left\{ 2mgI - H_* - h_*, \ H_* - h_* - \frac{3ml^2 \Delta_e^2}{h_1^2} = f_1(h_1) \right\} = f_0(h_1),$$

and $P(\varphi) = mgI (1 - \cos \varphi)$ is the potential energy.

Hence, $|\sin \varphi| \geq \sqrt{1 - \left( 1 - \frac{f_0(h_1)}{mgI} \right)^2}$ and

By definition of $t_2$ we have $\frac{d|\ddot{\psi}(t_2)|}{dt} > 0$, hence, $\dot{\psi}(t_2) = \text{sign} \dot{\psi}(t_2)$. From $f_3(h_1) > 0$ it follows that $|\frac{g}{t} \sin \varphi(t_2)| > \frac{\alpha(\varphi(t_2) + e(\varphi(t_2))}{mgI}$. Then sign $\dot{\psi}(t_2) = -\text{sign} \dot{\psi}(t_2)$. Therefore, sign $\dot{\psi}(t_2) = -\text{sign} \dot{\psi}(t_2)$. By definition of $\varphi_0$ it follows that $|\ddot{\psi}| > \frac{\Delta \sqrt{6}}{h_1}$ on the strip $\Gamma$ (see Fig. 3). Thus,

$$\dot{V}(t) \leq -\frac{\gamma}{2} (H(z) - H_*)^2 \psi^2 + \frac{\gamma \Delta_e^2}{2} \leq -\frac{\gamma \Delta_e^2}{4} = -\frac{\alpha}{2}.$$
Denote $\epsilon$ where $f$.

Since $H$.

Hence, $t$ and $z(t)$.

estimate $V(t_3) - V(t_1)$. From Cauchy–Schwarz inequality one has $\int_{t_2}^{t_3} |\dot{\varphi}|. From (12)

$\int_{t_2}^{t_3} |\dot{\varphi}| \leq \frac{1}{24} \gamma h_1^2 \left( \int_{t_2}^{t_3} \dot{\varphi}^2 dt \right)^{\frac{1}{2}}$.

Hence,

$V(t_3) - V(t_2) \leq -\frac{1}{24} \gamma h_1^2 \left( \frac{t_3}{t_2} \right)^2 \left( \int_{t_2}^{t_3} \dot{\varphi}^2 dt \right)^{\frac{1}{2}}$

$= -\frac{1}{24} \gamma h_1^2 \frac{1}{t_3 - t_2} (\varphi(t_3) - \varphi(t_2))^2 \leq -\frac{1}{6} \gamma h_1^2 \frac{1}{t_3 - t_2} \varphi_0^2$.

Since

$t_3 - t_2 \leq \frac{2 \varphi_0 h_1}{\Delta_e \sqrt{6}}$.

$V(t_3) - V(t_2) \leq -\frac{1}{2\sqrt{6}} h_1 \varphi_0 \Delta_e$.

Denote $e(z) = const = \frac{1}{2\sqrt{6}} \gamma h_1 \varphi_0 \Delta_e - \alpha A$. Therefore, if $f_4(h_1) > 0$ then

$V(t_3) - V(t_1) \leq -\epsilon < 0$.

Thus, all conditions of Lemma 1 are fulfilled. The statement of Theorem 1 follows.

4 Numerical example

Consider system (1), (6) with the following parameters: $m = 1, \quad l = 1, \quad g = 9.8, \quad h = 2mgl - 0.01 = 19.59, \quad H_* = 9.6$.

In Theorem 2 the initial parameter $h_*$ is such that $z(t) \in H^{-1}_{[H_* - h_*, H_* + h_*]}$ for all $t \geq 0$. Moreover, for small $\Delta_e$ the value of $\Delta_e$ is close to $h_*$, i.e. one can say that $h_*$ mainly plays the role of an initial condition parameter. Let $\gamma = 0.1$. The colored areas on Figs. 4 and 5 show those $h_*$ and $\Delta_e$ for which system of inequalities (11) is feasible. Furthermore, the color scales on Figs. 4 and 5 provide the values of minimum possible $\kappa_1$ and maximum possible $\kappa_2$ respectively.

Fig. 4. The dependence of minimum possible $\kappa_1$ on $\Delta_e$ and $h_*$. From Fig. 4 one can see that with increasing $h_*$ the attraction domain $H^{-1}_{[H_* - \kappa_1, H_* + \kappa_1]}$ also increases. Therefore, it is important to prioritize between the accuracy of convergence and the width of initial area.

Fig. 5. The dependence of maximum possible $\kappa_2$ on $\Delta_e$ and $h_*$.
Let $h_\ast = 8$. We show the influence of the control gain $\gamma$ on the initial and attraction domains. The colored areas on Figs. 6 and 7 show those $\gamma$ and $\Delta_e$ for which system of inequalities (11) is feasible. Note that from Theorem 2 one obtains that $\Delta_e < 6.5$, however, for any large $\gamma$ there exists a small enough $\Delta_e$ such that system (11) is feasible.

From Fig. 6 one can see that for given $\Delta_e$ the attraction domain $H_{-1}[H_\ast - \kappa_1, H_\ast + \kappa_1]$ is smaller for smaller $\gamma$. For example if $\Delta_e = 1.8$ then $\kappa_1 = 3.23$, $\kappa_2 = 7.94$ for $\gamma = 0.4$ (Fig. 8) and $\kappa_1 = 2.17$, $\kappa_2 = 7.999$ for $\gamma = 0.01$ (Fig. 9), therefore, the attraction domain is smaller in the second case (for $\gamma = 0.01$), however, the convergence time $T$ is larger also in the second case.

Note that in Figs. 8 and 9 the control error $e_u(z)$ was defined as the “worst case” error, namely,

$$ e_u(z) = -\text{sign}(U(z)) \gamma \Delta_e. $$

Indeed, since $\dot{V}(z) = -\gamma(H(z) - H_\ast)^2 \dot{\phi}^2 - U(z)e_u(z)$, the “worst case” error is $e_u(z)$ such that the second term is nonnegative and has the maximum absolute value.

Fig. 6. The dependence of minimum possible $\kappa_1$ on $\Delta_e$ and $\gamma$.

Fig. 7. The dependence of maximum possible $\kappa_2$ on $\Delta_e$ and $\gamma$.

Now consider a quantizer $q(z)$ such that

$$ Z_i = \{ \varphi_k < \varphi < \varphi_k + \tau_\varphi, \dot{\varphi}_k < \dot{\varphi} < \dot{\varphi}_k + \tau_{\dot{\varphi}} \} \cap X_h $$

and $z_i = (\varphi_k + \frac{\tau_\varphi}{2}, \dot{\varphi}_k + \frac{\tau_{\dot{\varphi}}}{2})$ (see Fig. 10), where $\tau_\varphi = 0.042$, $\tau_{\dot{\varphi}} = 0.0128$. Then $\Delta_1 = 0.021$, $\Delta_2 = 0.0064$.

Fig. 8. Phase portrait. $\Delta_e = 1.8$, $\gamma = 0.4$

Fig. 9. Phase portrait. $\Delta_e = 1.8$, $\gamma = 0.01$

Fig. 10. Quantizer regions

and $\Delta_e = 1.8$. Hence, one can compare the behavior of trajectories with the actual quantizer error $e_u(z)$ for $\gamma = 0.4$ (see Fig. 11) and $\gamma = 0.4$ (see Fig. 12) to the previous case where $e_u(z)$ was modeled as the “worst case” error.

Consider a more accurate quantizer with $\Delta_1 =$
Better results can be obtained for smaller $\gamma$ (although the convergence time $T$ will be increased). For example $x_1 = 0.259$, $x_2 = 7.9099$ for $\gamma = 0.1$, i.e., for any given initial condition $z(0) \in H^{-1}_{[1,6.001,17.5999]}$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H^{-1}_{[1,6.1,17.6]}$ for all $t \geq 0$ and there exists $T > 0$ such that $z(t) \in H^{-1}_{[9.341,9.859]}$ for all $t \geq T$ (see Fig. 14).

5 Epilogue

The problem of partial nonlinear control using quantized state feedback was considered by the example of controlling the pendulum’s energy, which contains all the difficulties that are typical for nonlinear partial stable systems. As the nominal feedback law, a control based on the speed gradient method was chosen. The main contribution lied in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds.

Numerical example shows that one can choose a smaller gain $\gamma$ to decrease the attraction domain, although convergence time in this case is increased.

Next steps in this research are the extension of the results to the case of systems with more than one degree of freedom (possibly with friction or dissipation) and to the presence of time delays in addition to quantization. Note that we have already considered time delays in cart-pendulum systems (see [1]) and obtained experimental results for a cart-pendulum system constructed from Lego Mindstorms NXT. Therefore, another task is to demonstrate the performance of the results on the pendulum with quantization by experiments.

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References


