Nonlinear observers robust to measurement disturbances in an ISS sense

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Abstract

This paper formulates and studies the concept of quasi-Disturbance-to-Error Stability (qDES) which characterizes robustness of a nonlinear observer to an output measurement disturbance. In essence, an observer is qDES if its error dynamics are input-to-state stable (ISS) with respect to the disturbance as long as the plant's input and state remain bounded. We develop Lyapunov-based sufficient conditions for checking the qDES property for both full-order and reduced-order observers. We use these conditions to show that several well-known observer designs yield qDES observers, while some others do not. Our results also enable the design of novel qDES observers, as we demonstrate with examples. When combined with a state feedback law robust to state estimation errors in the ISS sense, a qDES observer can be used to achieve output feedback control design with robustness to measurement disturbances. As an application of this idea, we treat a problem of stabilization by quantized output feedback.

Index Terms

Input-to-State Stability, Nonlinear Observer, Measurement Disturbance, Robustness, Quantization

I. Introduction

Nonlinear control theory has long been trying to cope with situations where state measurements available for feedback are incomplete or imprecise. By "incomplete measurements" we mean measured outputs of lower dimension than the state; by "imprecise," state measurements corrupted by disturbances. A common way to deal with incomplete measurements is to build an *observer* that generates an asymptotically convergent estimate of the full state. Many different nonlinear observer designs are available in the literature, and several of them will be discussed

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later in the paper. When the full state is measured but is subject to a measurement disturbance, one tries to design a feedback law that possesses some kind of robustness to the disturbance. It has become standard practice in the nonlinear control literature to take *input-to-state stability (ISS)*, introduced by Sontag in [27], as a benchmark robustness notion. Design of control laws guaranteeing ISS with respect to measurement disturbances is a difficult problem that has received considerable attention; again, we postpone an overview of the relevant results until later (see Remark 5 in Section VI).

The above discussion naturally leads to the following important question: how should one proceed in the face of *both* of the indicated challenges, i.e., when only output measurements are available and, moreover, they are affected by a measurement disturbance? As noted in [20], one can envision a solution in the form of a robust state feedback controller and a robust observer, where the observer's robustness is interpreted as ISS from the output measurement disturbance to the state estimation error while the controller's robustness is understood as ISS with respect to the state estimation error. Since a cascade connection of two ISS systems is ISS, the resulting closed-loop system will then be ISS with respect to the measurement disturbance. While some results on designing ISS controllers are available as already mentioned, surprisingly little is known about the second component of the approach just described, namely, constructing observers with robustness to measurement disturbances in an ISS sense. This is the gap that the present work is intended to fill. Our goals are actually three-fold: first, to formulate a suitable ISS-type robustness property of the observer; second, to derive conditions for checking this robustness property; and third, to identify observer designs (both known and new ones) satisfying these conditions.

Before we can describe in more detail our approach and results and their relationships to the existing nonlinear observer literature, we need to fix some basic terminology and notation. We consider a general nonlinear system ("plant")

$$\dot{x} = f(x, u), \qquad y = h(x, d) \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathcal{U} \subset \mathbb{R}^k$ is the control input taking values in a set $\mathcal{U}$ of admissible input vectors, $y \in \mathbb{R}^p$ is the measured output, and $d \in \mathbb{R}^q$ is the measurement disturbance. We call d an *additive* measurement disturbance if $h(x, d) = h_0(x) + d$ for some function h_0. It is assumed that f is locally Lipschitz and h is continuous, and the two signals $u(\cdot)$ and $d(\cdot)$ are assumed to be locally essentially bounded throughout the paper. A *state observer* for the plant (1) is a pair consisting of a dynamical system and a static map

$$\dot{z} = F(z, y, u), \qquad \hat{x} = H(z, y) \tag{2}$$

where $z \in \mathbb{R}^m$ is the observer state, $\hat{x} \in \mathbb{R}^n$ is the estimate of the plant state, F is locally Lipschitz, and H is continuous. The quality of state estimation is measured in terms of the *state estimation error* e defined as

$$e := \hat{x} - x = H(z, h(x, d)) - x. \tag{3}$$

Moreover, we call the observer (2) a *full-order observer* when $H(z, y) = z$ (so that $\hat{x} = z$, and thus, $m = n$), and a *reduced-order observer* when $m < n$.[1]

Unlike in the linear case, a nonlinear observer that makes the state estimation error e converge to 0 when $d \equiv 0$ does not automatically guarantee a graceful degradation of the quality of state estimation for nonzero d. An example has already been given in [25, Sec. 5], where a nonlinear full-order observer for a stable linear plant provides global asymptotic convergence of e to 0 when $d \equiv 0$, yet e can become unbounded in the presence of an arbitrarily small additive measurement disturbance. Therefore, robustness of the observer to measurement disturbances needs to be explicitly formulated and studied. The first obvious candidate for such a robustness property is ISS from d to e; for example, for a full-order observer this ISS property takes the form

$$|z(t) - x(t)| \leq \beta(|z(0) - x(0)|, t) \vee \gamma(\|d\|_{[0,t]}) \qquad \forall t \geq 0 \tag{4}$$

with a class $\mathcal{KL}$ function[2] β and a class $\mathcal{K}$ function γ, where $\|d\|_{[0,t]} := \text{ess.sup}_{0 \leq s \leq t} |d(s)|$ and $\vee$ is the binary operator taking the maximum, i.e., $a \vee b := \max\{a, b\}$.[3] Since in this context d is the disturbance and $e = z - x$ is the estimation error, it seems more appropriate to rename the above property of ISS from d to e as *disturbance-to-error stability (DES)*, which is what we will do from now on.

While DES is certainly a desirable feature for an observer, unfortunately it is quite a strong condition; this will be illustrated in Examples 1 and 2 in the next section. Also, the DES property is not invariant under coordinate transformations (see Section II-A). A necessary condition for the existence of a full-order DES observer, under additive measurement disturbances, has already

[1] In this paper we do not study observers with $m = n$ but $H(z, y) \neq z$ (such observers are rarely studied in the literature) or observers with $m > n$.

[2] A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$.

[3] Alternatively, the sum could be used instead of the maximum to arrive at an equivalent property, but the formulation in terms of the maximum is more convenient in this paper.

been presented by Sontag and Wang [29, Prop. 23]: it is the incremental output-to-state stability (denoted by iOSS in [29]) of the plant (1). In addition, a sufficient condition for the existence of a full-order DES observer was given in [3, Prop. 6.1], which is that for some output injection term $\mathcal{L}(\cdot,\cdot,\cdot)$ with $\mathcal{L}(\cdot,\cdot,0) \equiv 0$ the system $\dot{x} = f(x,u) + \mathcal{L}(x,u,y^* - h(x))$ is incrementally input-to-state stable for any $u(\cdot)$ with y^* being regarded as the input. In this case, a full-order DES observer is given simply by $\dot{z} = f(z,u) + \mathcal{L}(z,u,y - h(z))$ and $\hat{x} = z$. The necessary condition that the plant be incrementally OSS is already rather strong.[4] Design of DES observers has been studied only for limited cases; for example, a globally Lipschitz nonlinear system admits a full-order DES observer if a certain LMI is satisfied [1].

In an effort to identify a robustness property that is more reasonable than DES, in this paper we propose to work with the relaxed notion of a quasi-DES (qDES, in short) observer; its earlier variation was introduced in [26] under the name "quasi-ISS observer." The relaxation consists in the fact that an ISS bound is imposed only as long as both the control input and the plant state remain bounded. We will present a formal definition of qDES observer in Section II, followed by a few motivating examples and a discussion of its advantage over the DES observer—the coordinate invariant property. It is not uncommon to utilize boundedness of the plant's state and input for observer synthesis and analysis. For example, a nonlinear observer was designed in [23] based on *a priori* knowledge of bounds for the plant's state and input. In [25], robustness of a specific observer to measurement disturbances in the ISS sense for a special class of systems was verified whenever the plant's input and output are bounded *a posteriori* (i.e., the bounds were not used in the design of the observer). Following a similar line of thinking, a construction of a reduced-order qDES observer was presented in [26], and it was later extended to quasi-ISDS (input-to-state dynamical stability) and to large-scale systems in [5]. Full-order qDES observers, on the other hand, remain to be investigated.

In this paper we develop a general framework for studying qDES observers, which encompasses both the full-order and the reduced-order case. In Section III we present a characterization of qDES observers in terms of Lyapunov functions. A basis for this characterization is provided by the notion of "state-independent IOS (input-to-output stability)" and its variations studied in [30], [31], owing to the fact that the measurement disturbance d and the state estimation error e can be viewed as the input and the output, respectively, of the overall system with state

[4]For example, the system $\dot{x}_1 = 0$, $\dot{x}_2 = x_1 x_3$, $\dot{x}_3 = -x_1 x_2$, and $y = x_3$ on $\{(x_1,x_2,x_3) : x_1 > 0\}$ is not OSS (and therefore, not iOSS either) but is observable and admits a convergent state observer. Another example is $\dot{x} = u$ and $y = x^2$ which is not iOSS while it is OSS, and is instantaneously observable when $u \neq 0$.

(x, z). While our analysis is inspired by that in [30], [31], it incorporates several novel elements; most notably, the proposed characterization uses a $\limsup$-type condition which turns out to be convenient for qDES observer validation compared with more usual ISS-Lyapunov differential inequalities. The resulting qDES observer framework also represents a significant departure from the previously cited results on robust observers. All these aspects of our formulation will be further discussed and supported with examples in Section III.

Since the qDES property is significantly less restrictive than the DES property, it is not surprising that many known nonlinear observer designs from the literature actually yield qDES observers. In Section IV we derive, as corollaries of our main framework, some readily verifiable sufficient conditions for qDES in the case of full-order observers, and then use these conditions to demonstrate that three well-known observer designs—the linearized error dynamics observer from [18], the high-gain observer from [13], and the circle criterion observer from [4]—indeed have the qDES property. Of course, some of the other known observers are not qDES, as we illustrate with a reduced-order observer example in Section V. We then proceed to show how the construction of a reduced-order qDES observer from [26] is recovered within the proposed general framework.

Returning to our original motivation of using a robust observer in conjunction with an ISS controller to achieve robustness to output measurement disturbances, we expect that there will be a price to pay for the fact that the observer is just qDES and not DES. Indeed, additional analysis and possibly extra assumptions will be needed to verify that the control input and the plant state remain bounded, as otherwise the qDES property is not useful. In Section VI, we consider the quantized output feedback stabilization problem which served as the initial impetus for discussing ISS observers in [20]. In this problem, the output quantization error plays the role of the measurement disturbance, and it is bounded as long as the plant's output is bounded. This provides a very natural setting for using an ISS controller together with a qDES observer. If the plant's initial condition lies in a suitable compact set and if we have sufficiently many quantization regions so that the quantization error is small enough, we are able to show that the plant's state and input remain bounded and the system is practically stabilized. After this application example, we conclude the paper in Section VII.

II. QUASI-DES OBSERVER

To define the notion of quasi-DES observer, we introduce the notation

$$e_0 := e|_{d\equiv 0} = H(z, h(x, 0)) - x.$$

For full-order observers we have $e_0 = e = z - x$, but for reduced-order observers typically $e_0 \neq e$.

Definition 1 (Quasi-DES Observer): We say that the system (2) is a *quasi-Disturbance-to-Error-Stable (qDES) observer* for the plant (1) if, for each $K > 0$, there exist a class $\mathcal{KL}$ function β_K and a class $\mathcal{K}$ function γ_K such that

$$|e(t)| \leq \beta_K(|e_0(0)|, t) \vee \gamma_K(\|d\|_{[0,t]}) \qquad \text{for almost all } t \geq 0 \tag{5}$$

whenever $\|u\|_{[0,t]} \leq K$ and $\|x\|_{[0,t]} \leq K$.

It is noted that the first argument of the function β_K on the right-hand side of (5) is the initial value not of e but of the disturbance-free error variable e_0. This is because, if $e(0)$ were used in β_K instead of $e_0(0)$, then there might exist a particular disturbance d such that $d(t) = 0$ for $t > 0$ and that $d(0)$ is non-zero such that $e(0) = H(z(0), h(x(0), d(0))) - x(0) = 0$, making the right-hand side of (5) zero so that we must have $e \equiv 0$, which means that the condition would not be realistic. Similarly, we only ask the inequality (5) to hold for almost all t because the error variable $e(t)$ at a particular time t may become arbitrarily large with some large value of $d(t)$, even though the essential supremum $\|d\|_{[0,t]}$ is small, and the inequality (5) would be violated at such times.

As discussed in the Introduction, the qDES observer property means that, as long as the plant's input u and state x remain bounded, the state estimation error e is robust to the disturbance d in the ISS sense [27]. The functions β_K and γ_K in (5) quantify the convergence rate and the ISS gain, respectively. If these functions can be chosen to be independent of K, then the observer becomes a *DES observer*, without the term "quasi". If the measurement disturbance is absent (i.e., $d \equiv 0$), the DES observer becomes a so-called *globally convergent observer* meaning that $\lim_{t\to\infty} e(t) = 0$ for any initial conditions as long as the solution exists for all forward time. The qDES observer becomes a *globally convergent observer with bounded input/state* when $d \equiv 0$ because the error convergence is guaranteed with bounded inputs and states.

It should be noted that the boundedness of the plant state $x(t)$ and the input $u(t)$ is not assumed *a priori*, nor do their bounds affect the design process of the observer. Definition 1 just says that the property (5) holds whenever these bounds are fulfilled. The following two examples are intended to motivate why for nonlinear systems it is natural that the boundedness of x and u becomes of importance in the discussion of robustness.

Example 1: The gain from the measurement disturbance to the estimation error may be unbounded with respect to $\|u\|_{[0,t]}$ and $\|x\|_{[0,t]}$. To see this, consider the plant $\dot{x} = -x + x^2 u$

with $y = x + d$. Obviously, $\dot{z} = -z + y^2 u$, $\hat{x} = z$ is a globally convergent observer when $d \equiv 0$. When $d \not\equiv 0$, the error dynamics become $\dot{e} = -e + 2xud + ud^2$ with $e = z - x = e_0$. This system is ISS from d to e when $x(t)$ and $u(t)$ are bounded, and the ISS gain is an unbounded function of $\|u\|_{[0,t]}$ and $\|x\|_{[0,t]}$. Therefore, this observer is a qDES observer, but not a DES observer. ///

Example 2: This example illustrates that boundedness of $u(t)$ required for the property (5) may be needed to guarantee a *uniform* convergence rate for each K. Consider the plant

$$\dot{x} = \left(\frac{u^2}{1+u^2} - 1\right) x + u$$

for which an observer may be given as $\hat{x} = z$ and

$$\dot{z} = \left(\frac{u^2}{1+u^2} - 1\right) z + u.$$

Hence the error dynamics (with $e = z - x = e_0$) becomes

$$\dot{e} = \left(\frac{u^2}{1+u^2} - 1\right) e.$$

It is noted that its convergence rate depends on the size of $u(t)$, and the rate can become arbitrarily small with large $u(t)$. The convergence becomes uniform with the boundedness of $u(t)$, and thus, it is a qDES observer but not a DES observer (there is no β function that works for all K). ///

The following example presents a globally convergent full-order observer that is not qDES.

Example 3: Consider a plant given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2, & y_1 &= x_1 + d_1, \\ \dot{x}_2 &= x_1 x_3, & y_2 &= x_2 + d_2, \\ \dot{x}_3 &= -x_1 x_2 + u, & u &= \sin t. \end{aligned} \tag{6}$$

It is seen that if x_1 is constant, then the (x_2, x_3)-dynamics is a marginally stable linear system (in fact, a harmonic oscillator) with a periodic input. The solution $x(t)$, as well as the input $u(t)$, are in fact bounded because $x_1(t) \to 2$ and the bounded input $\sin t$ does not cause resonance.[5]

Now, an observer of the form

$$\dot{z}_1 = -z_1 + 2 - (z_1 - y_1) \tag{7a}$$

$$\dot{z}_2 = z_1 z_3 - y_1 (z_2 - y_2) \tag{7b}$$

$$\dot{z}_3 = -z_1 z_2 + u - y_1 (z_2 - y_2) \tag{7c}$$

[5]For detailed analysis, see the Appendix.

with $\hat{x} = z$ is a globally convergent observer if $d \equiv 0$. That is, with $e_i := z_i - x_i$, we obtain the error dynamics

$$\dot{e}_1 = -2e_1$$
$$\begin{bmatrix} \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} x_1 e_3 - x_1 e_2 + e_1 \cdot (e_3 + x_3) \\ -x_1 e_2 - x_1 e_2 - e_1 \cdot (e_2 + x_2) \end{bmatrix} = x_1 \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} + e_1 \begin{bmatrix} e_3 + x_3 \\ -e_2 - x_2 \end{bmatrix} \quad (8)$$

whose solution converges to zero (because $x_1(t) \to 2$ and $e_1(t) \to 0$ as $t \to \infty$, and (8) is a stable linear system when $x_1 \equiv 2$ and $e_1 \equiv 0$).[5]

Finally, suppose that $d_1(t) = -x_1(t)$ and $d_2(t) = 0$, which are bounded, and that $z_1(0) = 1$, for simplicity. (In fact, we have the same result with any $z_1(0)$ and with any bounded d_1 and d_2 such that $\lim_{t\to\infty} d_1(t) = -2$ and $\lim_{t\to\infty} d_2(t) = 0$.) Then, from (7), it is seen that $z_1(t) = 1$, and that

$$\begin{bmatrix} \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \quad (9)$$

since $y_1(t) = x_1(t) + d_1(t) = 0$. Note that this system has a resonance at the frequency of 1 rad/sec and has the input of frequency 1 rad/sec. It is a standard exercise to check that $|(z_2(t), z_3(t))| \to \infty$ as $t \to \infty$, which illustrates that (7) is not a qDES observer since $|u(t)|$, $|x(t)|$, and $|d(t)|$ are bounded. ///

A. Coordinate Invariance Property

Another benefit of the qDES observer over the DES observer is that the qDES property is coordinate-invariant. As a matter of fact, even if one obtains a DES observer in some coordinates, it may not be a DES observer in other coordinates. This phenomenon is in fact inherited from the deficiency that global error convergence may not be preserved when $x(t)$ is unbounded. Consider a global diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ which transforms the plant (1) into different coordinates, $\zeta = \Phi(x)$. Then, even though one has an observer whose estimate $\hat{\zeta}(t)$ converges to $\zeta(t)$ as $t \to \infty$, it is not guaranteed that $\hat{x}(t) = \Phi^{-1}(\hat{\zeta}(t))$ converges to $x(t)$, as seen in the following example.

Example 4: Consider a C^1 increasing function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(s) = \begin{cases} 2s, & |s| \le 1, \\ s^2 + 1, & s > 1, \\ -s^2 - 1, & s < -1 \end{cases}$$

and consider a nonlinear system whose state is $x \in \mathbb{R}^2$. If this system is converted by the diffeomorphism $(\zeta_1, \zeta_2) = (\phi^{-1}(x_1), x_2)$ into

$$\dot{\zeta}_1 = 2\zeta_1 + \zeta_2, \qquad y = \zeta_1,$$
$$\dot{\zeta}_2 = -\zeta_2,$$

then a choice of an observer might be

$$\dot{z}_1 = 2z_1 + z_2 - 3(z_1 - y), \quad \dot{z}_2 = -z_2, \quad \hat{\zeta} = z,$$

because its error dynamics in these coordinates are globally exponentially stable. With the initial conditions $\zeta(0) = (-1, 0)$ and $z(0) = (0, 0)$, the solutions are given by

$$\zeta_1(t) = -e^{2t}, \quad \zeta_2(t) = 0, \quad \hat{\zeta}_1(t) = -e^{2t} + e^{-t}, \quad \hat{\zeta}_2(t) = 0,$$

so that it is seen that the estimation errors $\hat{\zeta}_1(t) - \zeta_1(t) = e^{-t}$, $\hat{\zeta}_2(t) - \zeta_2(t) = 0$ converge to zero. However, in the original coordinates, it is seen that, for t large enough to have $\hat{\zeta}_1(t) < -1$ (as well as $\zeta_1(t) = -e^{2t} < -1$),

$$\hat{x}_1(t) - x_1(t) = \phi(\hat{\zeta}_1(t)) - \phi(\zeta_1(t)) = 2e^{t} - e^{-2t},$$

so the observer is not convergent. ///

Example 4 alerts us that the DES property is coordinate-dependent as well. On the other hand, by virtue of restricting the state $x(t)$ to be bounded, the qDES property (5) is invariant with respect to coordinate changes.

Proposition 1: The qDES property (5) is coordinate-invariant.

Proof: Let $\zeta = \Phi(x)$ and $\hat{\zeta} = \Phi(\hat{x})$, where Φ is a diffeomorphism on $\mathbb{R}^n$. Let L_r be a Lipschitz constant of Φ on the ball of radius r around the origin, which is non-decreasing as r increases without loss of generality. Then, the class $\mathcal{K}$ function $p_K(r) := L_{K+r} \cdot r$ satisfies $|\Phi(\hat{x}) - \Phi(x)| \leq p_K(|\hat{x} - x|)$ as long as $|x| \leq K$. Similarly, consider a class $\mathcal{K}$ function $q_K(r)$ such that $|\Phi^{-1}(\hat{\zeta}) - \Phi^{-1}(\zeta)| \leq q_K(|\hat{\zeta} - \zeta|)$ under the condition that $|x| \leq K$ (and thus, $|\zeta| = |\Phi(x)| \leq \bar{K}$ with some $\bar{K}$). Then, assuming that (5) holds in the ζ-coordinates, it is seen that, for almost all $t \geq 0$,

$$\begin{aligned}
|\hat{x}(t) - x(t)| &= |\Phi^{-1}(\hat{\zeta}(t)) - \Phi^{-1}(\zeta(t))| \leq q_K(|\hat{\zeta}(t) - \zeta(t)|) \\
&\leq q_K\big(\beta_K(|\hat{\zeta}(0)|_{d(0)=0} - \zeta(0)|, t) \vee \gamma_K(\|d\|_{[0,t]})\big) \\
&= q_K(\beta_K(|\hat{\zeta}(0)|_{d(0)=0} - \zeta(0)|, t)) \vee q_K(\gamma_K(\|d\|_{[0,t]})) \\
&\leq q_K(\beta_K(p_K(|\hat{x}(0)|_{d(0)=0} - x(0)|), t)) \vee q_K(\gamma_K(\|d\|_{[0,t]}))
\end{aligned}$$

which implies the property (5) for $\hat{x}$ and x. Similarly, it can be shown from (5) in the x-coordinates that

$$|\hat{\zeta}(t) - \zeta(t)| \leq p_K(\beta_K(q_K(|\hat{\zeta}(0)|_{d(0)=0} - \zeta(0)|), t)) \vee p_K(\gamma_K(\|d\|_{[0,t]})) \quad \text{for a.a. } t \geq 0,$$

which implies (5) in the ζ-coordinates. ■

III. Characterization of qDES Observers

In this section, a characterization of qDES observers in terms of a Lyapunov-type function is given.

Theorem 1: The system (2) is a qDES observer for the plant (1) if there exists a C^1 function $V : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ such that the following hypotheses hold:

H1. V satisfies

$$\alpha_1(|H(z, h(x, 0)) - x|) \leq V(z, x) \leq \lambda(|x|)\alpha_2(|H(z, h(x, 0)) - x|) \qquad \forall z, x$$

for some class $\mathcal{K}_\infty$ functions α_1 and α_2 and positive non-decreasing function $\lambda : \mathbb{R} \to \mathbb{R}_{>0}$.

H2. The time derivative $\dot{V}(z, x, u, d) := \frac{\partial V}{\partial z}(z, x)F(z, h(x, d), u) + \frac{\partial V}{\partial x}(z, x)f(x, u)$ of V along solutions of (1) and (2) satisfies

$$\dot{V}(z, x, u, d) \leq -W(z, x, u, d) + g(z, x, u, d) \qquad \forall z, x, u, d$$

where $W : \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ and $g : \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^q \to \mathbb{R}$ are continuous functions with the properties that

$$W(z, x, u, d) \geq \alpha_3(|H(z, h(x, 0)) - x|, |x| \vee |u|) \qquad \forall z, x, u, d \tag{10}$$

for some class $\mathcal{KL}$ function α_3,

$$g(z, x, u, 0) \leq 0 \qquad \forall z, x, u, \tag{11}$$

and, for each $K > 0$, there exists a continuous function $\theta_K : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ non-decreasing in the second argument such that

$$\frac{g(z, x, u, d)}{W(z, x, u, d)} \leq \theta_K(|H(z, h(x, 0)) - x|, |d|) \tag{12}$$

for all d, $|x| \leq K$, $|u| \leq K$, z with $|H(z, h(x, 0)) - x| \neq 0$, and

$$\limsup_{\xi \to \infty} \theta_K(\xi, r) < 1 \qquad \forall r \geq 0. \tag{13}$$

H3. The set $\{z : |H(z, h(x, 0)) - x| = \xi\}$ is compact[6] for each $\xi \geq 0$ and x.

H4. There exists a continuous function $\rho : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ such that $\rho(x, 0) = 0$ and $|H(z, h(x, d)) - H(z, h(x, 0))| \leq \rho(x, d)$ for all z, x, and d.

Remark 1: The hypothesis H1 basically says that V is upper- and lower-bounded in terms of e_0 (the upper bound also has the factor $\lambda(|x|)$). H2 is our main hypothesis, which restricts the evolution of V along solutions. Note that the condition (10) is weaker than $W(z, x, u, d) \geq \alpha_3^\circ(|H(z, h(x, 0)) - x|)$ with a class $\mathcal{K}$ function α_3° (since a class $\mathcal{KL}$ function α_3 always exists with such α_3°). H3 and H4 are essentially mild technical conditions characterizing the dependence of the map H on z. Note that H3 and H4 trivially hold if $(\partial H)/(\partial z)$ exists and is a constant matrix of full column rank, for example, if $H(z, y) = z$ (the case of full-order observer), or $H(z, y) = [y^\top, (z - l(y))^\top]^\top$ where l is a certain function of y (the case of reduced-order observer in Section V).

Proof: The goal is to construct a class $\mathcal{KL}$ function β_K and a class $\mathcal{K}$ function γ_K for each $K > 0$ such that (5) holds as long as $\|u\|_{[0,t]} \leq K$ and $\|x\|_{[0,t]} \leq K$. For this, let us first pick an arbitrary $K > 0$, and let

$$\Theta_K(r) := \limsup_{\xi \to \infty} \theta_K(\xi, r). \tag{14}$$

Then, $\Theta_K(r) < 1$ for all $r \geq 0$ and $K > 0$ from (13), and $\Theta_K(\cdot)$ is non-decreasing because so is $\theta_K(\xi, \cdot)$ for each ξ and K. Define

$$\bar{\Theta}_K(k) := \frac{1}{2} + \frac{1}{2}\Theta_K(k), \qquad k \in \mathbb{N},$$

where $\mathbb{N}$ is the set of natural numbers. Then, $\{\bar{\Theta}_K(k)\}$ is a non-decreasing sequence of k such that

$$\Theta_K(r) < \bar{\Theta}_K(k) < 1, \qquad k - 1 < r \leq k \tag{15}$$

because, for $r \in (k-1, k]$, $\Theta_K(r) < \frac{1}{2} + \frac{1}{2}\Theta_K(r) \leq \frac{1}{2} + \frac{1}{2}\Theta_K(k) = \bar{\Theta}_K(k) < 1$.

Let a sequence $\{m_K(k), k \in \mathbb{N}\}$ be such that

$$\xi \geq m_K(k) \quad \Rightarrow \quad \theta_K(\xi, k) \leq \bar{\Theta}_K(k), \tag{16}$$

whose existence follows from (14) and (15) with $r = k$. By the fact that $\theta_K(\xi, \cdot)$ is non-decreasing, we have

$$\xi \geq m_K(k) \quad \Rightarrow \quad \theta_K(\xi, r) \leq \bar{\Theta}_K(k), \qquad k - 1 < r \leq k, \quad k \in \mathbb{N}. \tag{17}$$

[6]For cases when this set is empty, we follow the convention that an empty set is compact.

By H2, this in turn implies that, for all z, x, u, d, and $k \in \mathbb{N}$ such that $|x| \leq K$, $|u| \leq K$, $k-1 < |d| \leq k$,

$$|H(z,h(x,0)) - x| \geq m_K(k) \quad \Rightarrow \quad \frac{g(z,x,u,d)}{W(z,x,u,d)} \leq \bar{\Theta}_K(k)$$
$$\Rightarrow \quad \dot{V} \leq -(1-\bar{\Theta}_K(k))W(z,x,u,d) \leq -(1-\bar{\Theta}_K(k))\alpha_3(|H(z,h(x,0)) - x|, |x| \vee |u|). \tag{18}$$

On the other hand, we note that, since g is continuous and $g(z,x,u,0) \leq 0$ for any z, x, and u, there exists a continuous function $\delta^*(z,x,u)$ such that, for each z, x, and u,

$$g(z,x,u,\delta) \leq \bar{\Theta}_K(1)\alpha_3(|H(z,h(x,0)) - x|, |x| \vee |u|) \qquad \forall |\delta| \leq \delta^*(z,x,u),$$

and that $\delta^*(z,x,u) > 0$ for all z, x, and u such that $|H(z,h(x,0)) - x| > 0$. By H3, the set $\{z : |H(z,h(x,0)) - x| = \xi\}$ is compact (or possibly empty) for each $\xi \geq 0$ and x. Let

$$n_K^*(\xi) := \min_{|x|\leq K, |u|\leq K} \; \min_{\{z:|H(z,h(x,0))-x|=\xi\}} \delta^*(z,x,u)$$

which is defined for ξ such that the set over which the minimum is being taken is nonempty. Using the continuity of h and H, it is easy to show that the function n_K^* is defined on a subinterval of $[0,\infty)$ and is lower semi-continuous.[7] Moreover, we have $n_K^*(\xi) > 0$ for all $\xi > 0$ in the domain of n_K^*. Thus there exists a class $\mathcal{K}$ function $n_K : [0, m_K(1)] \to \mathbb{R}_{\geq 0}$ such that $n_K(\xi) \leq n_K^*(\xi)$ wherever both functions are defined, and $n_K(m_K(1)) \leq 1$. Then, by construction, $g(z,x,u,d) \leq \bar{\Theta}_K(1)\alpha_3(|H(z,h(x,0)) - x|, |x| \vee |u|)$ for all z, x, u, and d such that $|x| \leq K$, $|u| \leq K$, and $|d| \leq n_K(|H(z,h(x,0)) - x|) \leq n_K(m_K(1))$. This implies that

$$m_K(1) \geq |H(z,h(x,0)) - x| \geq n_K^{-1}(|d|) \quad \Rightarrow \quad \dot{V} \leq -(1-\bar{\Theta}_K(1))\alpha_3(|H(z,h(x,0)) - x|, |x| \vee |u|). \tag{19}$$

Now, pick a class $\mathcal{K}_\infty$ function M_K such that

$$M_K(r) \geq \begin{cases} n_K^{-1}(r), & 0 \leq r \leq n_K(m_K(1)), \\ m_K(1), & n_K(m_K(1)) < r \leq 1, \\ m_K(k), & k-1 < r \leq k, \; k \geq 2, \end{cases} \tag{20}$$

[7]Lower semi-continuity means that $n_K^*(\xi) \leq \liminf_{\eta\to\xi} n_K^*(\eta)$ for all ξ. To see why this property holds, note that $n_K^*(\xi)$ cannot exceed the limit of the values $n_K^*(\eta_i)$ for any sequence $\{\eta_i\} \to \xi$ because the limit of (a subsequence of) the sequence of points (x_i, u_i, z_i) at which the minimum defining $n_k^*(\eta_i)$ is achieved is included in the set over which the minimum defining $n_k^*(\xi)$ is being taken.

and pick a continuous non-increasing function $\phi_K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$0 < \phi_K(r) \leq 1 - \bar{\Theta}_K(k), \qquad k-1 < r \leq k, \quad k \in \mathbb{N} \tag{21}$$

and $\phi_K(0) = \lim_{r\to 0^+} \phi_K(r)$. Then, from (18), (19), (20), and (21),

$$|H(z, h(x,0)) - x| \geq M_K(|d|) \quad \Rightarrow \quad \dot{V} \leq -\phi_K(|d|)\alpha_3(|H(z,h(x,0)) - x|, |x| \vee |u|) \tag{22}$$

with $|x| \leq K$ and $|u| \leq K$. Since

$$\begin{aligned} V(z,x) \geq \bar{M}_K(|d|) := \lambda(K)\alpha_2(M_K(|d|)) \quad &\Rightarrow \quad \lambda(|x|)\alpha_2(|H(z,h(x,0)) - x|) \geq \lambda(K)\alpha_2(M_K(|d|)) \\ &\Rightarrow \quad |H(z,h(x,0)) - x| \geq M_K(|d|) \end{aligned}$$

by H1, it follows from (22) that, for almost all $t \geq 0$ and for all essentially bounded $d(\cdot)$,

$$\begin{aligned} V(z(t),x(t)) \geq \bar{M}_K(\|d\|) \quad \Rightarrow \quad & \\ \dot{V}(z(t),x(t)) &\leq -\phi_K(|d(t)|)\alpha_3(|H(z(t),h(x(t),0)) - x(t)|, K) \\ &\leq -\phi_K(\|d\|)\alpha_3\left(\alpha_2^{-1}\left(\frac{V(z(t),x(t))}{\lambda(K)}\right), K\right) \end{aligned}$$

as long as $\|x\| \leq K$ and $\|u\| \leq K$, where $\|\cdot\| := \|\cdot\|_{[0,\infty)}$. In other words, under the assumption that $\|x\| \leq K$ and $\|u\| \leq K$,

$$V(z(t),x(t)) \geq \bar{M}_K(\|d\|) \quad \Rightarrow \quad \frac{dV(z(t),x(t))}{d(\phi_K(\|d\|)t)} \leq -\alpha_K(V(z(t),x(t))) \tag{23}$$

for almost all $t \geq 0$, where $\alpha_K(V) := \alpha_3\left(\alpha_2^{-1}(V/\lambda(K)), K\right)$ is a class $\mathcal{K}$ function. From this and the standard arguments as in, e.g., [27] (see also [31, Lemma A.4]), it follows that there exists a class $\mathcal{KL}$ function $\bar{\beta}_K$ such that

$$V(z(t),x(t)) \leq \bar{\beta}_K(V(z(0),x(0)), \phi_K(\|d\|)t) \vee \bar{M}_K(\|d\|). \tag{24}$$

It follows from H1 and the fact that ϕ_K is non-increasing that

$$\begin{aligned} |e_0(t)| &\leq \alpha_1^{-1}(V(z(t),x(t))) \\ &\leq \alpha_1^{-1}\big(\bar{\beta}_K(\lambda(K)\alpha_2(|e_0(0)|), \phi_K(\|d\|)t) \vee \bar{M}_K(\|d\|)\big) \\ &\leq \alpha_1^{-1}\big(\bar{\beta}_K(\lambda(K)\alpha_2(|e_0(0)|), \phi_K(|e_0(0)|)t) \vee \bar{\beta}_K(\lambda(K)\alpha_2(\|d\|), \phi_K(\|d\|)t) \vee \bar{M}_K(\|d\|)\big) \\ &\leq \alpha_1^{-1}\big(\bar{\beta}_K(\lambda(K)\alpha_2(|e_0(0)|), \phi_K(|e_0(0)|)t)\big) \vee \alpha_1^{-1}\big(\bar{\beta}_K(\lambda(K)\alpha_2(\|d\|), 0) \vee \bar{M}_K(\|d\|)\big). \end{aligned} \tag{25}$$

Finally, using H4 and taking a class $\mathcal{K}$ function $\bar{\rho}_K$ such that

$$\bar{\rho}_K(r) \geq \max_{|x|\leq K, |\delta|\leq r} \rho(x,\delta), \tag{26}$$

we have for almost all $t \geq 0$ that

$$\begin{aligned}|e(t)| &\leq |H(z(t), h(x(t), d(t))) - H(z(t), h(x(t), 0))| + |e_0(t)| \\ &\leq \bar{\rho}_K(\|d\|) + |e_0(t)| \leq 2\bar{\rho}_K(\|d\|) \vee 2|e_0(t)|.\end{aligned} \tag{27}$$

Therefore, after defining

$$\begin{aligned}\beta_K(r,t) &= 2\alpha_1^{-1}(\bar{\beta}_K(\lambda(K)\alpha_2(r), \phi_K(r)t) \\ \gamma_K(r) &= 2\bar{\rho}_K(r) \vee 2\alpha_1^{-1}\left(\bar{\beta}_K(\lambda(K)\alpha_2(r), 0) \vee \bar{M}_K(r)\right)\end{aligned}$$

and replacing $\|d\|$ with $\|d\|_{[0,t]}$ due to causality, the inequality (5) follows from (25) and (27). ■

Remark 2: It is seen from the proof (in particular from (23) and (24)) that the convergence rate of the error may get smaller as the size of $\|d\|$ gets larger. Note that in (25), in order to get to the desired form (5), the effect of $\|d\|$ is moved from the class $\mathcal{KL}$ function to the class $\mathcal{K}$ function, which may be seen as trading slower convergence for a larger gain.

Example 5: In order to demonstrate Theorem 1, a (reduced-order) qDES observer is presented in this example. Consider a plant given by

$$\begin{aligned}\dot{x}_1 &= -2x_1 - 2x_2, \qquad y = x_1 + d, \\ \dot{x}_2 &= \frac{x_2}{1+x_1^2} + u.\end{aligned} \tag{28}$$

For this plant, consider an observer given by

$$\begin{aligned}\dot{z} &= \frac{-z + \tan^{-1}(y) - 2y}{1+y^2} + u \\ \hat{x}_1 &= y \\ \hat{x}_2 &= z - \tan^{-1}(y).\end{aligned} \tag{29}$$

The last equation implies that $H(z, h(x,d)) = (x_1 + d, z - \tan^{-1}(x_1 + d))$, which satisfies H3 and H4 in Theorem 1. This construction is inspired by [24] as follows. With a new variable $\zeta := x_2 + \tan^{-1}(x_1)$, we have the dynamics

$$\dot{\zeta} = \frac{x_2}{1+x_1^2} + u + \frac{-2x_1 - 2x_2}{1+x_1^2} = \frac{-\zeta + \tan^{-1}(x_1) - 2x_1}{1+x_1^2} + u \tag{30}$$

which are incrementally GAS [3] for any x_1 and u. Therefore, a copy of the system works as a globally convergent observer with bounded input/state when $d \equiv 0$, which is (29). Indeed, with $\epsilon := z - \zeta$, the error dynamics in this coordinate is given by

$$\dot{\epsilon} = \frac{-(\epsilon + \zeta) + \tan^{-1}(x_1 + d) - 2(x_1 + d)}{1 + (x_1 + d)^2} - \frac{-\zeta + \tan^{-1}(x_1) - 2x_1}{1 + x_1^2}.$$

When $d \equiv 0$, it becomes $\dot{\epsilon} = -\epsilon/(1+x_1^2)$, which shows that $\epsilon(t) \to 0$ if $\|x_1\|_{[0,\infty)}$ is bounded.

Let the function V of Theorem 1 be

$$V(z,x) = \frac{1}{2}(H(z,h(x,0))-x)^\top(H(z,h(x,0))-x) = \frac{1}{2}\left((x_1-x_1)^2 + (z-\tan^{-1}(x_1)-x_2)^2\right) = \frac{1}{2}\epsilon^2$$

which satisfies H1 of Theorem 1 with $\lambda \equiv 1$. (The function V is in fact $\frac{1}{2}e_0^2$ because $e_0 = H(z,h(x,0)) - x = [0,\epsilon]^\top$.) This in turn yields, by adding and subtracting a term,

$$\dot{V} = \frac{-\epsilon^2}{1+x_1^2} + \left[\frac{-\epsilon(\epsilon+\zeta)+\epsilon\tan^{-1}(x_1+d)-2\epsilon(x_1+d)}{1+(x_1+d)^2} - \frac{-\epsilon(\epsilon+\zeta)+\epsilon\tan^{-1}(x_1)-2\epsilon x_1}{1+x_1^2}\right].$$

This suggests to take W and α_3 as

$$W(z,x,u,d) = \frac{\epsilon^2}{1+x_1^2} = \frac{(z-\tan^{-1}(x_1)-x_2)^2}{1+x_1^2},$$

$$\alpha_3(s,r) = \frac{s^2}{1+r^2} \quad \text{since } \frac{\epsilon^2}{1+x_1^2} \geq \alpha_3(|H(z,h(x,0))-x|, |x| \vee |u|) = \frac{\epsilon^2}{1+(|x| \vee |u|)^2}.$$

In addition, the bracket term above is taken as the function $g(z,x,u,d)$ of Theorem 1, which satisfies $g(z,x,u,0)=0$ of (11). We note that, with $\zeta = x_2 + \tan^{-1}(x_1)$,

$$\begin{aligned}
\frac{g(z,x,u,d)}{W(z,x,u,d)} &= \frac{1+x_1^2}{\epsilon^2}\left[\frac{-\epsilon^2-\epsilon\zeta+\epsilon\tan^{-1}(x_1+d)-2\epsilon x_1-2\epsilon d}{1+(x_1+d)^2} - \frac{-\epsilon^2-\epsilon\zeta+\epsilon\tan^{-1}(x_1)-2\epsilon x_1}{1+x_1^2}\right] \\
&\leq \frac{1+x_1^2}{\epsilon^2}\left[\frac{-\epsilon^2}{1+(|x_1|+|d|)^2} + \frac{|\epsilon|\left(|x_2+\tan^{-1}(x_1)| + |\tan^{-1}(x_1+d)| + 2|x_1| + 2|d|\right)}{1}\right. \\
&\qquad \left. + \frac{\epsilon^2}{1+x_1^2} + \frac{|\epsilon|(|x_2|+2|x_1|)}{1+x_1^2}\right] \\
&\leq \left(1 - \frac{1+x_1^2}{1+(|x_1|+|d|)^2}\right) + \frac{(1+x_1^2)\left(|x_2+\tan^{-1}(x_1)| + |\tan^{-1}(x_1+d)| + 2|x_1| + 2|d|\right)}{|\epsilon|} \\
&\qquad + \frac{|x_2|+2|x_1|}{|\epsilon|} \\
&\leq \left(1 - \frac{1}{1+(K+|d|)^2}\right) + \frac{(1+K^2)\left(|K+\tan^{-1}(K)| + \pi/2 + 2K + 2|d|\right)}{|\epsilon|} + \frac{K+2K}{|\epsilon|} \\
&=: \theta_K(|\epsilon|, |d|)
\end{aligned}$$

and this function θ_K is non-decreasing in $|d|$ and satisfies (13), so that H2 holds. Therefore, the observer (29) is a (reduced-order) qDES observer. ///

In the derivations of inequalities in Example 5, the upper bounds were not tight. Thanks to the $\limsup$ operation in (13), we do not need these bounds to be very accurate as long as the resulting function θ_K is smaller than 1 for large $|\epsilon|$ and is non-decreasing in $|d|$. The next example also emphasizes the importance of the non-decreasing property of θ_K with respect to its second argument $|d|$ in H2 of Theorem 1.

Example 6: Let us consider a full-order observer so that $H(z, y) = z$ and $e = z - x = e_0$. With $\mathrm{sat}(s) := \mathrm{sign}(s)\min\{|s|, 1\}$, suppose that $\dot{V} = -W(z, x, u, d) + g(z, x, u, d)$ with $g(z, x, u, d) = \mathrm{sat}(|d(z-x)|)e^{-(|d||z-x|-1)^2}(z-x)^2$ and $W(z, x, u, d) = (z-x)^2$ so that

$$\frac{g(z, x, u, d)}{W(z, x, u, d)} = \mathrm{sat}(|d||e_0|)e^{-(|d||e_0|-1)^2} =: \vartheta_K(|e_0|, |d|). \tag{31}$$

The function ϑ_K is continuous and nonnegative, and satisfies the condition (13) since

$$\limsup_{\xi \to \infty} \vartheta_K(\xi, r) = 0 < 1 \qquad \forall r \geq 0, \ K > 0,$$

but is not non-decreasing in its second argument. In fact, ϑ_K takes the value of 1 on the curve $\{(d, e_0) : |d||e_0| = 1\}$ in the (d, e_0)-plane, and is less than 1 away from this curve, and therefore, there is no function $\theta_K \geq \vartheta_K$ that satisfies (13) and is non-decreasing in the second argument. Note that it is not possible to find a class $\mathcal{K}$ function M_K such that

$$|e_0| \geq M_K(|d|) \quad \Rightarrow \quad \dot{V} < 0$$

as required in the proof of Theorem 1. Indeed, for an arbitrary class $\mathcal{K}$ function M_K, let $d^* > 0$ be the solution to $M_K(d^*) = 1/d^*$ (which always exists). Then, for any d with $0 < |d| < d^*$, there is an e_0 such that $|e_0| \geq M_K(|d|)$ and $|d||e_0| = 1$. With such e_0 and d, we have $g/W = 1$ in (31), and thus $\dot{V} = 0$. ///

IV. Full-order qDES Observers

Now we use Theorem 1 to derive more easily verifiable sufficient conditions that guarantee qDES property in the case of full-order observers. If f in (1) and F in (2) are continuously differentiable, then a globally convergent observer can always be written as

$$\dot{z} = F(z, y, u) = f(z, u) + \mathcal{L}(z, y, u), \qquad \hat{x} = H(z, y) = z, \tag{32}$$

where $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^p \times \mathcal{U} \to \mathbb{R}^n$ is C^1 and $\mathcal{L}(z, y, u)$ becomes the zero vector whenever $h(z, 0) = y$. (See [29, Lemma 21] or [21] for a proof of this fact.) Then, with $d \equiv 0$, the problem of designing a globally convergent full-order observer can be thought of as a search for a function $V(x, e_0)$ (with $e_0 = z - x$) and a vector $\mathcal{L}(z, h(x, 0), u)$ such that

$$\alpha_1(|e_0|) \leq V(x, e_0) \leq \alpha_2(|e_0|) \tag{33}$$

and the time derivative of V along (1) and $\dot{e}_0 = f(e_0 + x, u) - f(x, u) + \mathcal{L}(e_0 + x, h(x, 0), u)$ satisfies

$$\frac{\partial V}{\partial x}(x, e_0)f(x, u)+\frac{\partial V}{\partial e_0}(x, e_0)[f(e_0+x, u)-f(x, u)]+\frac{\partial V}{\partial e_0}(x, e_0)\mathcal{L}(e_0+x, h(x, 0), u) \leq -\alpha_3^\circ(|e_0|) \tag{34}$$

for all e_0, x, and u, where α_1 and α_2 are class $\mathcal{K}_\infty$ functions and α_3° is a class $\mathcal{K}$ function.

Corollary 1: With a given $\mathcal{L}$ for which (34) holds, the system (32) is a full-order qDES observer for (1) if, for each $K > 0$, there is a nonnegative continuous function $G_K : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is non-decreasing in its second argument and satisfies

$$\frac{\partial V}{\partial e_0}(x, e_0)\mathcal{L}(e_0 + x, h(x, d), u) - \frac{\partial V}{\partial e_0}(x, e_0)\mathcal{L}(e_0 + x, h(x, 0), u) \leq G_K(|e_0|, |d|), \tag{35}$$

$$\limsup_{\xi \to \infty} \frac{G_K(\xi, r)}{\alpha_3^\circ(\xi)} < 1 \qquad \forall r \geq 0 \tag{36}$$

for all e_0, d, $|x| \leq K$, and $|u| \leq K$.

Proof: With $d \not\equiv 0$, the time derivative of $V(x, e_0)$ is given, similarly to (34), by

$$\dot{V} = \frac{\partial V}{\partial x}(x, e_0)f(x, u) + \frac{\partial V}{\partial e_0}(x, e_0)[f(e_0 + x, u) - f(x, u)] + \frac{\partial V}{\partial e_0}(x, e_0)\mathcal{L}(e_0 + x, h(x, d), u).$$

By adding and subtracting $\frac{\partial V}{\partial e_0}(x, e_0)\mathcal{L}(e_0 + x, h(x, 0), u)$ on the right-hand side of this formula, and using (34), we can verify the hypotheses H1–H4 of Theorem 1. Indeed, H3 and H4 hold since $H(z, h(x, 0)) = z$ (see Remark 1) while H1 holds by (33) with $e_0 = z - x$ and $\lambda \equiv 1$. H2 holds with $W(z, x, u, d) = \alpha_3^\circ(|z - x|)$, $g(z, x, u, d) = \frac{\partial V}{\partial e_0}(x, z - x)\mathcal{L}(z, h(x, d), u) - \frac{\partial V}{\partial e_0}(x, z - x)\mathcal{L}(z, h(x, 0), u)$ (the left-hand side of (35)), and $\theta_K(|e_0|, |d|) = G_K(|e_0|, |d|)/\alpha_3^\circ(|e_0|)$. ■

Corollary 2: With a given $\mathcal{L}$ for which (34) holds and h being C^1, the system (32) is a full-order qDES observer for (1) if the following conditions are satisfied:

1) $\frac{\partial \mathcal{L}}{\partial y}(z, y, u)$ is independent of z.
2) For each $K > 0$, there is a function α_K such that $|\frac{\partial V}{\partial e_0}(x, e_0)| \leq \alpha_K(|e_0|)$ for all e_0 and $|x| \leq K$, and

$$\limsup_{\xi \to \infty} \frac{\alpha_K(\xi)}{\alpha_3^\circ(\xi)} = 0. \tag{37}$$

Proof: Note that

$$\mathcal{L}(z, h(x, d), u) - \mathcal{L}(z, h(x, 0), u) = \int_0^1 \frac{\partial \mathcal{L}}{\partial y}(z, h(x, sd), u)\frac{\partial h}{\partial d}(x, sd)ds \cdot d =: \phi(x, u, d)$$

in which ϕ does not depend on z and is continuous. Let

$$G_K(\xi, r) = \alpha_K(\xi) \cdot \max_{|x| \leq K, |u| \leq K, |\delta| \leq r} |\phi(x, u, \delta)|. \tag{38}$$

Then, the assumptions of Corollary 1 hold. ■

Remark 3: A condition of the type (37), for the special case when G_K in (35) decomposes into a product of an e_0-dependent and an e_0-independent term as in (38), appeared in [26] (see Section V as well), and a similar condition was used in the context of ISS controller design in [28].

We now illustrate that several of the nonlinear observers in the literature are already qDES observers, even though this property has not been explored, to the authors' knowledge. Thanks to Corollary 1 and Corollary 2, verification of the qDES observer property becomes quite a simple task as seen in the following. Note that, since the qDES property is coordinate invariant, we can verify it in any convenient coordinates.

A. Linearized Error Dynamics

The observer presented in [18] is based on the technique of "linearized error dynamics." Here we just illustrate its qDES property in a particular coordinate system where the plant (1) is written as

$$\dot{x} = Ax + f(Cx, u), \quad y = Cx + d,$$

where (A, C) is a detectable matrix pair. With a matrix L such that $A - LC$ is Hurwitz, the observer is given by

$$\dot{z} = Az + f(y, u) + L(y - Cz), \quad \hat{x} = z$$

which corresponds to (32) with $\mathcal{L}(z, y, u) = f(y, u) - f(Cz, u) + L(y - Cz)$. Then, with $e_0 = z - x$, the error dynamics can be written as

$$\dot{e}_0 = (A - LC)e_0 + Ld + f(Cx + d, u) - f(Cx, u).$$

With $V = e_0^\top P e_0$, where $P > 0$ is the solution to $P(A - LC) + (A - LC)^\top P = -I$, we have

$$\dot{V} = -|e_0|^2 + 2e_0^\top PLd + 2e_0^\top P(f(Cx + d, u) - f(Cx, u)).$$

Hence, taking $\alpha_3^\circ(|e_0|) := |e_0|^2$ and choosing G_K as

$$G_K(|e_0|, |d|) := 2|e_0|\|PL\||d| + 2|e_0|\|P\| \max_{|x| \le K, |u| \le K, |\delta| \le |d|} |f(Cx + \delta, u) - f(Cx, u)|, \quad (39)$$

where $\|\cdot\|$ denotes the maximum singular value of a matrix, the inequalities (35) and (36) are easily verified. Then, Corollary 1 ensures the qDES property. Note that the maximum in (39) need not be actually computed to verify the assumptions of Corollary 1.

B. High-gain Observer

The observer from [13] is applicable to the plant given by

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1, u), \qquad y = x_1 + d \\
\dot{x}_2 &= x_3 + f_2(x_1, x_2, u), \\
&\vdots \\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_1, \ldots, x_{n-1}, u), \\
\dot{x}_n &= f_n(x, u)
\end{aligned}$$

where f_i is globally Lipschitz in $(x_1, \ldots, x_i)$ with its Lipschitz constant independent of $u \in \mathcal{U}$. The observer has the form

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} f_1(z_1, u) \\ \vdots \\ f_n(z, u) \end{bmatrix} + L(y - z_1), \qquad \hat{x} = z$$

where the injection gain L is designed by a nested high-gain technique (see [13]). In fact, it is shown in [13] that, with $e_0 = z - x$ and $V(e_0) = e_0^\top P e_0$, where P is a certain positive definite matrix, we have

$$\dot{V} \leq -\alpha e_0^\top P e_0, \qquad \alpha > 0$$

when there is no disturbance ($d \equiv 0$). Hence, (34) holds with $\alpha_3^\circ(|e_0|) = \alpha \lambda_{\min}(P)|e_0|^2$, where $\lambda_{\min}(\cdot)$ stands for the smallest eigenvalue of a matrix. And, the injection term $\mathcal{L}(z, y, u)$ is $L(y - z_1)$ so that $\frac{\partial \mathcal{L}}{\partial y} = L$ which is obviously independent of z. Since $|\frac{\partial V}{\partial e_0}| = 2\|P\||e_0| =: \alpha_K(|e_0|)$ satisfies (37), Corollary 2 verifies the qDES property.

C. Circle Criterion Observer

The circle criterion observer [4] is designed for a system given by

$$\dot{x} = Ax + \sum_{i=1}^{r} b_i \gamma_i(h_i x) + f(Cx, u), \qquad y = Cx + d$$

where b_i is the i-th column of a matrix $B \in \mathbb{R}^{n \times r}$, h_i is the i-th row of a matrix $H \in \mathbb{R}^{r \times n}$, and $\gamma_i : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, $i = 1, \ldots, r$. Assume that there exist $L \in \mathbb{R}^{n \times p}$, $M \in \mathbb{R}^{r \times p}$, and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r) > 0$ such that the system

$$\dot{\eta} = (A - LC)\eta - Bv, \qquad \nu = \Lambda(H + MC)\eta$$

with input v and output ν is strictly positive real (SPR), or, equivalently, there exist $P > 0$, L, M, and $\Lambda > 0$ such that

$$(A - LC)^\top P + P(A - LC) \le -\alpha I, \quad B^\top P + \Lambda(H + MC) = 0$$

with some $\alpha > 0$. Then, the observer given in [4] is

$$\dot{z} = Az - L(Cz - y) + \sum_{i=1}^{r} b_i \gamma_i(h_i z + m_i(Cz - y)) + f(y, u)$$

in which m_i is the i-th row of M.

With $V = e_0^\top P e_0$, we get

$$\begin{aligned}\dot{V} \le & -\alpha|e_0|^2 + 2e_0^\top PLd + 2e_0^\top P(f(Cx + d, u) - f(Cx, u)) \\ & + 2e_0^\top P \sum_{i=1}^{r} b_i \left(\gamma_i\big(h_i(e_0 + x) + m_i(C(e_0 + x) - (Cx + d))\big) - \gamma_i(h_i x)\right).\end{aligned}$$

Since $Pb_i = -\lambda_i(h_i + m_i C)^\top$, the inequality becomes

$$\begin{aligned}\dot{V} \le & -\alpha|e_0|^2 + [2e_0^\top PLd + 2e_0^\top P(f(Cx + d, u) - f(Cx, u))] \\ & - \sum_{i=1}^{r} 2\lambda_i e_0^\top (h_i + m_i C)^\top \left(\gamma_i\big((h_i + m_i C)e_0 - m_i d + h_i x\big) - \gamma_i(h_i x)\right).\end{aligned} \tag{40}$$

Here, we present a technical lemma (whose proof is in the Appendix).

Lemma 1: For any non-decreasing function $\gamma(\cdot)$ and any given numbers a, b, and c,

$$a(\gamma(a - b + c) - \gamma(c)) \ge 0 \qquad \text{if } |a| \ge |b|.$$

With the lemma, we conclude that the summation in (40) is nonnegative when $d = 0$. (This is seen with $a = (h_i + m_i C)e_0$, $b = m_i d = 0$, and $c = h_i x$.) Therefore, let $W(z, x, u, d) = \alpha|e_0|^2$ and take $g(z, x, u, d)$ as the remaining terms in (40). Then, it is seen that (11) of H2 in Theorem 1 holds (possibly with strict inequality). On the other hand, since the terms inside the brackets in (40) are the same as in Section IV-A, let us take $G_{0,K}(|e_0|, |d|)$ as the function G_K in (39), which dominates the terms in the brackets. For the terms in the summation of (40), we claim that there exist nonnegative functions $G_{i,K}(|d|)$, $i = 1, \dots, r$ such that

$$-2\lambda_i e_0^\top (h_i + m_i C)^\top \left(\gamma_i\big((h_i + m_i C)e_0 - m_i\delta + h_i x\big) - \gamma_i(h_i x)\right) \le G_{i,K}(|d|) \tag{41}$$

for all e_0, $|x| \le K$, $|\delta| \le |d|$, and all i. Indeed, applying Lemma 1 with $a = (h_i + m_i C)e_0$, $b = m_i\delta$, and $c = h_i x$, it is seen that the left-hand side of (41) becomes nonpositive if $|(h_i + m_i C)e_0| \ge |m_i\delta|$. Hence, with

$$G_{i,K}(|d|) := \max\left\{ \max_{\substack{|a| \le |b| \le |m_i||d| \\ |x| \le K}} -2\lambda_i a(\gamma_i(a - b + h_i x) - \gamma_i(h_i x)), 0 \right\},$$

(41) follows. Finally, let $G_K(|e_0|,|d|) := G_{0,K}(|e_0|,|d|) + \sum_{i=1}^{r} G_{i,K}(|d|)$. Then (13) follows since

$$\limsup_{|e_0|\to\infty} \frac{G_K(|e_0|,|d|)}{\alpha|e_0|^2} = 0 < 1 \quad \forall K > 0,\ d.$$

All conditions H1, H2, H3, and H4 in Theorem 1 are verified, hence the qDES property is ensured.

V. Reduced-order qDES Observers

We have seen in the previous section that a few observer designs automatically yield qDES observers. On the other hand, the following example shows that this is not the case for the so-called I&I (immersion and invariance) observer design [15], [16].

Example 7: Consider a plant given by

$$\begin{aligned} \dot{x}_1 &= (1 - 2e^{x_1^2})x_1 + u, \qquad y = x_1 + d, \\ \dot{x}_2 &= (x_1^2 - 1)x_2 + u, \end{aligned}$$

where $u \in \mathcal{U} := [-1, 1]$, and a reduced-order observer

$$\begin{aligned} \dot{z} &= F(z, y, u) = (y^2 - 1)z + u \\ \hat{x} &= H(z, y) = \begin{bmatrix} y \\ z \end{bmatrix}. \end{aligned} \tag{42}$$

This observer serves as a globally convergent reduced-order observer when $d \equiv 0$, which can be verified with

$$V(x, \epsilon) = \left(1 - \frac{1}{2}e^{-x_1^2}\right)\epsilon^2, \qquad \epsilon := z - x_2.$$

Indeed, the function V satisfies that $0.5|\epsilon|^2 \le V(x, \epsilon) \le |\epsilon|^2$ for all x and ϵ, and

$$\dot{V} = [(1 + ux_1)e^{-x_1^2} - 2]\epsilon^2$$

in which the bracket term is less than or equal to $(1 + \sqrt{3})/2 - 2$ for all x_1 and $u \in \mathcal{U}$. (The maximum of $f_u(x_1) := (1 + ux_1)e^{-x_1^2}$ occurs at $x_1^* = (-1 + \sqrt{1 + 2u^2})/(2u)$ for any $u \in \mathcal{U}$. Then, $f_u(x_1) \le f_u(x_1^*) = (1 + \sqrt{1 + 2u^2})/2 \cdot e^{-x_1^{*2}} \le (1 + \sqrt{3})/2$ for any x_1 and $u \in \mathcal{U}$.) Hence, $\epsilon(t)$ exponentially converges to zero and $\lim_{t\to\infty}(\hat{x}(t) - x(t)) = 0$. The reduced-order observer (42) in fact is inspired by [25, Remark 4], and satisfies all the conditions of [16, Proposition 1] so that it can be classified as an I&I observer.

However, this observer does not have the qDES property. We first note that $x(t)$, as well as $u(t)$, is bounded. This can be easily seen from $d|x_1|/dt \le -(2e^{|x_1|^2} - 1)|x_1| + |u|$ except when

$x_1 = 0$, and so, $\limsup_{t\to\infty} |x_1(t)| < 1$. Then, from the x_2-dynamics, it is seen that $|x_2(t)|$ is also bounded. Now suppose that $d(t) = -x_1(t) + 2$, which is a bounded disturbance. Then, we have $\dot{z} = 3z + u$, and $z(t)$ may diverge while $u(t)$ is bounded. This shows that (42) is not a qDES observer. ///

Motivated by the observation in Example 7 that the function V is dependent not only on the error variable ϵ but also on the plant state x explicitly, we present a sufficient condition for a reduced-order observer to be a qDES observer based on a state-independent error Lyapunov function. For this, let us first suppose that the plant (1) has a linear output as in

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = f(x, u) \\ y &= x_1 + d \end{aligned} \tag{43}$$

where $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n-p}$, and $d \in \mathbb{R}^p$. When the system (1) is not in the form (43), it may be converted into (43) by a diffeomorphism $\Phi(x)$. This is indeed possible if the output map has the form $h(x, d) = h_0(x) + d$ (i.e., the disturbance is additive) where h_0 is C^1 with locally Lipschitz partial derivatives and if h_0 admits a complementary map $\phi : \mathbb{R}^n \to \mathbb{R}^{n-p}$ with the same regularity as h_0 so that $\Phi(x) = [h_0(x)^\top, \phi(x)^\top]^\top$ is a desired diffeomorphism converting (1) into (43) with a locally Lipschitz right-hand side. Thanks to Proposition 1, the qDES property is preserved under such a coordinate change.

Assumption 1: There exist a C^1 function $l : \mathbb{R}^p \to \mathbb{R}^{n-p}$ whose partial derivatives are locally Lipschitz, a C^1 function $V : \mathbb{R}^{n-p} \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions α_1, α_2, α_3°, and α_4 such that for all $\epsilon \in \mathbb{R}^{n-p}$, $\chi_1 \in \mathbb{R}^p$, $\chi_2 \in \mathbb{R}^{n-p}$, and $u \in \mathcal{U}$,

$$\alpha_1(|\epsilon|) \le V(\epsilon) \le \alpha_2(|\epsilon|), \qquad \left|\frac{\partial V}{\partial \epsilon}(\epsilon)\right| \le \alpha_4(|\epsilon|), \tag{44}$$

$$\begin{aligned} \frac{\partial V}{\partial \epsilon}(\epsilon)\Big(&[f_2(\chi_1, \epsilon + \chi_2, u) + \frac{\partial l}{\partial \chi_1}(\chi_1) f_1(\chi_1, \epsilon + \chi_2, u)] \\ &- [f_2(\chi_1, \chi_2, u) + \frac{\partial l}{\partial \chi_1}(\chi_1) f_1(\chi_1, \chi_2, u)]\Big) \le -\alpha_3^\circ(|\epsilon|), \end{aligned} \tag{45}$$

and

$$\limsup_{\xi\to\infty} \frac{\alpha_4(\xi)}{\alpha_3^\circ(\xi)} = 0. \tag{46}$$

Under Assumption 1, a reduced-order qDES observer can be constructed (based on the design of [15], [24]) as in the following result, which appeared in [26] and is reproduced here for completeness.

Corollary 3: Under Assumption 1, the system

$$\begin{aligned}\dot{z} &= f_2(y, z - l(y), u) + \frac{\partial l}{\partial y}(y) f_1(y, z - l(y), u),\\ \hat{x}_1 &= y,\\ \hat{x}_2 &= z - l(y),\end{aligned} \tag{47}$$

where $z \in \mathbb{R}^{n-p}$ is the observer state, is a reduced-order qDES observer for (43).

Proof: Define $\zeta := x_2 + l(x_1)$. Then, the plant (43) is globally converted into

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \zeta - l(x_1), u),\\ \dot{\zeta} &= f_2(x_1, \zeta - l(x_1), u) + \frac{\partial l}{\partial x_1}(x_1) f_1(x_1, \zeta - l(x_1), u) =: \bar{f}(x_1, \zeta, u),\\ y &= x_1 + d,\end{aligned} \tag{48}$$

where the shortcut notation $\bar{f}$ is introduced for convenience. With $\bar{f}$, the observer (47) can be simply written as $\dot{z} = \bar{f}(y, z, u)$.

Let $\epsilon := z - \zeta$. Then, since $H(z, y) = [y^\top, (z - l(y))^\top]^\top$, the conditions H3 and H4 in Theorem 1 hold (see Remark 1). Moreover, since $H(z, x_1) - x = [0^\top, \epsilon^\top]^\top$, the function $V(\epsilon)$ in Assumption 1 can play the role of $V(z, x)$ in Theorem 1 and satisfies H1 with α_1 and α_2 of (44). The time derivative of V along (47) and (48) is

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial \epsilon}(\epsilon)\left(\bar{f}(y, \epsilon + \zeta, u) - \bar{f}(x_1, \zeta, u)\right)\\ &= \frac{\partial V}{\partial \epsilon}(\epsilon)\left(\bar{f}(y, \epsilon + \zeta, u) - \bar{f}(y, \zeta, u)\right) + \frac{\partial V}{\partial \epsilon}(\epsilon)\left(\bar{f}(y, \zeta, u) - \bar{f}(x_1, \zeta, u)\right).\end{aligned}$$

The first term on the right-hand side, which corresponds to $-W$ of Theorem 1, is less than or equal to $-\alpha_3^\circ(|\epsilon|)$ by (45). (Indeed, the inequality (45) can be rewritten as

$$\frac{\partial V}{\partial \epsilon}(\epsilon)\left(\bar{f}(\chi_1, \epsilon + \chi_2, u) - \bar{f}(\chi_1, \chi_2, u)\right) \leq -\alpha_3^\circ(|\epsilon|),$$

which holds for all independent variables ϵ, u, χ_1, and χ_2. Hence, y and ζ in the previous equation can be considered as χ_1 and χ_2, respectively. This is in fact true thanks to the state-independence of the function V.) Now treating the second term, which vanishes when $d = 0$, as the function g of Theorem 1, we obtain that

$$\begin{aligned}\frac{g}{W} &\leq \frac{1}{\alpha_3^\circ(|\epsilon|)}\left|\frac{\partial V}{\partial \epsilon}(\epsilon)\right|\left|\bar{f}(y, \zeta, u) - \bar{f}(x_1, \zeta, u)\right|\\ &\leq \frac{\alpha_4(|\epsilon|)}{\alpha_3^\circ(|\epsilon|)}\max_{|x|\leq K, |u|\leq K, |\delta|\leq r}|\bar{f}(x_1 + \delta, x_2 + l(x_1), u) - \bar{f}(x_1, x_2 + l(x_1), u)| =: \theta_K(|\epsilon|, r)\end{aligned}$$

for all $\epsilon \neq 0$, $|x| \leq K$, $|u| \leq K$, and $|d| \leq r$. Then H2 of Theorem 1 holds from (46). ■

Remark 4: Assumption 1 automatically holds if, for (43) with $d \equiv 0$, there exists a globally convergent full-order observer $\dot{z} = F(z, y, u)$, $\hat{x} = z$ that admits a quadratic positive definite error Lyapunov function

$$\mathcal{V}(e_1, e_2) = \frac{1}{2} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^\top \begin{bmatrix} P_1 & P_2^\top \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \qquad e_i := z_i - x_i, \quad i = 1, 2$$

such that

$$\dot{\mathcal{V}} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^\top \begin{bmatrix} P_1 & P_2^\top \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} F_1(z, x_1, u) - f_1(x, u) \\ F_2(z, x_1, u) - f_2(x, u) \end{bmatrix} \leq -\alpha \left| \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right|^2$$

with $\alpha > 0$. Since $F_i(z, x_1, u) = f_i(x_1, z_2, u)$, $i = 1, 2$ when $z_1 = x_1$ (by an argument similar to the one showing that $\mathcal{L}$ in (32) becomes zero when the estimated output equals the actual output of the plant; see the paragraph below (32)), the above inequality can be rewritten when $e_1 = 0$ as

$$\begin{aligned} \dot{\mathcal{V}}|_{e_1=0} &= e_2^\top P_2(f_1(x_1, z_2, u) - f_1(x_1, x_2, u)) + e_2^\top P_3(f_2(x_1, z_2, u) - f_2(x_1, x_2, u)) \\ &= e_2^\top P_3[(f_2(x_1, z_2, u) - f_2(x_1, x_2, u)) + P_3^{-1} P_2(f_1(x_1, z_2, u) - f_1(x_1, x_2, u))] \leq -\alpha |e_2|^2, \end{aligned}$$

in which P_3 is positive definite since $\mathcal{V}$ is positive definite. This inequality implies Assumption 1 with $\epsilon = e_2$, $V(\epsilon) = \frac{1}{2}\epsilon^\top P_3 \epsilon$, $\chi = x$, and $l(x_1) = P_3^{-1} P_2 x_1$. The utility of this observation lies in the fact that most nonlinear observer designs in the literature are based on quadratic error Lyapunov functions.

Example 8: Let us demonstrate a construction of a reduced-order qDES observer via Corollary 3. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 + 4x_2^3 + 2u \\ \dot{x}_2 &= x_2^3 + u \\ y &= x_1 + d \end{aligned} \tag{49}$$

which is taken from [6]. This system is already in the form (43), and Assumption 1 is satisfied with $V(\epsilon) = \epsilon^2/2$, $l(\chi_1) = -(1/4)\chi_1$, and $\alpha_3^\circ(s) = (1/2)s^2$. Indeed, the left-hand side of (45) becomes

$$\epsilon\Big([(\epsilon+\chi_2)^3+u-\frac{1}{4}(\chi_1+2(\epsilon+\chi_2)+4(\epsilon+\chi_2)^3+2u)]-[\chi_2^3+u-\frac{1}{4}(\chi_1+2\chi_2+4\chi_2^3+2u)]\Big) = -\frac{1}{2}\epsilon^2$$

which verifies the claim. Therefore, the reduced-order qDES observer (47) becomes

$$\begin{aligned}\dot{z} &= -\frac{1}{4}y - \frac{1}{2}(z + \frac{1}{4}y) + \frac{1}{2}u \\ \hat{x}_1 &= y \\ \hat{x}_2 &= z + \frac{1}{4}y.\end{aligned} \tag{50}$$

///

VI. Application: Quantized Output Feedback Control

A. ISS Controller plus Observer Set-up

Consider again the plant (1) and the observer (2) which we assume to be qDES. For simplicity, we confine ourselves in this section to the full-order observer case (see [26] for related developments in the reduced-order observer case). So, here we assume that $\hat{x} = z$ and the state estimation error is $e = z - x = e_0$.

Next, suppose that a "nominal" controller (i.e., a controller that we would apply if the state x were directly available for control) is given in the form of a static feedback $u = k(x)$. This naturally leads us to define a dynamic output feedback controller by the law

$$u = k(z) = k(x + e)$$

together with the observer dynamics (2). We impose the following assumption on the feedback law k.

Assumption 2: The system $\dot{x} = f(x, k(x + e))$ is input-to-state stable (ISS) with respect to the input e, i.e., its solutions satisfy

$$|x(t)| \leq \hat{\beta}(|x(0)|, t) \vee \hat{\gamma}\left(\|e\|_{[0,t]}\right) \tag{51}$$

for a class $\mathcal{KL}$ function $\hat{\beta}$ and a class $\mathcal{K}$ function $\hat{\gamma}$.

In other words, our state feedback law should provide ISS with respect to a state measurement error, which in our case is the observer's state estimation error.

Remark 5: The existence of feedback laws providing ISS with respect to measurement errors is studied in several references. As was demonstrated by way of counterexamples in [10] and later in [8], not every stabilizable nonlinear system, even affine in controls, is input-to-state stabilizable with respect to measurement errors by means of static feedback. In [9] and [11, Chapter 6], static feedback laws guaranteeing ISS with respect to measurement errors were designed for the class of single-input plants in strict feedback form, via backstepping and "flattened" Lyapunov functions. In that work, the function $g(x)$ multiplying the control was assumed to be sign-definite and known. For the case when the sign of $g(x)$ is unknown, a time-varying feedback solution was developed for one-dimensional systems and then extended to feedback passive systems of any dimension in [12]. In [8], a time-varying feedback was designed to handle affine systems for which $g(x)$ is allowed to have zero crossings, but only in one dimension. In [14], small-gain techniques were applied to a class of systems with unknown parameters and unmodeled dynamics. In [22], a hybrid control solution was developed for systems possessing an output function whose dynamics take the form considered in [12] and with respect to which the system is minimum phase (in a suitable sense); this class covers the counterexample from [10] but not the one from [8]. The papers [6] and [7] identified a class of static state feedbacks guaranteeing ISS with respect to measurement errors, which consist of inverse optimal feedbacks with certain additional structure.

The overall closed-loop system consisting of the plant, the observer, and the control law is

$$\begin{aligned} \dot{x} &= f(x, k(z)) \\ \dot{z} &= F(z, h(x, d), k(z)). \end{aligned} \tag{52}$$

Combining the ISS property (51) of the controller with the qDES property (5) of the observer (recall that here $e_0 = e$) and applying a standard ISS cascade argument (cf. [27]), we can show that the closed-loop system is *quasi-ISS*[8] in the sense that, for each $K > 0$,

$$\left| \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \right| \leq \bar{\beta}_K \left(\left| \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \right|, t \right) \vee \bar{\gamma}_K \left(\|d\|_{[0,t]} \right) \tag{53}$$

as long as $\|x\|_{[0,t]} \leq K$ and $\|k(z)\|_{[0,t]} \leq K$, where $\bar{\beta}_K$ is a class $\mathcal{KL}$ function and $\bar{\gamma}_K$ is a class $\mathcal{K}$ function. (The cascade argument establishes this quasi-ISS property in the (x, e)-coordinates, and hence the same property holds in the $(x, z) = (x, x + e)$-coordinates, albeit with different $\bar{\beta}_K$ and $\bar{\gamma}_K$ functions.) We note for future use the obvious fact that

$$\bar{\beta}_K(s, 0) \geq s \qquad \forall\, s \geq 0. \tag{54}$$

B. Quantizer as Disturbance Generator

By an *output quantizer* we mean a piecewise constant function $q : \mathbb{R}^p \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^p$. Consider now a plant with state dynamics as in (1) but with quantized output

[8]The terminology of quasi-ISS is used differently in [2].

measurements:

$$\dot{x} = f(x, u), \qquad y = q(h_0(x))$$

where $h_0 : \mathbb{R}^n \to \mathbb{R}^p$ is a continuous map. If we introduce the quantization error

$$d := q(h_0(x)) - h_0(x)$$

then the output of this plant can be written as

$$y = h_0(x) + d =: h(x, d)$$

and this fits into our set-up (1) with an additive measurement disturbance. As in [19], [20], we assume that there exist positive numbers M and Δ (called the quantizer's *range* and *error bound*) such that the following condition holds:

$$|h_0(x)| \le M \quad \Longrightarrow \quad |d| \le \Delta. \tag{55}$$

Since the quantizer saturates outside a bounded region in the output space (the ball of radius M around the origin), we must work on this bounded region and the qDES formulation will turn out to be adequate.

Suppose, as in Section VI-A, that we are given a full-order observer in the form (2) which is qDES, and a static control law $k(\cdot)$ which fulfills Assumption 2 (ISS with respect to the state estimation error). As we showed earlier, the closed-loop system (52) then possesses the quasi-ISS property expressed by (53). Take κ_y to be some class $\mathcal{K}_\infty$ function such that

$$|h_0(x)| \le \kappa_y(|x|) \qquad \forall\, x. \tag{56}$$

Similarly, take κ_u to be some class $\mathcal{K}_\infty$ function such that

$$|k(z)| \le \kappa_u(|z|) \qquad \forall\, z. \tag{57}$$

Let

$$K := \kappa_y^{-1}(M) \vee \kappa_u(\kappa_y^{-1}(M)). \tag{58}$$

We are now ready to state the following result, which provides an ultimate bound on the solutions of the closed-loop system starting in a suitable region. (A similar result but for the reduced-order observer case appeared in [26].)

Proposition 2: With $\bar{\beta}_K$ and $\bar{\gamma}_K$ coming from (53), M and Δ as in (55), κ_y and κ_u coming from (56) and (57), and K defined in (58), assume that

$$\bar{\gamma}_K(\Delta) < \kappa_y^{-1}(M). \tag{59}$$

Suppose that the initial condition of the closed-loop system (52) satisfies

$$\left|\begin{bmatrix} x(0) \\ z(0) \end{bmatrix}\right| < E_0 \tag{60}$$

where $E_0 > 0$ is such that

$$\bar{\beta}_K(E_0, 0) = \kappa_y^{-1}(M). \tag{61}$$

Then the corresponding solution satisfies

$$\limsup_{t\to\infty} \left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \le \bar{\gamma}_K(\Delta). \tag{62}$$

Proof: Note first of all that E_0 indeed exists and satisfies

$$E_0 \le \kappa_y^{-1}(M) \tag{63}$$

by virtue of (61) and (54). As long as the inequality

$$\left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \le \kappa_y^{-1}(M)$$

remains true, we have the following:

- $|x(t)| \le \kappa_y^{-1}(M) \le K$ by (58);
- $|u(t)| \le \kappa_u(\kappa_y^{-1}(M)) \le K$ by (57) and (58) again;
- $|d(t)| \le \Delta$ by (55) because $|h_0(x)| \le M$ by (56).

The time

$$T := \sup\left\{ t \ge 0 : \left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| < \kappa_y^{-1}(M) \right\} \le \infty$$

is well defined thanks to (60) and (63). For $t \in [0, T]$, we have from the above calculations that

$$\left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \le \bar{\beta}_K\left(\left|\begin{bmatrix} x(0) \\ z(0) \end{bmatrix}\right|, t\right) \vee \bar{\gamma}_K(\Delta) < \bar{\beta}_K(E_0, 0) \vee \kappa_y^{-1}(M) = \kappa_y^{-1}(M) \tag{64}$$

by virtue of (53), (59), (60), and (61). If T were finite, this would be a contradiction, hence $T = \infty$ and the above analysis is valid for all time. Since $\bar{\beta}_K$ is a class $\mathcal{KL}$ function, for every $\epsilon > 0$ there exists a time $T(\epsilon)$ such that

$$\bar{\beta}_K\left(\left|\begin{bmatrix} x(0) \\ z(0) \end{bmatrix}\right|, t\right) \le \epsilon \qquad \forall\, t \ge T(\epsilon)$$

which in view of the first inequality in (64) gives

$$\left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \le \epsilon \vee \bar{\gamma}_K(\Delta) \qquad \forall\, t \ge T(\epsilon).$$

This proves (62). ■

Remark 6: For the ultimate bound (62) to guarantee contraction, we need to know that $\bar{\gamma}_K(\Delta) < E_0$. In light of (61) this is equivalent to

$$\bar{\beta}_K(\bar{\gamma}_K(\Delta), 0) < \kappa_y^{-1}(M) \tag{65}$$

which is a strengthening of (59). Note that $\bar{\gamma}_K$ depends on K which in turn depends on M, i.e., M affects both sides of the inequality (65). With a fixed M, we can always satisfy (65) by making Δ small enough. In other words, (65) basically says that we must have sufficiently many quantization regions so that the quantizer's error bound is small enough. The same comments apply to the condition (59).

The above result is especially useful in situations where the quantization can be *dynamic*, in the sense that the parameters of the quantizer can be changed on-line by the control designer [19]. We can then improve on the ultimate bound (62) by using a "zooming" strategy. In the context of observer-based quantized output feedback, this idea is developed in more detail in [20] for full-order DES observers and in [26] for reduced-order qDES observers; the case of full-order qDES observers considered here can be treated similarly.

VII. Conclusion

We proposed and studied the notion of a qDES observer, which captures robustness of a nonlinear observer to output measurement disturbances. We developed a general framework for studying both full-order and reduced-order qDES observers, based on Lyapunov functions. Three well-known observer designs (the linearized error dynamics, high-gain, and circle criterion observers) were shown to already possess the qDES property, and novel qDES observers for several systems were constructed. Our results were illustrated on numerous examples. As an application, we presented and analyzed a quantized output feedback control design that relies on an ISS state feedback controller and a qDES observer. Future work will focus on identifying interesting classes of nonlinear systems to which our qDES observer methodology can be applied.

Appendix

Detailed discussion about Example 3: Boundedness of the solution $x(t)$ to (6) is seen as follows. First, from (6), we have $x_1(t) = 2+(x_1(0)-2)e^{-t}$ which is bounded. Let $\alpha(t,s) := \int_s^t x_1(\tau)d\tau = 2(t-s) + (e^{-s} - e^{-t})(x_1(0) - 2)$. Then, the state-transition matrix of

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & x_1(t) \\ -x_1(t) & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

is obtained by

$$\Phi(t,s) := \begin{bmatrix} \cos(\alpha(t,s)) & \sin(\alpha(t,s)) \\ -\sin(\alpha(t,s)) & \cos(\alpha(t,s)) \end{bmatrix}.$$

Hence, with $\bar{x} := [x_2, x_3]^\top$, we have, from (6),

$$\bar{x}(t) = \Phi(t,0)\bar{x}(0) + \int_0^t \Phi(t,\tau) \begin{bmatrix} 0 \\ \sin\tau \end{bmatrix} d\tau = \Phi(t,0)\bar{x}(0) + \int_0^t \begin{bmatrix} \sin(\alpha(t,\tau))\sin\tau \\ \cos(\alpha(t,\tau))\sin\tau \end{bmatrix} d\tau.$$

Then, boundedness of $\bar{x}(t)$ follows since $\Phi(t,0)$, $\int_0^t \sin(\alpha(t,\tau))\sin\tau d\tau$, and $\int_0^t \cos(\alpha(t,\tau))\sin\tau d\tau$ are bounded; for example, $\int_0^t \sin(\alpha(t,\tau))\sin\tau d\tau = \sin(2t - e^{-t}(x_1(0)-2)) \int_0^t \cos(2\tau - e^{-\tau}(x_1(0)-2))\sin\tau d\tau - \cos(2t - e^{-t}(x_1(0)-2)) \int_0^t \sin(2\tau - e^{-\tau}(x_1(0)-2))\sin\tau d\tau$ is bounded.

To show that $\lim_{t\to\infty} e(t) = 0$, we refer to (8), from which $\lim_{t\to\infty} e_1(t) = 0$ is straightforward. For e_2 and e_3, it is observed that

$$\begin{bmatrix} \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \left[x_1(t) \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e_1(t) \\ -e_1(t) & 0 \end{bmatrix} \right] \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} e_1(t)x_3(t) \\ e_1(t)x_2(t) \end{bmatrix}.$$

From [17, Example 9.6], the system

$$\begin{bmatrix} \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \left[x_1(t) \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e_1(t) \\ -e_1(t) & 0 \end{bmatrix} \right] \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}$$

is an exponentially stable linear system after the time when $x_1(t)$ becomes positive. Moreover, we know that $e_1(t)$ converges to zero while $x_2(t)$ and $x_3(t)$ are bounded. Thus, $e_2(t)$ and $e_3(t)$ converge to zero.

Proof of Lemma 1: If $a \geq 0$ then inequality $|a| \geq |b|$ implies $a \geq |b|$ and so $a - b \geq 0$. Since γ is non-decreasing, the claim follows. When $a < 0$, it follows from $|a| \geq |b|$ that $-a \geq |b|$ so that $a - b \leq 0$ and the claim again follows.

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