ROBUST OBSERVERS and PECORA–CARROLL SYNCHRONIZATION with LIMITED INFORMATION

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TALK OUTLINE

• Observers robust to measurement disturbances in ISS sense: formulation and Lyapunov condition
  [Shim–L, TAC, 2016]

• Application to robust synchronization: Lorenz chaotic system
  (this paper)
TALK OUTLINE

• Observers robust to measurement disturbances in ISS sense: formulation and Lyapunov condition

• Application to robust synchronization: Lorenz chaotic system
INPUT—to—STATE STABILITY (ISS)  [Sontag '89]

System $\dot{x} = f(x, d)$ is ISS if its solutions satisfy

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\|d\|_{[0,t]}\right)$$

where $\gamma \in \mathcal{K}_\infty$, $\beta(\cdot, t) \in \mathcal{K}_\infty$, $\beta(r, \cdot) \searrow 0$

ISS $\iff$ existence of ISS Lyapunov function:

pos. def., rad. unbdd, $C^1$ function $V$ satisfying

$$|x| \geq \rho(|d|) \Rightarrow \dot{V} < 0 \quad (\rho \in \mathcal{K}_\infty)$$

or equivalently

$$\dot{V} \leq -\alpha(|x|) + \chi(|d|) \quad (\alpha, \chi \in \mathcal{K}_\infty)$$
**ROBUST OBSERVER DESIGN PROBLEM**

Plant: \( \dot{x} = f(x, u), \quad y = h(x, d) \quad (x \in \mathbb{R}^n) \)

Observer: \( \dot{z} = F(z, y, u), \quad \hat{x} = H(z, y) \quad (z \in \mathbb{R}^m) \)

Full-order observer: \( \hat{x} = z, \quad m = n \); reduced-order: \( m < n \)

State estimation error: \( e := \hat{x} - x = H(z, h(x, d)) - x \)

Robustness issue: can have \( e \to 0 \) when \( d \equiv 0 \)

yet \( e \not\to \infty \) for arbitrarily small \( d \neq 0 \)
DISTURBANCE–to–ERROR STABILITY (DES)

Plant: \[ \dot{x} = f(x, u), \quad y = h(x, d) \]
Observer: \[ \dot{z} = F(z, y, u), \quad \hat{x} = H(z, y) \]
Estimation error: \[ e := \hat{x} - x \]

ISS-like robustness notion: call observer DES if
\[ |e(t)| \leq \beta(|e(0)|, t) + \gamma \left( \|d\|_{[0,t]} \right) \]
\[ \beta \in \mathcal{KL}, \quad \gamma \in \mathcal{K}_{\infty} \]

Known conditions for this [Sontag–Wang ’97, Angeli ’02] are very strong

Also, DES is coordinate dependent as global error convergence is coordinate dependent: \[ z \rightarrow x \not\Rightarrow \Phi(z) \rightarrow \Phi(x) \]

Path toward less restrictive, coordinate-invariant robustness property: impose DES only as long as \( x, u \) are bounded
QUASI–DISTURBANCE–to–ERROR STABILITY (qDES)

\[
\begin{align*}
\dot{x} &= f(x, u), \quad y = h(x, d) \\
\dot{z} &= F(z, y, u), \quad \hat{x} = H(z, y) \\
e &= \hat{x} - x
\end{align*}
\]

Definition: observer is quasi-Disturbance-to-Error Stable (qDES) if \( \forall K > 0 \ \exists \beta_K \in \mathcal{K}\mathcal{L}, \ \gamma_K \in \mathcal{K}\infty \) such that

\[
|e(t)| \leq \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})
\]

whenever \( \|u\|_{[0,t]}, \|x\|_{[0,t]} \leq K \)

The qDES property is invariant to coordinate changes
REduced-Order qDES ObsErvers

Plant (after a coordinate change):

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, u) \\
\dot{x}_2 &= f_2(x_1, x_2, u) \\
y &= x_1 + d
\end{align*}
\]

 Observer:

\[
\begin{align*}
\dot{z} &= f_2(y, z, u) \\
\hat{x}_1 &= y \\
\hat{x}_2 &= z
\end{align*}
\]

\[
e := z - x_2, \quad V = V(e)
\]

\[
\dot{V} = \frac{\partial V}{\partial e}[f_2(x_1, x_2 + e, u) - f_2(x_1, x_2, u)]
\]

Assume this is \(-\alpha_3(|e|)\), then we have an asymptotic observer: \(e \to 0 \text{ when } d \equiv 0\)
REDUCED-ORDER qDES OBSERVERS

Plant (after a coordinate change):
\[
\dot{x}_1 = f_1(x_1, x_2, u) \\
\dot{x}_2 = f_2(x_1, x_2, u) \\
y = x_1 + d
\]

Observer:
\[
\dot{z} = f_2(y, z, u) \\
\hat{x}_1 = y \\
\hat{x}_2 = z
\]

\[e := z - x_2, \quad V = V(e)\]

\[
\dot{V} = \frac{\partial V}{\partial e} [f_2(y, x_2 + e, u) - f_2(x_1, x_2, u)]
\]

\[
\dot{V} = \frac{\partial V}{\partial e} [f_2(y, x_2 + e, u) - f_2(y, x_2, u)] + \frac{\partial V}{\partial e} [f_2(y, x_2, u) - f_2(x_1, x_2, u)]
\]

assumed to be \[\leq -\alpha_3(|e|)\]

upper-bounded by \[\phi_K(|d|)\]

assume this has norm \[\leq \alpha_4(|e|)\]

Then
\[
\dot{V} \leq -\alpha_3(|e|) + \alpha_4(|e|)\phi_K(|d|)
\]

whenever \[\|u\|[0,t], \|x\|[0,t] \leq K\]
REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):
\[ \dot{x}_1 = f_1(x_1, x_2, u) \]
\[ \dot{x}_2 = f_2(x_1, x_2, u) \]
\[ y = x_1 + d \]

Observer:
\[ \dot{z} = f_2(y, z, u) \]
\[ \hat{x}_1 = y \]
\[ \hat{x}_2 = z \]

Asymptotic ratio condition for qDES [L–Shim, TAC, 2015]:
\[ \limsup_{r \to \infty} \frac{\alpha_4(r) \phi_K(s)}{\alpha_3(r)} < 1 \quad \forall s \iff \lim_{r \to \infty} \frac{\alpha_4(r)}{\alpha_3(r)} = 0 \]

If we have \( \alpha \in \mathcal{K}_\infty \) such that \( \alpha_3(r) \geq \alpha(r) \alpha_4(r) \)
then
\[ \dot{V} \leq -[\alpha(|e|) - \phi_K(|d|)] \cdot \alpha_4(|e|) \]
\[ \dot{V} \leq \text{when } 3(|e|) + -\alpha_4(\phi_K)(|d|) \cdot \dot{d}(|d|) \]

Can estimate ISS gain but only if \( \alpha \) is known
• Observers robust to measurement disturbances in ISS sense: formulation and Lyapunov condition

• Application to robust synchronization: Lorenz chaotic system
**ROBUST PECORA–CARROLL SYNCHRONIZATION**

Leader

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

\[d \rightarrow y \rightarrow x_1 + y \rightarrow \hat{z} = f_2(y, z)\]

Follower

\[e := z - x_2\]

Robust synchronization: \( \forall K > 0 \ \exists \beta_K \in \mathcal{KL}, \gamma_K \in \mathcal{K}_\infty \) s.t.

\[
|e(t)| \leq \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})
\]

whenever \( \|x\|_{[0,t]} \leq K \)

Equivalently: follower is a reduced-order qDES observer for leader

Sufficient condition from before: \( \exists V = V(e) \) s.t.

\[
\frac{\partial V}{\partial e} \leq \alpha_4(|e|), \quad \frac{\partial V}{\partial e}(e)(f_2(x_1, z) - f_2(x_1, x_2)) \leq -\alpha_3(|e|), \quad \text{and}
\]

\[
\lim_{r \to \infty} \frac{\alpha_4(r)}{\alpha_3(r)} = 0 \quad \text{(asymptotic ratio condition)}
\]
APPLICATION EXAMPLE

Lorenz system
Can show \( x \) is bounded using \( V(x) = x_1^2 + x_2^2 + (x_3 - \sigma - \theta)^2 \)

Can show qDES from \( d \) to \( e := \begin{pmatrix} z_2 - x_2 \\ z_3 - x_3 \end{pmatrix} \) using \( V(e) = e_2^2 + e_3^2 \)

For \( d \) arising from time sampling and quantization, we can derive an explicit bound on synchronization error which is inversely proportional to data rate (see paper for details)
CONCLUSIONS

Summary:
• qDES observer concept and Lyapunov condition
• Asymptotic ratio characterization of ISS
• Robust version of Pecora-Carroll synchronization scheme
• Application example: Lorenz system with sampled and quantized measurements

Other applications:
• Quantized output feedback control [with H. Shim, TAC 2016]
• Synchronization of electric power generators [with A. Domínguez-Garcia, ongoing work]