Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

Stabilizing a nonlinear system with limited information feedback

Daniel Liberzon
Coordinated Science Laboratory
Univ. of Illinois at Urbana-Champaign
Urbana, IL, U.S.A.
liberzon@uiuc.edu

***Abstract*— This paper is concerned with the problem of stabilizing a nonlinear continuous-time system by using sampled encoded measurements of the state. We demonstrate that global asymptotic stabilization is possible if a suitable relationship holds between the number of values taken by the encoder, the sampling period, and a system parameter, provided that a feedback law achieving input-to-state stability with respect to measurement errors can be found.**

I. INTRODUCTION

In recent years, extensive research activity has been devoted to the question of how much information a feedback controller really needs in order to stabilize a given system. Questions of this kind are motivated by applications where communication capacity is limited (e.g., a large number of systems sharing the same network cable or wireless medium, microsystems with a large number of sensors and actuators on a single chip) as well as situations where security considerations compel one to transmit as little information as possible. Among the many references on this subject, the ones particularly close in spirit to the present work are [18], [3], [17], [10], [14], [2], [7], [9].

All results developed in the aforementioned papers are limited to linear systems. The work reported here is a first step towards understanding information-based control aspects for nonlinear systems. Specifically, we extend the result and the control scheme described in [9] to nonlinear systems, characterizing the amount of information sufficient for global asymptotic stabilization. An underlying assumption is the existence of a feedback law which stabilizes the system in the case of perfect information and, moreover, provides robustness with respect to measurement errors in the sense of *input-to-state stability* (ISS) as defined in [15]. This assumption is quite restrictive in general, although some results on designing such control laws are available; see [5], [6], [4], [8].

The set-up considered in this paper is as follows. The system to be stabilized is

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control variable, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function satisfying $f(0, 0) = 0$. Control inputs considered in this paper are piecewise Lipschitz continuous. The term "limited information feedback" refers to the following scenario:

SAMPLING. Measurements are to be received by the controller at discrete times $0, \tau, 2\tau, \ldots,$ where $\tau > 0$ is a fixed *sampling period*.

ENCODING. At each of the above sampling times, the measurement received by the controller must be a number in the set $\{0, 1, \ldots, N\}$, where N is a fixed positive integer.

Thus the data available to the controller is a stream of integers

$$q_0(x(0)), \quad q_1(x(\tau)), \quad q_2(x(2\tau)), \quad \ldots$$

where $q_k(\cdot) : \mathbb{R}^n \to \{0, 1, \ldots, N\}$ is, for each k, some *encoding function*. For different values of k we can use different encoding functions. As we will see, it is natural to use the previous values $q_i(x(i\tau))$, $i = 0, \ldots, k-1$ to define the function $q_k(\cdot)$. We assume that the controller knows the initial encoding function $q_0(\cdot)$ as well as the rule that defines $q_k(\cdot)$ on the basis of the previously received encoded measurements, so that for each k the function q_k is known to the controller. In other words, there is a communication protocol satisfying the above constraints upon which the process and the controller agree in advance.

We find it convenient to use the infinity norm $\|x\|_\infty := \max\{|x_i| : 1 \le i \le n\}$ on $\mathbb{R}^n$. We let $B^n_\infty(x_0, r)$ denote a ball with respect to this norm with radius r and center x_0, i.e., the hypercube centered at x_0 with edges $2r$:

$$B^n_\infty(x_0, r) := \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \le r\}.$$

II. CONTROL STRATEGY AND ASSUMPTIONS

In this section we describe the proposed control strategy based on limited information feedback, stating and briefly discussing necessary assumptions along the way. Our first goal is to obtain an upper bound on the size of the state. We do this by "zooming out", i.e., expanding the support of the encoding function, fast enough to dominate the growth of the state for the uncontrolled system (no feedback is applied at this stage). The following assumption is needed to execute this task.

ASSUMPTION 1. *The unforced system*

$$\dot{x} = f(x, 0) \tag{2}$$

is forward complete. This means that for every initial state $x(0)$ the solution of (2), which we denote by $\xi(x(0),\cdot)$, is defined for all $t \geq 0$.

Set the control u equal to 0. Let $\mu_0 := 1$. Pick a sequence $\mu_1, \mu_2, \ldots$ that increases fast enough to dominate the rate of growth of $\|x(t)\|_\infty$ at the times $\tau, 2\tau, \ldots$; for example, define $\mu_1 := 2\max_{\|x(0)\|_\infty \leq \tau,\, t\in[0,\tau]} \|\xi(x(0),t)\|_\infty$, $\mu_2 := 2\max_{\|x(0)\|_\infty \leq 2\tau,\, t\in[0,2\tau]} \|\xi(x(0),t)\|_\infty$, and so on. This construction guarantees the existence of an integer $k_0 \geq 0$ such that $\|x(k_0\tau)\|_\infty \leq \mu_{k_0}$. For $k = 0,1,\ldots$, define the encoding function q_k by the formula

$$q_k(x) := \begin{cases} 1 & \text{if } x \in B^n_\infty(0,\mu_k) \\ 0 & \text{otherwise} \end{cases}$$

Then we can take k_0 to be the smallest k for which $q_k(x(k\tau)) = 1$. We have thus obtained the bound

$$\|x(k_0\tau)\|_\infty \leq E_0 := \mu_{k_0} \tag{3}$$

using the encoded state measurements with $N = 1$. (Such binary encoding can be realized by a quantizer taking 3^n values; cf. [3].)

The inequality (3) means that the state of the system at time $t = k_0\tau$ lies in $B^n_\infty(0,E_0)$. In other words, $\hat{x}(k_0\tau) := 0$ can be viewed as an estimate of $x(k_0\tau)$ with estimation error of infinity norm at most E_0. Our goal now is to generate state estimates with estimation errors approaching 0 as $t \to \infty$, while using these estimates to compute the feedback law.

ASSUMPTION 2. *The system* (1) *admits a locally Lipschitz feedback law* $u = k(x)$ *which satisfies* $k(0) = 0$ *and renders the closed-loop system input-to-state stable (ISS) with respect to measurement errors.* Written in terms of the infinity norm and for piecewise continuous inputs (which is sufficient for our purposes), this condition means that there exist functions[1] $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $x(t_0)$ and every piecewise continuous signal e the corresponding solution of the system

$$\dot{x} = f(x, k(x+e)) \tag{4}$$

satisfies

$$\begin{aligned}\|x(t)\|_\infty \leq \beta(\|x(t_0)\|_\infty, t-t_0) \\ + \gamma\big(\sup\nolimits_{s\in[t_0,t]} \|e(s)\|_\infty\big) \qquad \forall\, t \geq t_0.\end{aligned} \tag{5}$$

Take κ to be some class $\mathcal{K}_\infty$ function with the property that $\kappa(r) \geq \max_{\|x\|_\infty \leq r} \|k(x)\|_\infty$ for all $r \geq 0$. Then

$$\|k(x)\|_\infty \leq \kappa(\|x\|_\infty) \qquad \forall\, x. \tag{6}$$

Let L be the Lipschitz constant for f on the region

$$\{(x,u) : \|x\|_\infty \leq D,\ \|u\|_\infty \leq \kappa(D)\} \tag{7}$$

[1]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

where

$$D := \beta(E_0, 0) + \gamma\big(\sqrt[n]{N}E_0\big) + \sqrt[n]{N}E_0. \tag{8}$$

Define

$$\Lambda := e^{L\tau} \geq 1. \tag{9}$$

For $t \in [k_0\tau, k_0\tau+\tau)$, let $u(t) = 0$. At time $t = k_0\tau + \tau$, consider the box $B^n_\infty(0, \Lambda E_0)$.

ASSUMPTION 3. *The number* $\sqrt[n]{N}$ *is an odd integer.* This assumption is made mostly for notational convenience. If $\sqrt[n]{N}$ is not an integer, we can work with some $N' \leq N$ such that $\sqrt[n]{N'}$ is an integer. The reason for taking this integer to be odd is to ensure that the control strategy described below preserves the equilibrium at the origin. By making slight modifications, we can also achieve this property when the above integer is even.

Assumption 3 allows us to define the encoding function q_{k_0+1} as follows: divide $B^n_\infty(0,\Lambda E_0)$ into N equal hypercubic boxes, numbered from 1 to N in some specific way, and let $q_{k_0+1}(x)$ be the number of the box that contains x if $x \in B^n_\infty(0,\Lambda E_0)$, and 0 otherwise. In case x lies on the boundary of several boxes, the value $q_{k_0+1}(x)$ can be chosen arbitrarily among the candidates. If $q_{k_0+1}(x(k_0\tau+\tau)) > 0$, then the encoded measurement specifies a box with edges at most $2\Lambda E_0/\sqrt[n]{N}$ which contains $x(k_0\tau+\tau)$. Letting $\hat{x}(k_0\tau+\tau)$ be the center of this box, we obtain

$$\|\hat{x}(k_0\tau+\tau) - x(k_0\tau+\tau)\|_\infty \leq \Lambda E_0/\sqrt[n]{N}.$$

If $q_{k_0+1}(x(k_0\tau+\tau)) = 0$, we interpret this as an error and return to the "zooming-out" stage described earlier.

For $t \in [k_0\tau+\tau, k_0\tau+2\tau)$, we apply the control law

$$u(t) = k(\hat{x}(t)) \tag{10}$$

where $\hat{x}(\cdot)$ is the solution of the "copy" of the system (1), given by

$$\dot{\hat{x}} = f(\hat{x}, u)$$

with the initial condition $\hat{x}(k_0\tau+\tau)$ specified before. At time $t = k_0\tau + 2\tau$, consider the box $B^n_\infty(\hat{x}(k_0\tau+2\tau^-), \Lambda^2E_0/\sqrt[n]{N})$. To define the encoding function q_{k_0+2}, divide this box into N equal hypercubic boxes and let $q_{k_0+2}(x)$ be the number of the box that contains x or, if $x \notin B^n_\infty(\hat{x}(k_0\tau+2\tau^-), \Lambda^2E_0/\sqrt[n]{N})$, let $q_{k_0+2}(x) = 0$. If $q_{k_0+2}(x(k_0\tau+2\tau)) > 0$, then the encoded measurement singles out a box with edges at most $2\Lambda^2E_0/(\sqrt[n]{N})^2$ which contains $x(k_0\tau+2\tau)$. Let $\hat{x}(k_0\tau+2\tau)$ be the center of this box to obtain

$$\|\hat{x}(k_0\tau+2\tau) - x(k_0\tau+2\tau)\|_\infty \leq \Lambda^2E_0/(\sqrt[n]{N})^2$$

and continue. If $q_{k_0+2}(x(k_0\tau+2\tau)) = 0$, go back to the "zooming-out" stage.

Repeating the above procedure, we see that as long as the encoded measurements received by the controller are positive, the upper bounds on the norm of the estimation error $\|\hat{x}-x\|_\infty$ at the sampling times $k_0\tau$, $k_0\tau+\tau$, $k_0\tau+2\tau, \ldots$ form a geometric progression with ratio $\Lambda/\sqrt[n]{N}$. The goal of forcing the estimation error to approach 0 motivates our final assumption.

ASSUMPTION 4. *We have*

$$\Lambda < \sqrt[n]{N}.$$

In view of the definition of Λ via the formula (9), this inequality characterizes the trade-off between the amount of information provided by the encoder at each sampling time and the required sampling frequency. This relationship depends explicitly on the Lipschitz constant L which, as we will see, can be interpreted as a measure of expansiveness of the system (1).

III. MAIN RESULT

Theorem 1 *Under Assumptions 1–4, the control law described in Section II globally asymptotically stabilizes the system* (1).

PROOF. We first show that $\|x(t)\|_\infty < D$ and $\|\hat{x}(t)\|_\infty < D$ for all $t \geq k_0\tau$, where D is defined by (8) and E_0 is defined by (3). Suppose that this is not true. Then, since x is continuous with $\|x(k_0\tau)\|_\infty \leq E_0 < D$ and $\hat{x}$ is continuous from the right with $\hat{x}(k_0\tau) = 0$, the time

$$\bar{t} := \min\{t > k_0\tau : \max\{\|x(\bar{t})\|_\infty, \|\hat{x}(\bar{t})\|_\infty\} \geq D\} \quad (11)$$

is well defined. We have $\|x(t)\|_\infty < D$ and $\|\hat{x}(t)\|_\infty < D$ for all $t \in [k_0\tau, \bar{t})$. The formulas (10) and (6) imply that (x, u) and $(\hat{x}, u)$ stay inside the region (7) on the interval $[k_0\tau, \bar{t})$. Let us label the estimation error as

$$e := \hat{x} - x. \quad (12)$$

We know that $\|e(k_0\tau)\|_\infty \leq E_0$. Combining the equation (valid between the sampling times)

$$\dot{e} = f(\hat{x}, u) - f(x, u)$$

with the formula

$$\|f(\hat{x}, u) - f(x, u)\|_\infty \leq L\|e\|_\infty$$

and applying the Bellman-Gronwall lemma, we conclude that for every interval $(t_1, t_2] \subset [k_0\tau, \bar{t})$ not containing any sampling times we have

$$\|e(t_2)\|_\infty \leq e^{L(t_2-t_1)}\|e(t_1)\|_\infty \leq \Lambda\|e(t_1)\|_\infty$$

where the last inequality follows from (9). This in turn guarantees that at each sampling time $k\tau \in [k_0\tau, \bar{t})$, we have $q_k(x(k\tau)) > 0$ and the upper bound on $\|e\|_\infty$ is divided by $\sqrt[n]{N}$. Invoking Assumption 4, we arrive at the bound $\|e(t)\|_\infty \leq \Lambda E_0$ for all $t \in [k_0\tau, \bar{t})$. If $\bar{t}$ is not a sampling time, then e is continuous at $\bar{t}$; if $\bar{t}$ is a sampling time, then e can only decrease at $\bar{t}$. In either case, we actually have $\|e(t)\|_\infty \leq \Lambda E_0$ for all $t \in [k_0\tau, \bar{t}]$. Now, Assumption 2 expressed by the formula (5) with $t_0 = k_0\tau$ implies that $\|x(t)\|_\infty \leq \beta(E_0, 0) + \gamma(\Lambda E_0) < D$ for all $t \in [k_0\tau, \bar{t}]$, where the last inequality follows from Assumption 4. Using (12), we also obtain $\|\hat{x}(t)\|_\infty \leq \beta(E_0, 0) + \gamma(\Lambda E_0) + \Lambda E_0 < D$ for all $t \in [k_0\tau, \bar{t}]$. This yields a contradiction with the definition (11) of $\bar{t}$.

We have thus established that all of the above estimates are valid with $\bar{t} = \infty$. In particular, the upper bound on $\|e\|_\infty$ is divided by $\sqrt[n]{N}$ at the sampling times $k_0\tau+\tau$, $k_0\tau+2\tau$, $\ldots$ and grows by the factor of Λ on every interval between these times. By Assumption 4, we have $e(t) \to 0$ as $t \to \infty$. The evolution of x is governed by the system (4), and in view of the ISS property of this system with respect to e we conclude that x converges to 0 as well.

It remains to prove that the origin is a stable equilibrium of the closed-loop system in the sense of Lyapunov. The fact that it is an equilibrium is clear from the conditions $f(0, 0) = 0$ and $k(0) = 0$ and from Assumption 3 (the latter ensures that if $x \equiv 0$ then $\hat{x} \equiv 0$ because one of the boxes is always centered at the origin). Take an arbitrary $\varepsilon > 0$. We need to find a $\delta > 0$ such that the solutions of (4) with initial conditions in $B^n_\infty(0, \delta)$ remain in $B^n_\infty(0, \varepsilon)$ for all time. From the Lyapunov-like characterization of ISS proved in [16] we know that there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x, e \in \mathbb{R}^n$ we have

$$\alpha_1(\|x\|_\infty) \leq V(x) \leq \alpha_2(\|x\|_\infty) \quad (13)$$

and

$$\|x\|_\infty \geq \rho(\|e\|_\infty) \;\Rightarrow\; \frac{\partial V}{\partial x} f(x, k(x+e)) \leq -\alpha_3(\|x\|_\infty).$$

These formulas are easily seen to imply that the solutions of (4) starting in the region

$$\mathcal{R} := \{x : V(x) \leq \alpha_1(\varepsilon)\} \quad (14)$$

remain in this region as long as

$$\|e\|_\infty \leq c := \rho^{-1} \circ \alpha_2^{-1} \circ \alpha_1(\varepsilon).$$

Choose a sufficiently large integer $\bar{k} \geq 0$ such that

$$\Lambda^{\bar{k}+1} E_0/(\sqrt[n]{N})^{\bar{k}} \leq c.$$

Our previous analysis implies that $\|e(t)\|_\infty \leq c$ for all $t \geq k_0\tau + \bar{k}\tau$. Moreover, in view of Assumption 1 and the fact that the origin is an equilibrium of the unforced system (2), there exists a class $\mathcal{K}_\infty$ function ν such that all solutions of (2) satisfy $\|\xi(x(0), t)\|_\infty \leq \nu(\|x(0)\|_\infty)$ for all $t \in [0, \bar{k}\tau]$. (Just take some function $\nu \in \mathcal{K}_\infty$ satisfying $\nu(r) \geq \max_{\|x(0)\|_\infty \leq r,\, t \in [0, \bar{k}\tau]} \|\xi(x(0), t)\|_\infty$ for all $r \geq 0$.) Choose a sufficiently small $\delta > 0$ such that

$$\nu(\delta) < \min\left\{\Lambda^{\bar{k}-1}/(\sqrt[n]{N})^{\bar{k}-1}, \alpha_2^{-1} \circ \alpha_1(\varepsilon)\right\}.$$

This inequality and Assumption 4 ensure that if $\|x(0)\|_\infty \leq \delta$, then $k_0 = 0$, $\hat{x}(t) = 0$ for all $t \in [0, \bar{k}\tau)$, and $\|x(t)\|_\infty < \alpha_2^{-1} \circ \alpha_1(\varepsilon)$ for all $t \in [0, \bar{k}\tau]$. Since the last inequality implies $V(x(t)) < \alpha_1(\varepsilon)$ by virtue of (13), we see that the solutions of the closed-loop system with initial conditions in $B_\infty^n(0, \delta)$ stay in the region $\mathcal{R}$ defined by (14) for all $t \in [0, \bar{k}\tau]$. In light of the analysis given before for $t \geq k_0\tau + \bar{k}\tau = \bar{k}\tau$, we conclude that these solutions stay in $\mathcal{R}$ forever. To finish the proof, note that $\mathcal{R} \subset B_\infty^n(0, \varepsilon)$ by (13). □

Remark 1 Instead of binary encoding functions, we could let q_k, $k \leq k_0$ be encoding functions taking $N + 1$ values, similar to the functions q_k, $k \geq k_0 + 1$ but defined for the boxes $B_\infty^n(0, \mu_k)$. While this would make the "zooming-out" stage slightly more complicated, it has the advantage that the upper bound on $\|e(k_0\tau)\|_\infty$ would then be divided by $\sqrt[n]{N}$. This would allow us to eliminate $\sqrt[n]{N}$ from the formula for D and obtain the bound $\|e(t)\|_\infty < E_0$ for all $t \geq k_0\tau$, thus rendering Λ independent of N and making the relationship expressed by Assumption 4 more transparent (although Λ would still implicitly depend on the initial state through E_0). Another way to achieve the same goal is to leave the functions q_k, $k \leq k_0$ as they are but observe that by virtue of Assumption 1, $x(k_0\tau + \tau)$ belongs to a bounded region whose size is determined by E_0 and which could be used to define D and to bound $\|e(t)\|_\infty$ for $t \geq k_0\tau + \tau$. In both cases, the functions q_k, $k \geq k_0 + 1$ would need to be redefined appropriately. □

Remark 2 In Theorem 1, we were only concerned with the behavior of the state x. Since the control law applied to the system for $t \geq k_0\tau$ is a dynamic one with the (discontinuous) state $\hat{x}$, it is more accurate to refer to stability properties of the resulting composite system. Global asymptotic stability of this overall system (appropriately defined to take into account the constraint $\hat{x}(k_0\tau) = 0$ imposed by the construction) can be easily deduced from the above proof. □

IV. CONCLUSIONS

We studied the problem of stabilizing a nonlinear system using sampled encoded measurements of its state. The result and the method of proof presented here are extensions of those described in [9] for the case of linear systems. Our sufficient condition for global asymptotic stabilization involves a relationship between the number of values taken by the encoder and the sampling frequency, and relies on the assumption of input-to-state stabilizability with respect to measurement errors. The stabilizing control law takes the form of a "certainty equivalence" discontinuous dynamic feedback. Similar techniques can be used for the case when *control signals* rather than state measurements are encoded, assuming ISS with respect to *actuator* errors.

The limited information feedback strategy proposed in this paper is not intended for practical implementation. Our main goal was to identify the difficulties associated with extending relevant existing results from linear to nonlinear systems, as well as possible tools that can be used to overcome them. Compared with the analysis given in [9], the key complications in the nonlinear context are that the propagation of the estimation error is not as straightforward to characterize and that a strong form of robustness with respect to this error is needed from the controller.

Regarding the latter restriction, Claudio De Persis has kindly pointed out to the author that it can be relaxed in at least two ways. First, using the exponential decay of $e(t)$ and the results of [1], one can replace ISS in Assumption 2 by the weaker integral-ISS property plus an additional technical condition on the corresponding nonlinear gain; see [12], [11]. Second, if one assumes only that the control law globally asymptotically stabilizes the system in the absence of measurement errors, then asymptotic stabilization with encoded state feedback can still be achieved at the expense of increasing the sampling rate according to the size of the initial condition [13].

V. ACKNOWLEDGMENT

This work was supported by NSF ECS-0134115 CAR and DARPA/AFOSR MURI F49620-02-1-0325 Awards.

VI. REFERENCES

[1] M. Arcak, D. Angeli, and E. D. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. Control Optim.*, 40:1888–1904, 2002.

[2] J. Baillieul. Feedback designs in information-based control. In B. Pasik-Duncan, editor, *Stochastic Theory and Control*, volume 280 of *Lecture Notes in Control and Information Sciences*, pages 35–57. Springer, Berlin, 2002.

[3] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control*, 45:1279–1289, 2000.

[4] N. C. S. Fah. Input-to-state stability with respect to measurement disturbances for one-dimensional systems. *ESAIM J. Control, Optimization and Calculus of Variations*, 4:99–122, 1999.

[5] R. A. Freeman. Global internal stabilizability does not imply global external stabilizability for small sensor disturbances. *IEEE Trans. Automat. Control*, 40:2119–2122, 1995.

[6] R. A. Freeman and P. V. Kokotović. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser, Boston, 1996.

[7] J. P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. In *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, 2002.

[8] Z.-P. Jiang, I. Mareels, and D. Hill. Robust control of uncertain nonlinear systems via measurement feedback. *IEEE Trans. Automat. Control*, 44:807–812, 1999.

[9] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 48:304–307, 2003.

[10] G. N. Nair and R. J. Evans. Exponential stabilisability of finite-dimensional linear systems with limited data rates. *Automatica*, 39:585–593, 2003.

[11] C. De Persis. n-bit stabilization of n-dimensional nonlinear feedforward systems. 2003. Preprint.

[12] C. De Persis. Stabilization under input constraints via communication channel. *SIAM J. Control Optim.*, 2003. Submitted.

[13] C. De Persis and A. Isidori. Stabilizability by state feedback implies stabilizability by encoded state feedback. 2003. Preprint.

[14] I. R. Petersen and A. V. Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proc. 40th IEEE Conf. on Decision and Control*, pages 304–309, 2001.

[15] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34:435–443, 1989.

[16] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems Control Lett.*, 24:351–359, 1995.

[17] S. Tatikonda. *Control Under Communication Constraints*. PhD thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA, 2000.

[18] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints II: stabilization with limited information feedback. *IEEE Trans. Automat. Control*, 44:1049–1053, 1999.