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Towards ISS Disturbance Attenuation for Randomly Switched Systems

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Abstract— **We are concerned with input-to-state stability (ISS) of randomly switched systems. We provide preliminary results dealing with sufficient conditions for stochastic versions of ISS for randomly switched systems without control inputs, and with the aid of universal formulae we design controllers for ISS-disturbance attenuation when control inputs are present. Two types of switching signals are considered: the first is characterized by a statistically slow-switching condition, and the second by a class of semi-Markov processes.**

Index Terms— **randomly switched systems, input-to-state stability, multiple ISS-Lyapunov functions, universal formula for feedback stabilization.**

I. INTRODUCTION

SINCE its introduction in [1] the concept of input-to-state stability (ISS) has received widespread attention on both theoretical and practical fronts; see [2] for a recent detailed discussion. The ISS property characterizes behavior of the state trajectory of a deterministic nonlinear system perturbed by bounded disturbance inputs; as such it provides a framework for robustness analysis of nonlinear systems. Initially stated for deterministic inputs, various extensions of the ISS property have been made for inputs modeled as random processes, one of which is exponential input-to-state stability [3]. The ISS property has been employed in constructive ways for stability analysis, stabilizing feedback controller synthesis, adaptive control schemes, etc.

With the growing interest in the theory and applications of hybrid systems, considerable effort has been directed towards understanding the behavior of switched systems. A switched system has two ingredients: a family of subsystems, and a switching signal which specifies the active subsystem at each instant of time. An important control-theoretic issue is that of stability and stabilization of these systems, and a number of interesting techniques have evolved over the past two decades to deal with this; for a discussion see, e.g., [4, Chapters 2, 3]. More recently, looking beyond stability, robustness and ISS properties of deterministic switched systems have received attention; see [5] and the references therein. There appears to be a common thread of slow switching in these results. That is to say, if the constituent subsystems are each ISS and the switching is sufficiently slow, then the switched system is also ISS.

In this article we are concerned with ISS of randomly switched nonlinear systems, i.e., ISS properties of switched systems whose switching signal is a random process. We provide preliminary results dealing with sufficient conditions for a stochastic version of ISS of these systems. Two types of switching signals are considered; the first is characterized by a statistically slow switching condition, and the second is a class of semi-Markov processes. For these classes of switching signals it is difficult to apply traditional approaches which rely on an infinitesimal (or extended) generator [6], for either there is little information available about the parameters of the switching signal, or there is strong dependence on past history of the process.

The approach pursued here employs multiple ISS-Lyapunov functions in the spirit of our earlier works [7], [8] on stability analysis of randomly switched systems without inputs. Our approach highlights the interaction of deterministic dynamical systems with the stochastic switching signal. The switching signals considered here are adopted from these articles, but the analysis in the presence of inputs as we carry out here is more involved.

With the analysis results in hand, we turn to control synthesis. Two types of controller architectures are considered: in the first case the controllers depend on both the switching signal and the state, and in the second case the controllers depend only on the state. The technical tools are off-the-shelf universal formulae for ISS disturbance attenuation [9] and our analysis results. To the best of our knowledge this is the first time that ISS under random switching is being studied.

The article is organized as follows. In §II we fix notations and define our property of interest. The analysis results are given in §III, a proof of one result is sketched in §IV, and the synthesis results are presented in §V. We conclude in §VI with a short discussion of the case of Markovian switching signals.

II. PRELIMINARIES

Let $\mathbb{R}_{\geqslant 0} := [0, \infty[$, $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$, and $\|f\|_A$ denote the essential supremum norm of the function f on the set $A \subseteq \mathbb{R}_{\geqslant 0}$. Recall that a function $\alpha : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{K}$ if α is monotone increasing, continuous, and $\alpha(0) = 0$. Also, α belongs to class-$\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$, and $\alpha \nearrow \infty$. A function $\beta : \mathbb{R}^2_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{KL}$ if $\beta(\cdot, s) \in \mathcal{K}$ for each s and $\beta(r, \cdot) \searrow 0$ for each r. We let $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$ for $x, y \in \mathbb{R}$.

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a probability space [10], with Ω the set of events, $\mathfrak{F}$ a sigma-algebra on Ω, and P a probability measure on $(\Omega, \mathfrak{F})$. We let $\mathsf{E}\big[\,\cdot\,\big]$ denote mathematical expectation and $\mathsf{E}^{\mathfrak{F}'}\big[\,\cdot\,\big]$ (or $\mathsf{E}\big[\,\cdot\,\big|\mathfrak{F}'\big]$) denote conditional mathematical

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expectation given a sigma-subalgebra $\mathfrak{F}'$ of $\mathfrak{F}$. We let $\mathsf{P}^{\mathfrak{F}'}(\,\cdot\,)$ (or $\mathsf{P}(\,\cdot\,|\mathfrak{F}')$) denote conditional probability given $\mathfrak{F}'$.

A. Randomly switched systems with disturbance inputs

Let $\mathcal{P} := \{1, \ldots, \mathrm{N}\}$ be a finite index set, and for each $i \in \mathcal{P}$ let us consider the system

$$\dot{x} = f_i(x, d) \tag{1}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a continuously differentiable vector field, $f_i(0, 0) = 0$. We allow $d : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}^k$ to be a measurable, locally essentially bounded function of time; this ensures local existence and uniqueness of solutions of (1). Let σ be a càdlàg stochastic process (i.e., a stochastic process whose sample paths are continuous from the right and possess limits from the left) on $(\Omega, \mathfrak{F})$ taking values in $\mathcal{P}$. We assume that for each $t \geqslant 0$ and each $\omega \in \Omega$ there exists a strictly positive number $\epsilon(t, \omega)$ such that $\sigma(t + s, \omega) = \sigma(t, \omega)$ on $[t, t + \epsilon(t, \omega)[$. Under this condition we know [11, Theorem T26, p. 304] that the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ generated by σ is right-continuous, and we augment $\mathfrak{F}_0$ with all P-null sets. We say that σ is a random switching signal, and it generates the randomly switched system from the family (1) given by

$$\dot{x} = f_\sigma(x, d), \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0, \tag{2}$$

where $x_0 \in \mathbb{R}^n$.

B. Input-to-state stability

Input-to-state stability (ISS) was formulated for a single system in [1]; let us state the definition corresponding to the i-th member of the family defined above.

The system (1) is *input-to-state stable* if there exist functions $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}_\infty$ such that for every $x_0 \in \mathbb{R}^n$ and measurable and locally bounded input d, the estimate

$$\|x(t)\| \leqslant \beta_i(\|x_0\|, t) + \gamma_i\big(\|d\|_{[0,t[}\big) \tag{3}$$

holds for all $t \geqslant 0$ along solutions of (1).

Definition 1: The system (2) *satisfies an* ISS *in* $\boldsymbol{L}_1$ *estimate at switching instants* if there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for every $x_0 \in \mathbb{R}^n$, every measurable and essentially bounded input d, the estimate

$$\mathsf{E}\big[\alpha(\|x(\tau_\nu)\|)\big] \leqslant \beta(\|x_0\|, \nu) + \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \tag{4}$$

holds for all $\nu \in \mathbb{N}$ along solutions of (2). ◇

Notice that the expectation on the left hand side involves a class-$\mathcal{K}_\infty$ function α. In the absence of randomness this statement in terms of α is equivalent to the statement that $\|x(t)\| \leqslant \beta'(\|x_0\|, t) + \gamma'\big(\|d\|_{[0,t[}\big)$ for some functions $\beta' \in \mathcal{KL}$ and $\gamma' \in \mathcal{K}_\infty$, where we have employed a weak triangle inequality for class-$\mathcal{K}_\infty$ functions.[1] In the context of randomly switched systems, however, without further assumptions on α one cannot conclude that $\mathsf{E}\big[\,\|x(t)\|\,\big] \leqslant \beta'(\|x_0\|, t) + \gamma'\big(\|d\|_{[0,t[}\big)$ from (4). However, it is often the case that we get polynomial functions of the state inside the expectation, which in general yields stronger bounds. For instance, if the function α is quadratic, it is convex, and an application of Jensen's inequality[2] leads to the last inequality.

Let us also note that Definition 1 does not claim ISS of every sample path of the system (2); the qualitative and quantitative aspects of this definition do not concern information about individual trajectories.

Suppose that (1) is ISS for each $i \in \mathcal{P}$. Then by definition there exist functions $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}_\infty$ such that (3) holds along solutions of the i-th subsystem. However, without further stipulations on σ, in general it is not true that the switched system generated by σ from the family $\{f_i\}_{i\in\mathcal{P}}$ retains the ISS property (i.e., there will exist *unique* functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that (4) holds for any trajectory of the switched system (2)). In §III we consider different classes of switching signals for which we give sufficient conditions for different types of ISS-type estimates.

III. Analysis Results

A. Statistically slow switching

We assume no more structure of the switching signal than a slow switching condition, which is reminiscent of the switching rate of a Poisson counter. A similar condition was employed in the main theorem of [7], where we dealt with stability under no disturbance inputs and slow switching. We also assume that each member of the family of subsystems is ISS. First a piece of notation: let $N_\sigma(t_2, t_1)$ denote the number of jumps made by σ on the interval $]t_1, t_2] \subseteq \mathbb{R}_{\geqslant 0}$, $t_1 \leqslant t_2$.

Recall that we have $\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, \mathsf{P}\big)$ as a complete filtered probability space satisfying the usual conditions. A $[0, \infty]$-valued random variable τ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-*stopping time* if $\{\tau \leqslant t\} \in \mathfrak{F}_t$ for each $t \geqslant 0$. A random variable τ' is an $(\mathfrak{F}_t)_{t\geqslant 0}$-*optional time* if $\{\tau' < t\} \in \mathfrak{F}_t$ for each $t > 0$. It is quite clear that an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time is a $(\mathfrak{F}_t)_{t\geqslant 0}$-stopping time. It is a standard result that if $(\mathfrak{F}_t)_{t\geqslant 0}$ is a right-continuous filtration, every $(\mathfrak{F}_t)_{t\geqslant 0}$-stopping time is also an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time. For details see, e.g., [10].

Definition 2: The switching signal σ is said to belong to *class G* if the following condition holds: there exist $\overline{\lambda}, \widetilde{\lambda} > 0$ and $k_0 \in \mathbb{N} \cup \{0\}$, such that for every $(\mathfrak{F}_t)_{t\geqslant 0}$-stopping time t' and for all $k \geqslant 0$:

$$\mathsf{P}^{\mathfrak{F}_{t'}}\big(N_\sigma(t' + s, t') = k\big) \leqslant \mathrm{e}^{-\widetilde{\lambda} s} \frac{\big(\overline{\lambda} s\big)^k}{k!}. \qquad ◇$$

Note that if $\overline{\lambda} = \widetilde{\lambda}$ and $k_0 = 0$, then Definition 2 gives the jump rate of a stationary Poisson process.

We have the following

Lemma 3: If σ belongs to class G, then $(\tau_i)_{i\in\mathbb{N}}$ is almost surely divergent.

One can prove this by estimating the expected value of $N_\sigma(t, 0)$ from the bound in Definition 2 for a fixed $t \geqslant 0$ (since each fixed t is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time), which is readily seen to be finite. See also [13, Chapter 3] for an alternative argument.

[1] The weak triangle inequality for a function $\gamma \in \mathcal{K}_\infty$ is: $\gamma(r_1 + r_2) \leqslant \gamma(2(r_1 \vee r_2)) \leqslant \gamma(2r_1) + \gamma(2r_2)$.

[2] Jensen's inequality [12] states that if X is an integrable random variable and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, then $\phi\big(\mathsf{E}[X]\big) \leqslant \mathsf{E}\big[\phi(X)\big]$.

Our results employ a family of ISS-Lyapunov functions; the following assumption collects the properties we require from them. The analysis will proceed with the aid of ISS-Lyapunov-like functions.[3] The following assumption collects the properties we shall require from them.

Assumption 4: Suppose that there exist continuously differentiable functions $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$, functions $\alpha_1, \alpha_2, \chi \in \mathcal{K}_\infty$, and numbers $\mu \geqslant 1$, $\lambda_i \in \Lambda \subseteq \mathbb{R}$, $i \in \mathcal{P}$, such that for all $(i, x, d) \in \mathcal{P} \times \mathbb{R}^n \times \mathbb{R}^k$ we have

(Vd1) $\alpha_1(\|x\|) \leqslant V_i(x) \leqslant \alpha_2(\|x\|)$;

(Vd2) $\dfrac{\partial V_i}{\partial x}(x) f_i(x,d) \leqslant -\lambda_i V_i(x) + \chi(\|d\|)$;

(Vd3) $V_i(x) \leqslant \mu V_j(x)$. ◇

Note that if we allow Λ to include negative numbers, then not all λ_i's need to be positive, which in turn means that not all subsystems are required to be ISS.

The function V_i in (Vd1) and (Vd2) above is called an ISS-*Lyapunov function* for the i-th subsystem. If Λ consists of positive real numbers, (Vd2) is equivalent to each subsystem being ISS. Let us note that conventionally ISS-Lyapunov functions are defined in a little different way, for instance, the right-hand side of (VL2) is $-\alpha'(\|x\|) + \chi'(\|d\|)$, or the right-hand side of (VL2) is $-\alpha'(\|x\|)$, for $\alpha', \chi' \in \mathcal{K}_\infty$, but they are equivalent to (VL2), as proved in [14].

Theorem 5: Consider the switched system (2), and suppose that

(G1) σ belongs to class G;

(G2) Assumption 4 holds with $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$;

(G3) $\mu < \big(\widetilde{\lambda} + \lambda_\circ\big)/\overline{\lambda}$.

Then there exists a monotonically nondecreasing sequence $(T_i)_{i\in\mathbb{N}}$ of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times with $\lim_{t\to\infty} T_i = \infty$ a.s., and functions $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$, such that

$$\mathsf{E}\big[\alpha(\|x(t)\|)\mathbf{1}_{\{t\in[T_{i-1},T_i[\}\cap\{T_{i-1}<\infty\}}\big] \leqslant \beta(\|x_0\|, t) \vee \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \quad (5)$$

for all $t \geqslant 0$ and $i \in \mathbb{N}$.

The proof of Theorem 5 is rather long, and may be found in [13, Chapter 3]; we sketch the main steps in §IV. See also §III-C below for a discussion.

B. A class of semi-Markov switching signals

In this subsection we assume σ possesses more structure than being statistically slow-switching. Let $S_i := \tau_i - \tau_{i-1}$ for $i \in \mathbb{N}$ be the i-th holding time, $(\tau_i)_{i\in\mathbb{N}}$ being the sequence of switching instants.

Definition 6: The switching signal σ is said to belong to *class UH* if it satisfies:

(UH1) the sequence $(S_i)_{i\in\mathbb{N}}$ of holding times is a collection of i.i.d uniform-(T) random variables;

(UH2) the sequence $\big(\sigma(\tau_i)\big)_{i\in\mathbb{N}\cup\{0\}}$ of values is i.i.d with $\mathsf{P}\big(\sigma(\tau_i) = i\big) = q_j$ for some $q_j \in]0,1[$, $j \in \mathcal{P}$;

(UH3) the two sequences $(S_i)_{i\in\mathbb{N}}$ and $\big(\sigma(\tau_i)\big)_{i\in\mathbb{N}}$ are mutually independent. ◇

[3]Notice that since we do not always require, (as in §III-B) every subsystem to be ISS, these functions are not ISS-Lyapunov functions in the strict sense of the term.

The class UH of switching signals is simply a representative example of the class of semi-Markov switching signals that we can treat in our framework; see [13] for other classes of switching signals and related discussion.

Lemma 7: For switching signals of class UH, the sequence $(\tau_i)_{i\in\mathbb{N}}$ is almost surely divergent.

The above lemma can be established by appealing to the Strong Law of Large Numbers [15, Chapter 2]; see also [13, Chapter 2] for alternative arguments.

Theorem 8: Consider the switched system (2). Suppose that

(U1) σ belongs to class UH;

(U2) Assumption 4 holds with $\Lambda = \mathbb{R}$;

(U3) $\displaystyle\sum_{j\in\mathcal{P}} \frac{\mu q_j\big(1 - \mathrm{e}^{-\lambda_j T}\big)}{\lambda_j T} < 1$.

Then (2) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants.

C. Discussion

The results above fall short of being satisfactory. Indeed, perhaps the most natural adaptation of the ISS concept to the stochastic case would involve bounds of the type

$$\mathsf{E}\big[\alpha(\|x(t)\|)\big] \leqslant \beta(\|x_0\|, t) + \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \quad (6)$$

for all $x_0 \in \mathbb{R}^n$, $t \geqslant 0$, and essentially bounded inputs d. However, the technical difficulties, particularly in the absence of Markovian assumptions on σ, are formidable. Let us consider switching signals belonging to class G. If B denotes the ball around the origin whose radius is $\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)$, and B' is a larger concentric ball, then the solution trajectory $x(\cdot)$ enters B and exits B' at random instants, as defined in (7); the sequence $(T_i)_{i\in\mathbb{N}}$ in (5) is actually this set of random instants. There is no further structure which prevents the number of exit/entry times from increasing at least linearly with time t (the linearity follows at once from the observation that the set of vector fields $\{f_i\}_{i\in\mathcal{P}}$ is locally Lipschitz, and $\|d\|_{\mathbb{R}_{\geqslant 0}} < \infty$. It is also clear that estimates for the probability distribution of the holding times are not available. Hence "gain-margin" type arguments appear to be the only mode of attack, as we pursue in §IV. As asserted in Theorem 5, it is possible to get bounds on the expectation of the state at some given time, restricted to each of these random excursion intervals, but gluing these estimates to get a uniform bound for a given time t is a difficult problem, and in our case it is yet unsolved.

On the other hand, in the case of switching signals of class UH, the holding times are explicitly characterized, but the chief issue is that of obtaining an estimate for $\mathsf{E}\big[\alpha(\|x(t)\|)\big]$ from an ISS estimate in $\boldsymbol{L}_1$ at switching instants. To wit, there can potentially be indefinitely many jumps of σ before and after a given time t; therefore countably many simultaneous interpolations are needed to get an estimate of $\mathsf{E}\big[\alpha(\|x(t)\|)\big]$, and such an interpolation is again a difficult problem. Unlike in the deterministic case, one is necessarily forced to work with random intervals.

Let us also note that ISS-type estimates "in probability" for diffusion processes have appeared in the literature, for instance, in [16, Theorem 4.2], and more recently in [17, §2]. Although the system models in the above references differ from ours, the essential technical difficulties remain the same. Unfortunately, these difficulties were not realized in the aforesaid references, and the claims made in both of them are still open.

IV. Proofs

Proof of Theorem 5 (Sketch). The argument is divided into five steps for convenience. We shall employ the equivalent "gain-margin" characterization [2] of ISS of the individual subsystems; see [13, chapter 3] for a more detailed proof.

Step 1. Let us fix an essentially bounded disturbance input signal d with $\|d\|_{\mathbb{R}_{\geqslant 0}} > 0$, an initial condition $x_0 \in \mathbb{R}^n$, and define the open sets $C_1 := \{z \in \mathbb{R}^n \mid \|z\| < \rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\}$ and $C_2 := \{z \in \mathbb{R}^n \mid \|z\| < \eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\}$, where $\eta > 0$ is chosen such that $\alpha_1(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})) > 2\alpha_2(\rho(\|d\|_{\mathbb{R}_{\geqslant 0}}))$. Let us suppose that $x_0 \notin C_1$, the other case being similar. We define the following sequence of random times taking values in $[0,\infty]$:

$$\begin{aligned}
&\check{t}_1 := \inf\{t > 0 \mid x(t) \in C_1\},\\
&\hat{t}_1 := \inf\{t > \check{t}_1 \mid x(t) \in \mathbb{R}^n \setminus C_2\},\\
&\ldots\\
&\check{t}_{i+1} := \inf\{t > \hat{t}_i \mid x(t) \in C_1\} \quad \text{for } i \in \mathbb{N},\\
&\hat{t}_{i+1} := \inf\{t > \check{t}_{i+1} \mid x(t) \in \mathbb{R}^n \setminus C_2\} \quad \text{for } i \in \mathbb{N},
\end{aligned} \tag{7}$$

where it is understood that if any $\check{t}_i$ or $\hat{t}_i$ is ∞, then each of the definitions which follow it in the above sequence is set to ∞. We note that both $\check{t}_i$ and $\hat{t}_i$ are $[0,\infty]$-valued $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times.

Step 2. Pointwise on $\{t, \tau_i \in [0, \check{t}_1[\}$ we have $x(t), x(\tau_i) \in \mathbb{R}^n \setminus C_1$, and from (Vd2)-(Vd3) we get

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[0,\check{t}_1[\}}\big] = \alpha_2(\|x_0\|)\,\mathrm{e}^{-\left(\lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda}\right)t}.$$

Therefore,

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[0,\check{t}_1[\}}\big] \leqslant \beta(\|x_0\|, t) \quad \forall\, t \geqslant 0,$$

where $\beta(r,s) := \alpha_2(r)\,\mathrm{e}^{-\lambda s}$, $\lambda := \lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda} > 0$ by (Gd3).

Step 3. Pointwise on $\{t, \tau_i \in [\check{t}_j, \hat{t}_j[\} \cap \{\check{t}_j < \infty\}$ for $i, j \in \mathbb{N}$ we have $x(t), x(\tau_i) \in C_2$ by (7) and continuity of $x(\cdot)$. Employing (Vd1) leads to

$$\forall\, t \in [\check{t}_j, \hat{t}_j[\quad V_{\sigma(t)}(x(t)) \leqslant \alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big).$$

whenever $\hat{t}_j < \infty$. Taking expectations we arrive at

$$\begin{aligned}
&\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{\hat{t}_j<\infty\}\cap\{t\in[\check{t}_j,\hat{t}_j[\}}\Big]\\
&\leqslant \alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)\mathsf{P}\big(\{t \in [\check{t}_j, \hat{t}_j[\} \cap \{\hat{t}_j < \infty\}\big).
\end{aligned}$$

Step 4. Pointwise on $\{t, \tau_i \in [\hat{t}_j, \check{t}_{j+1}[\} \cap \{\hat{t}_j < \infty\}$ for $i, j \in \mathbb{N}$ we have

$$\begin{aligned}
&\frac{\partial V_{\sigma(t)}}{\partial x}(x(t)) f_{\sigma(t)}(x(t), d(t)) \leqslant -\lambda_\circ V_{\sigma(t)}(x(t)),\\
&\quad \forall\, k \in \mathcal{P} \quad V_{\sigma(\tau_i)}(x(\tau_i)) \leqslant \mu V_k(x(\tau_i))
\end{aligned} \tag{8}$$

in view of (Vd2)-(Vd3). Therefore,

$$\begin{aligned}
&\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\hat{t}_j<\infty\}}\Big]\\
&\leqslant \mathsf{E}\Bigg[\sup_{s\geqslant 0} V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Bigg].
\end{aligned} \tag{9}$$

It can be shown that the process $\big(V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j + s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\big)_{s\geqslant 0}$ is a nonnegative $\big(\mathfrak{F}_{\hat{t}_j+s}\big)_{s\geqslant 0}$-potential. Further detailed calculations lead to

$$\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\hat{t}_j<\infty\}}\Big] \leqslant \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big),$$

where we let $\gamma(r) := (1 + 1/\delta)\alpha_2(\eta\rho(r))$.

Step 5. It remains to define the sequence $(T_i)_{i\in\mathbb{N}}$ of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times. Letting $T_{2k-1} := \check{t}_k$ and $T_{2k} := \hat{t}_k$, $k \in \mathbb{N}$, we see from Steps 2 through 4 that

$$\begin{aligned}
&\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[T_{i-1},T_i[\}\cap\{T_{i-1}<\infty\}}\big]\\
&\qquad \leqslant \beta(\|x_0\|, t) \vee \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big),
\end{aligned}$$

which proves the claim. □

Proof of Theorem 8 (Sketch). Fix $\nu \in \mathbb{N}$, and let $k' := \mu\left[\sum_{j\in\mathcal{P}} \frac{q_j}{\lambda_j}\left[1 - \frac{1-\mathrm{e}^{-\lambda_j T}}{\lambda_j T}\right]\right] \Big/ \left[1 - \sum_{j\in\mathcal{P}} \frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right]$. In view of (Vd2), pointwise on $\{s \in [\tau_i, \tau_{i+1}[\}$, $i \in \mathbb{N}$, and applying (Vd3) at $t = \tau_{i+1}$,

$$\begin{aligned}
V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) &\leqslant \mu V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\\
&\quad + \frac{\mu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_{\sigma(\tau_i)}}\Big(1 - \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\Big).
\end{aligned}$$

Iterating the above inequality from $i = 0$ through $i = \nu - 1$, we get

$$\begin{aligned}
V_{\sigma(\tau_\nu)}(x(\tau_\nu)) &\leqslant \mu^\nu V_{\sigma(0)}(x_0) \prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\\
&\quad + \mu^\nu \chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \sum_{i=0}^{\nu-1} \frac{\mu^{-i}}{\lambda_{\sigma(\tau_i)}} \left(\prod_{j=i+1}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right.\\
&\qquad \left. - \prod_{j=i}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right).
\end{aligned} \tag{10}$$

The expectation of the first term on the right-hand side of (10) can be evaluated as

$$\begin{aligned}
&\mathsf{E}\left[\mu^\nu V_{\sigma(0)}(x_0) \prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\right]\\
&\leqslant \alpha_2(\|x_0\|) \left(\sum_{j\in\mathcal{P}} \frac{\mu q_j\left(1 - \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^\nu,
\end{aligned} \tag{11}$$

by utilizing (Vd1) and (UH1)-(UH3). Also, from (UH3) we have

$$\begin{aligned}
&\mathsf{E}\left[\frac{\left(\prod_{j=i+1}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right)}{\lambda_{\sigma(\tau_i)}}\right]\\
&= \prod_{j=1+1}^{\nu-1} \mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_{j+1})}S_{j+1}}\Big]\,\mathsf{E}\left[\frac{1 - \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\right].
\end{aligned} \tag{12}$$

Now for each $j \in \mathbb{N}$ we have

$$\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_j)} S_{j+1}}\Big] = \sum_{k\in\mathcal{P}} \frac{q_k\left(1-\mathrm{e}^{-\lambda_k T}\right)}{\lambda_k T}, \tag{13}$$

and for each $i \in \mathbb{N}$,

$$\mathsf{E}\left[\frac{1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\right] = \sum_{k\in\mathcal{P}} \frac{q_k}{\lambda_k}\left(1-\frac{1-\mathrm{e}^{-\lambda_k T}}{\lambda_k T}\right). \tag{14}$$

Substituting the right-hand sides of (14) and (13) back into (12) and simplifying, we see that

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big] \leqslant \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}} \frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu} + k'\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big). \tag{15}$$

Now, letting $\gamma(r) := k'\chi(r)$ and $\beta(r,s) := \alpha_2(r)\eta^s$, where $\eta := \sum_{j\in\mathcal{P}} \frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}$, an application of (Vd1) on the left-hand side of (15) immediately proves the assertion. $\square$

V. Control synthesis for ISS disturbance attenuation

We look at two different controller architectures, namely, one in which the controller is mode-dependent, and the other in which the controller is mode-independent. That is to say, in the first case, u is a function of both the state x and the switching signal σ, while in the second case u is just a function of x.

A. Mode-dependent controllers

Consider the affine-in-control switched system perturbed by a disturbance signal

$$\dot{x} = f_\sigma(x,d) + \sum_{i=1}^{m} g_{\sigma,i}(x)u_i, \qquad x(0)=x_0, \quad t \geqslant 0, \tag{16}$$

where $x \in \mathbb{R}^n$ is the state, u_i, $i = 1,\ldots,m$, are the (scalar) control inputs, $f_j : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $g_{j,i} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth maps for each $j \in \mathcal{P}$, $i \in \{1,\ldots,m\}$. Let $\mathcal{U}$ be the set where the control $u := [u_1,\ldots,u_m]^{\mathrm{T}}$ takes its values. For the moment we let $\mathcal{U}$ be a subset of $\mathbb{R}^m$ containing the origin. With a feedback control function $k_\sigma(x) := [u_{\sigma,1}(x),\ldots,u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed-loop system stands as

$$\dot{x} = f_\sigma(x,d) + \sum_{i=1}^{m} g_{\sigma,i}(x)k_{\sigma,i}(x), \qquad x(0)=x_0, \quad t \geqslant 0, \tag{17}$$

We let the switching signal σ be a stochastic process as defined in §II, and let $x_0 \neq 0$.

Our goal is to choose a control function k_σ so that (17) satisfies some ISS in $\boldsymbol{L}_1$ estimate at switching instants. We shall appeal to our analysis results of §III and universal formulae for ISS disturbance attenuation to achieve this objective.

Universal feedback control functions attaining ISS disturbance attenuation for nonlinear systems affected by disturbances and possessing control inputs were constructed in [9]. The results in that article rely on universal formulae for asymptotic feedback stabilization of nonlinear systems; applications include systems in which the control takes values in various restricted control sets, and a universal formula is available. In our illustrative result below we utilize off-the-shelf universal feedback control functions for ISS disturbance attenuation from [9]. The next proposition is a typical illustration of such a result.

Let us define the map $\varphi : \mathbb{R}\times\mathbb{R}^m \longrightarrow \mathbb{R}$ given by

$$\varphi(a,b) := \begin{cases} -\dfrac{a+\sqrt{a^2+\|b\|^4}}{\|b\|^2}\,b & \text{if } b \neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

the function $\widetilde{W}_j(x) := \big[\mathrm{L}_{g_{j,1}}V_j(x),\ldots,\mathrm{L}_{g_{j,m}}V_j(x)\big]$, and a map $\overline{W}_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, with values chosen such that it is smooth away from 0 and continuous at 0, and

$$\max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_j}{\partial x}(x)f_j(x,d) - \chi(\|d\|)\right\} + \lambda_j V_j(x) \leqslant \overline{W}_j(x) \leqslant \max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_j}{\partial x}(x)f_j(x,d) - \chi(\|d\|)\right\} + 2\lambda_j V_j(x) \tag{18}$$

for all $x \in \mathbb{R}^n$, $j \in \mathcal{P}$.

Proposition 9: Consider the system (16) with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ belongs to class UH, and

(Cd1) (Vd1) of Assumption 4 holds;
(Cd2) (Vd3) of Assumption 4 holds;
(Cd3) $\exists\,\alpha,\chi \in \mathcal{K}_\infty$, $\exists\,\lambda_j \in \Lambda = \mathbb{R}$, $j \in \mathcal{P}$, such that $\forall\,x \in \mathbb{R}^n\smallsetminus\{0\}$, $\forall\,d \in \mathbb{R}^k$ and $\forall\,j \in \mathcal{P}$ we have

$$\inf_{u\in\mathcal{U}}\left\{\frac{\partial V_j}{\partial x}(x)f_j(x,d) + 3\lambda_j V_j(x) + \sum_{i=1}^{m}\mathrm{L}_{g_{j,i}}V_j(x)u_i\right\} \leqslant \chi(\|d\|);$$

(Cd4) $\forall\,\varepsilon > 0\;\exists\,\delta > 0$ such that if $x(\neq 0)$ satisfies $\|x\| < \delta$, then $\exists\,u \in \mathbb{R}^m$, $\|u\| < \varepsilon$, such that $\forall\,j \in \mathcal{P}$

$$\max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_j}{\partial x}(x)f_j(x,d) - \chi(\|d\|)\right\} + \sum_{i=1}^{m}\mathrm{L}_{g_{j,i}}V_j(x)u_i \leqslant -\lambda_j V_j(x);$$

(Cd5) (U3) of Theorem 8 hold.

Then under the feedback control function

$$k_\sigma(x) = \varphi\left(\overline{W}_\sigma(x), \widetilde{W}_\sigma^{\mathrm{T}}(x)\right) \tag{19}$$

the system (17) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants,

The proof relies heavily on the proof of [9, Theorem 3], see [13, Chapter 3] for details.

B. Mode-independent controllers.

Consider the affine in control switched system (16). Let $k(x) = [k_1(x), \ldots, k_m(x)]^{\mathrm{T}}$ be a feedback control function, with which the closed-loop system stands as

$$\dot{x} = f_\sigma(x, d) + \sum_{i=1}^{m} g_{\sigma,i}(x) k_i(x), \quad x(0) = x_0, \ \ t \geqslant 0. \quad (20)$$

We let the switching signal σ be a stochastic process as defined in §II, and let $x_0 \neq 0$.

Our objective is to choose a control function k such that (20) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants, for some class-$\mathcal{K}_\infty$ function α.

Proposition 10: Consider the system (16) with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ belongs to class UH, and

(CUd1) (Vd1) and (Vd3) of Assumption 4 holds;

(CUd2) there exists a control function $k : \mathbb{R}^n \longrightarrow \mathcal{U}$, such that $\frac{\partial V_i}{\partial x}(x)\big(f_i(x,d) + g_i(x)k(x)\big) \leqslant -\lambda_i V_i(x) + \chi(\|d\|)$ for every $i \in \mathcal{P}$, $x \in \mathbb{R}^n$;

(CUd3) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i\big(1 - \mathrm{e}^{-\lambda_i T}\big)}{\lambda_i T} < 1.$

Then under k the system (16) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants.

The assertion follows immediately by first observing that the closed-loop system is (20), and then applying Theorem 8 to (20).

VI. Remarks on Markovian Switching Signals

We have established sufficient conditions for different ISS-type properties of the randomly switched system (2) under different classes of switching signals. Let us reiterate that for switching signals of class G and UH considered here, it is difficult to write an infinitesimal generator, since there is either too little information about the parameters of the switching signal, or a strong dependence on its past history. For switching signals coming from continuous-time Markov chains it is possible to employ the infinitesimal (or extended) generator to derive conditions for stability. We get stronger bounds by this route, as shown in [13, Chapter 3]. Indeed, we have

Theorem 11: Consider the system (2), and suppose that σ is a continuous-time Markov chain with generator matrix $Q = [q_{i,j}]_{\mathrm{N}\times\mathrm{N}}$. Moreover, suppose that there exist functions $V : \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $V(i, \cdot)$ is continuously differentiable for each i, $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$, and a constant $\lambda_\circ > 0$, such that

- $\alpha_1(\|x\|) \leqslant V(i,x) \leqslant \alpha_2(\|x\|)$,
- $\mathcal{L}V(i,x) \leqslant -\lambda_\circ V(i,x)$ whenever $\|x\| \geqslant \rho(\|d\|)$.

Then the inequality in (6) holds for some $\beta \in \mathcal{KL}$ and some $\alpha, \gamma \in \mathcal{K}_\infty$.

For definitions of Markov chains, (local) martingales, and martingale problems, see, e.g., [10]. The operator $\mathcal{L}$ is defined in terms of an appropriate martingale problem as follows. Let $h : \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that there exists a measurable function $\widetilde{h} : \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the process

$$\left(h(\sigma(t), x(t)) - h(\sigma_0, x_0) - \int_0^t \widetilde{h}(\sigma(s), x(s))\,\mathrm{d}s\right)_{t\geqslant 0}$$

is a mean-zero $(\mathfrak{F}_t)_{t\geqslant 0}$-local martingale. We define $\mathcal{L}h(i,x) := \widetilde{h}(i,x)$, where $\mathcal{L}$ is the extended generator [6] corresponding to the Markov process $(\sigma(t), x(t))_{t\geqslant 0}$. Of course, finding the class of functions h for which such a $\widetilde{h}$ exists is a nontrivial matter, but it is usually not difficult to find a subclass. Often the operator $\mathcal{L}$ is defined in terms of a differentiation operation, namely,

$$\mathcal{L}h(i,x) = \lim_{h\downarrow 0} \frac{\mathsf{E}\big[h(\sigma(t+h), x(t+h))\big|A_t\big] - h(i,x)\big)}{h},$$

where $A_t := \big\{(\sigma(t), x(t)) = (i,x)\big\}$, and $h : \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function that is pointwise continuously differentiable on the set $\mathcal{P}$.

A similar approach relying on the solution to appropriate martingale problems can be adopted if σ is a general marked point process [11] with suitable stochastic jump intensities, and will be reported elsewhere. Another interesting direction of work concerns establishing ISS-type estimates "in probability" of (2), such as those formulated in [16], [17].

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