OUTPUT-INPUT STABILITY:
A NEW VARIANT OF THE
MINIMUM-PHASE PROPERTY FOR
NONLINEAR SYSTEMS

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MOTIVATION

$$\dot{x} = f(x,u)\,, \qquad y = h(x)$$

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stability
(no outputs)

$u = 0 \Rightarrow x \to 0$

linear: stable
eigenvalues

ISS:

$$|x(t)| \le \beta(|x(0)|,t) + \gamma(\sup_{0\le s\le t} |u(s)|)$$

$$\dot{V} \le -\alpha(|x|) + \chi(|u|)$$

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detectability
(no inputs)

$y = 0 \Rightarrow x \to 0$

linear: stable
unobserv. modes

$$|x(t)| \le \beta(|x(0)|,t) + \gamma(\sup_{0\le s\le t} |y(s)|)$$

$$\dot{V} \le -\alpha(|x|) + \chi(|y|)$$

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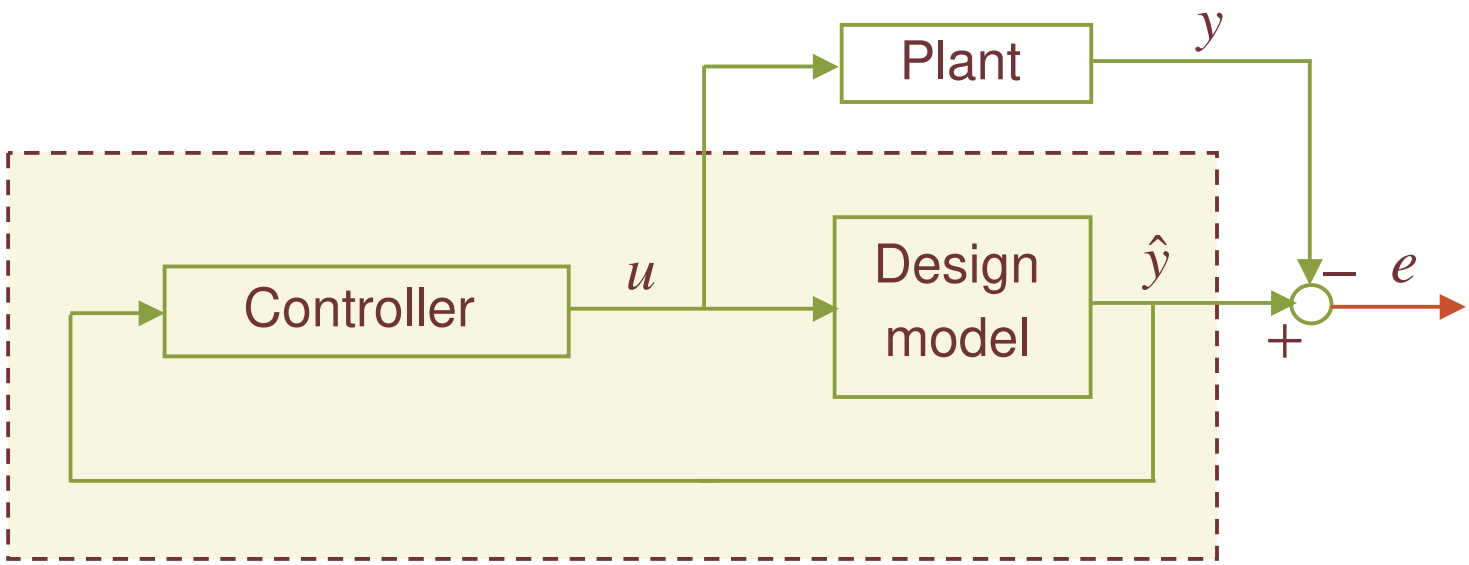
minimum
phase

$y = 0 \Rightarrow u, x \to 0$

linear: stable zeros,
stable inverse: $y^{(r)} \mapsto u$

? ? ?

MOTIVATION: Adaptive Control



If:
- the system in the box is output-stabilized
- the plant is minimum-phase

Then the closed-loop system is detectable through e ("tunable" – Morse '92)

DEFINITION

Call the system

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

output-input stable if $\exists$ integer N and functions $\beta \in KL,\ \gamma \in K_\infty$ s.t. $\forall x(0), u \in C^{N-1}$

$$\left\| \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\| \leq \beta(|x(0)|, t) + \gamma(\sup_{0 \leq s \leq t} |y^N(s)|)$$

where $y^N := (y, \dot{y}, \ldots, y^{(N)})$

Example: $\dot{y} = u$

Reduces to min phase for linear systems (MIMO too)

DISCUSSION

Split into two:

$$\left|x(t)\right| \leq \beta(\left|x(0)\right|, t) + \gamma(\left|\sup_{0 \leq s \leq t} y^N(s)\right|)$$

This is detectability (w.r.t. extended output)

Have a Lyapunov sufficient condition

$$\left|u(t)\right| \leq \beta(\left|x(0)\right|, t) + \gamma(\left|\sup_{0 \leq s \leq t} y^N(s)\right|)$$

This is related to relative degree

RELATIVE DEGREE

Call r a uniform relative degree if $y, \dot{y}, \ldots, y^{(r-1)}$ don't depend on u and

$$|u| \leq \rho_1(|x|) + \rho_2(|y^{(r)}|)$$

for some $\rho_1, \rho_2 \in K_\infty$

For affine systems: $y^{(r)} = a(x) + b(x)u$
this reduces to the usual definition $(b(x) \neq 0)$

However, the system $\dot{y} = \arctan u$
doesn't have a relative degree

SOME RESULTS

- Detectability w.r.t. extended output: $|x(t)| \le \beta(|x(0)|, t) + \gamma(|\sup_{0 \le s \le t} y^N(s)|)$

  plus relative degree: $|u| \le \rho_1(|x|) + \rho_2(|y^{(r)}|)$

  imply output-input stability: $\left|\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}\right| \le \beta(|x(0)|, t) + \gamma(\sup_{0 \le s \le t} |y^N(s)|)$

- Output-input stability implies detectability

- Main result: for SISO systems analytic in controls, output-input stability implies relative degree

  Example: affine systems $y^{(r)} = a(x) + b(x)u$

  Not true for MIMO systems !

FEEDBACK DESIGN

$$\dot{\xi}_1 = \xi_2$$
$$\vdots$$
$$\dot{\xi}_r = a(\xi,\eta) + b(\xi,\eta)u$$

$y = \xi_1$, r — relative degree

Apply u to have $\dot{\xi} = A\xi$ with A stable

If the system is output-input stable then

$$|x(t)| \le \beta(|x(0)|,t) + \gamma(|\sup_{0\le s\le t} y^N(s)|)$$

implies $x \to 0$

No global normal form is needed

With global normal form:

$$\dot{\eta} = q(\eta,\xi)$$

output-input stability corresponds to ISS internal dynamics

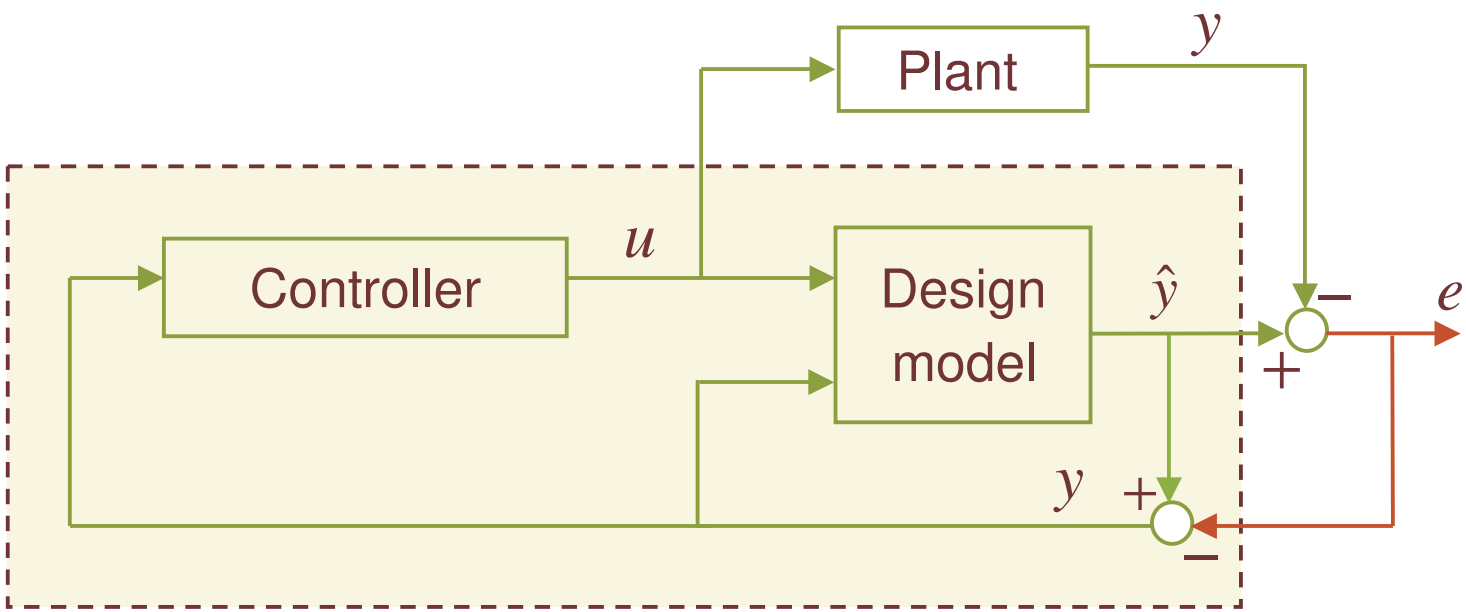
CASCADE SYSTEMS

If:
- Σ_1 is detectable (IOSS)
- Σ_2 is output-input stable ($N=r$)

Then the cascade system is detectable (IOSS) w.r.t. u and extended output y^r

For linear systems recover usual detectability (observability decomposition)

ADAPTIVE CONTROL



If:
- the plant is output-input stable ($N=r$)
- the system in the box is input-to-output stable (IOS) from e^{r-1} to $\hat{y}^r$

Then the closed-loop system is detectable through e^r ("weakly tunable")