OUTPUT-INPUT STABILITY: A NEW VARIANT OF THE MINIMUM-PHASE PROPERTY FOR NONLINEAR SYSTEMS

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Abstract: This paper studies the notion of output-input stability, which is a variant of the minimum-phase property for general smooth nonlinear control systems. In the spirit of the "input-to-state stability" philosophy, the definition of the new concept requires the state and the input of the system to be bounded by a suitable function of the output and derivatives of the output, modulo a decaying term depending on initial conditions. The class of output-input stable systems includes all affine systems in global normal form whose internal dynamics are input-to-state stable and also all left-invertible linear systems whose transmission zeros have negative real parts. A characterization of output-input stability for SISO systems is given in terms of suitable relative degree and detectability concepts.

Keywords: Nonlinear control systems, minimum-phase systems.

1. INTRODUCTION

A linear single-input/single-output (SISO) system is said to be *minimum-phase* if the numerator polynomial of its transfer function has all its zeros in the open left half of the complex plane. This property can be given a simple interpretation that involves the *relative degree* of the system, which equals the difference between the degrees of the denominator and the numerator of the transfer function. Namely, if a linear system of relative degree r is minimum-phase, then the "inverse" system, driven by the r-th derivative of the output of the original system, is stable. For left-invertible, multi-input/multi-output (MIMO) systems, in place of the zeros of the numerator one appeals to the so-called *transmission zeros* (Morse, 1973).

For nonlinear systems that are affine in controls, a major contribution of Byrnes and Isidori (1988) was to define the minimum-phase property in terms of the new concept of *zero dynamics*. The zero dynamics are the internal dynamics of the system under the action of an input that holds the output constantly at zero. The system is called *minimum-phase* if the zero dynamics are (globally) asymptotically stable. In the SISO case, a unique input capable of producing the zero output is guaranteed to exist if the system has a uniform relative degree, which is now defined to be the number of times one has to differentiate the output until the input appears. Extensions to MIMO systems are discussed in (Byrnes and Isidori, 1991; Isidori, 1995).

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In view of the need to work with the zero dynamics, the above definition of a minimum-phase nonlinear system prompts one to look for a change of coordinates that transforms the system into a certain normal form. It has also been recognized that just asymptotic stability of the zero dynamics is sometimes insufficient for control design purposes, so that additional requirements need to be placed on the internal dynamics of the system. One such common requirement is that the internal dynamics be *input-to-state stable* with respect to the output and its derivatives up to order $r-1$, where r is the relative degree (see, e.g., Praly and Jiang, 1993). These remarks suggest that while the current notion of a minimum-phase nonlinear system is important and useful, it is also of interest to develop alternative (possibly stronger) concepts which can be applied when asymptotic stability of zero dynamics is inadequate.

In this paper we present the notion of *output-input stability*, which does not rely on zero dynamics or normal forms and is not restricted to affine systems. Loosely speaking, we call a system output-input stable if its state and input eventually become small when the output and derivatives of the output are small. Conceptually, the new notion relates to the existing concept of a minimum-phase nonlinear system in much the same way as input-to-state stability (ISS) relates to global asymptotic stability under zero inputs (0-GAS), modulo the duality between inputs and outputs. An important outcome of this parallelism is that the tools that have been developed for studying ISS and related concepts can be employed to study output-input stability.

It follows from our definition that if a system has a uniform relative degree (in an appropriate sense) and is detectable through the output and its derivatives up to some order, uniformly over all inputs that produce a given output, then it is output-input stable. For SISO systems that are real analytic in controls, the converse also holds, providing a useful equivalent characterization of output-input stability (Theorem 1). The class of output-input stable systems as defined here includes all left-invertible linear systems whose transmission zeros have negative real parts (Theorem 2) and all affine systems in global normal form with input-to-state stable internal dynamics. An application of the new notion to adaptive control is discussed in (Liberzon *et al.*, 2000).

2. DEFINITION

We consider nonlinear control systems of the general form

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned} \tag{1}$$

where the state x takes values in $\mathbb{R}^n$, the input u takes values in $\mathbb{R}^m$, the output y takes values in $\mathbb{R}^l$ (for some positive integers n, m, and l), and the functions f and h are smooth (C^∞). Admissible input (or "control") signals are locally essentially bounded, Lebesgue measurable functions $u:[0,\infty)\to\mathbb{R}^m$. For every initial condition $x(0)$ and every input $u(\cdot)$, there is a maximally defined solution $x(\cdot)$ of the system (1), and the corresponding output $y(\cdot)$. Note that whenever the input function u is $k-1$ times continuously differentiable, where k is a positive integer, the derivatives $\dot{y}$, $\ddot{y}$, …, $y^{(k)}$ are well defined.

We will let $\|\cdot\|_{[a,b]}$ denote the essential supremum norm of a signal restricted to an interval $[a,b]$, i.e. $\|z\|_{[a,b]} := \operatorname{ess\,sup}\{|z(s)| : a\le s\le b\}$, where $|\cdot|$ is the standard Euclidean norm. When some vectors $v_1\in\mathbb{R}^{n_1},\dots,v_k\in\mathbb{R}^{n_k}$ are given, we will often use the simplified notation $(v_1;\dots;v_k)$ for the "stack" vector $(v_1^T,\dots,v_k^T)^T\in\mathbb{R}^{n_1+\dots+n_k}$. Given an $\mathbb{R}^l$-valued signal z and a nonnegative integer k, we will denote by $\mathbf{z}^k$ the $\mathbb{R}^{l(k+1)}$-valued signal

$$\mathbf{z}^k := (z_1;\dot{z}_1;\dots;z_1^{(k)};\dots;z_l;\dot{z}_l;\dots;z_l^{(k)})$$

provided that the indicated derivatives exist.

Definition 1. We will call the system (1) *output-input stable* if there exist a positive integer N, a class $\mathcal{KL}$ function[3] β, and a class $\mathcal{K}_\infty$ function γ such that for every initial state $x(0)$ and every $N-1$ times continuously differentiable input u the inequality

$$\left|\begin{pmatrix}u(t)\\ x(t)\end{pmatrix}\right| \le \beta(|x(0)|,t)+\gamma(\|\mathbf{y}^N\|_{[0,t]}) \tag{2}$$

holds for all t in the domain of the corresponding solution of (1). □

The inequality (2) can be interpreted in terms of two separate properties of the system. The first one is that if the output and its derivatives are small, then the input becomes small. Roughly speaking, this means that the system has a stable left inverse in the input-output sense. However, no explicit construction of such a left inverse is necessary. The second property is that if the output and its derivatives are small, then the state becomes small. This signifies that the system is (zero-state) detectable through the output and its derivatives, uniformly with respect to inputs; we will call such systems "weakly uniformly 0-detectable" (see Section 4). Thus output-input stable systems form a subclass of weakly uniformly 0-detectable ones.

[3] Recall that a function $\alpha:[0,\infty)\to[0,\infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0)=0$. If $\alpha\in\mathcal{K}$ is unbounded, then it is said to be of *class* $\mathcal{K}_\infty$. A function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t\ge0$ and $\beta(s,t)$ decreases to 0 as $t\to\infty$ for each fixed $s\ge0$.

Detectability is a state-space concept, whose attractive feature is that it can be characterized by Lyapunov-like dissipation inequalities.

In view of the bound on the magnitude of the state, a more complete name for the property introduced here would perhaps be "(differential) output-to-input-and-state stability", or DOSIS, but we choose to call it output-input stability for simplicity. The conditions imposed by Definition 1 capture intrinsic properties of the system, which are independent of a particular coordinate representation. They are consistent with the intuition provided by the concept of a minimum-phase linear system. In fact, for SISO linear systems the output-input stability property reduces precisely to the classical minimum-phase property, as we now show.

Example 1. Consider a stabilizable and detectable linear SISO system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned} \tag{3}$$

Let r be its relative degree. This means that we have $c^T b = c^T A b = \cdots = c^T A^{r-2} b = 0$ but $c^T A^{r-1} b := g \neq 0$. From the formula $y^{(r)}(t) = c^T A^r x(t) + gu(t)$ we immediately obtain

$$|u(t)| \leq \frac{|(A^r)^T c||x(t)| + |y^{(r)}(t)|}{|g|}. \tag{4}$$

Moreover, it is well known that there exists a linear change of coordinates $x \mapsto (\xi, \eta)$, where $\xi \in \mathbb{R}^r$, $\eta \in \mathbb{R}^{n-r}$, and $\xi_1 = y$, which transforms the system (3) into the normal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_r &= d^T \xi + f^T \eta + gu \\ \dot{\eta} &= P\xi + Q\eta \end{aligned}$$

and (3) is minimum-phase (in the classical sense) if and only if Q is a stable matrix. Stability of Q is equivalent to the existence of positive constants λ and μ such that for all initial states and all inputs u we have $|\eta(t)| \leq e^{-\lambda t}|\eta(0)| + \mu\|\xi\|_{[0,t]}$. Combining the last inequality with (4), one easily arrives at (2) with $N = r$. On the other hand, if (2) holds, then we know that $y \equiv 0$ implies $x \to 0$, and so (3) must be minimum-phase. Incidentally, note that when $N = r$ in (2), the smoothness of u becomes superfluous, because y is automatically r times (a.e.) differentiable for every admissible input u. $\square$

The above remarks suggest that the concepts of relative degree and detectability are related to the output-input stability property. In the subsequent sections, we will explore to what extent the situation described in Example 1 carries over to (possibly MIMO) nonlinear systems.

We conclude this section with another motivating observation, important from the point of view of control design. It is formally expressed by the following proposition.

Proposition 1. Assume that the system (1) is output-input stable. Suppose that we are given a feedback law $u = k(x)$ with the following properties: there exist a system

$$\dot{\xi} = F(\xi), \qquad \xi \in \mathbb{R}^q \tag{5}$$

with a globally asymptotically stable equilibrium at the origin, a map $H : \mathbb{R}^q \to \mathbb{R}^l$ with $H(0) = 0$, and a class $\mathcal{K}_\infty$ function χ such that for every initial condition $x(0)$ for the system

$$\dot{x} = f(x, k(x)) \tag{6}$$

there exists an initial condition $\xi(0)$ for the system (5) with $|\xi(0)| \leq \chi(|x(0)|)$ for which we have $h(x(t)) = H(\xi(t))$ along the corresponding solutions. Then the origin is a globally asymptotically stable equilibrium of the closed-loop system (6).

As an example, consider the affine system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{7}$$

Suppose that this system has a uniform relative degree r (in the sense of Isidori, 1995) and is output-input stable. It may or may not have a global normal form (construction of a global normal form requires additional properties besides relative degree). However, relabeling y as ξ_1 gives

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_r &= b(x) + a(x)u \end{aligned}$$

where $a(x) := L_g L_f^{r-1} h(x)$ and $b(x) := L_f^r h(x)$. We can then apply a state feedback law which brings this to the form $\dot{\xi} = F(\xi)$, where the last system has a globally asymptotically stable equilibrium at the origin. (One possible choice is a linearizing feedback

$$u = -\frac{1}{a(x)}\big(b(x) + p_{r-1}\xi_r + \cdots + p_0\xi_1\big)$$

where the numbers p_i, $i = 0, \ldots, r-1$ are such that polynomial $s^r + p_{r-1}s^{r-1} + \cdots + p_1 s + p_0$ has all its roots in the left half-plane.) In view of Proposition 1, the entire system becomes globally asymptotically stable. Note that this is true for *every* such feedback law, which for minimum-phase systems is generally not the case and special care needs to be taken in designing a stabilizing feedback (Isidori, 1995, Section 9.2).

3. RELATIVE DEGREE

Let us first consider the case when the system (1) is SISO, i.e. when $m = l = 1$. For each $k = 0, 1, \ldots$

define, recursively, the functions $H_k : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ by the formulas $H_0 := h$ and

$$H_{k+1}(x, u) := \frac{\partial H_k}{\partial x} f(x, u_0) + \sum_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j} u_{j+1} \quad (8)$$

where the arguments of H_k are $(x, u_0, \ldots, u_{k-1})$. The significance of the functions H_k lies in the fact that if the input $u(\cdot)$ is in C^{k-1} for some positive integer k, then along each solution $x(\cdot)$ of (1) the corresponding output has a continuous k-th derivative satisfying

$$y^{(k)}(t) = H_k\big(x(t), u(t), \ldots, u^{(k-1)}(t)\big).$$

In particular, suppose that H_k is independent of $u_0, \ldots, u_{k-1}$ for all k less than some positive integer r. Then H_r depends only on x and u_0, as given by

$$H_r(x, u_0) = \frac{\partial H_{r-1}}{\partial x}(x)\, f(x, u_0)\,.$$

As an example, for the affine system (7) we have

$$H_r(x, u_0) = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u_0\,.$$

In this case we see that for every initial condition and every input, $y^{(r-1)}$ exists and is an absolutely continuous function of time, and we have

$$y^{(r)}(t) = H_r(x(t), u(t))$$

for almost all t in the domain of the corresponding solution. The converse is also true, namely, if $y^{(r-1)}$ exists and is absolutely continuous for all initial states and all inputs, then H_k must be independent of $u_0, \ldots, u_{k-1}$ for all $k < r$.

Definition 2. We will say that a positive integer r is the (uniform) *relative degree* of the system (1) if the following two conditions hold:

(1) For each $k < r$, the function H_k is independent of $u_0, \ldots, u_{k-1}$.
(2) There exist two class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that

$$|u_0| \le \rho_1(|x|) + \rho_2\left(|H_r(x, u_0)|\right) \quad (9)$$

for all $x \in \mathbb{R}^n$ and all $u_0 \in \mathbb{R}$. $\square$

If there exists such an integer r, then it is unique, and cannot exceed the dimension of the system. For affine systems, the above definition is consistent with the usual one:

Proposition 2. Consider the affine system (7). A positive integer r is the relative degree of (7) in the sense of Definition 2 if and only if $L_g L_f^k h(x) = 0$ for all x and all integers $k < r-1$, $L_g L_f^{r-1} h(x) \neq 0$ for all x, and $L_f^r h(0) = 0$.

Proposition 2 implies, in particular, that for the SISO affine system (7) with $f(0) = 0$ Definition 2 reduces to the standard definition of uniform relative degree as given, e.g., in (Isidori, 1995). The definition of relative degree proposed here is not restricted to affine systems. As a simple example, the system $\dot{y} = u^2$ has relative degree 1 according to Definition 2. This case is also covered by the definition of relative degree for not necessarily affine systems given in (Nijmeijer and van der Schaft, 1990, p. 417). In general, however, our definition is more restrictive; for example, the system $\dot{y} = \arctan u$ would have relative degree 1 in the context of (Nijmeijer and van der Schaft, 1990), but the bound (9) does not hold.

The above concept extends in a straightforward fashion to the case when the system (1) is MIMO, i.e. when m and l are not necessarily equal to 1. For each $i \in \{1, \ldots, l\}$, let H_0^i be the i-th component of h, and define the functions $H_k^i : \mathbb{R}^n \times (\mathbb{R}^m)^k \to \mathbb{R}$, $k = 1, 2, \ldots$ recursively by the formula (8) with H_k^i, H_{k+1}^i instead of H_k, H_{k+1}. We will say that a set of positive integers $\{r_1, \ldots, r_l\}$ is a (uniform) *relative degree* of the system (1) if the following two conditions hold:

(1) For each $i \in \{1, \ldots, l\}$ and each $k < r_i$, the function H_k^i is independent of $u_0, \ldots, u_{k-1}$.

(2) There exist two class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that

$$|u_0| \le \rho_1(|x|) + \rho_2\left(\left|\left(H_{r_1}^1(x, u_0); \ldots; H_{r_l}^l(x, u_0)\right)\right|\right)$$

for all $x \in \mathbb{R}^n$ and all $u_0 \in \mathbb{R}^m$.

Note that for MIMO systems the relative degree is not necessarily unique (example: $\dot{y}_1 = u_1^2 + u_2^2$, $\dot{y}_2 = y_2$; $\{1, k\}$ is a relative degree for every $k > 0$).

4. DETECTABILITY

Consider a general system of the form

$$\dot{x} = f(x, u).$$

We recall from (Sontag, 1989) that this system is called *input-to-state stable* (ISS) if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ and every input u the corresponding solution satisfies the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t]})$$

for all $t \ge 0$.

Given a system with both inputs and outputs

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \quad (10)$$

we will say that it is *0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such for every $x(0)$ and every u the corresponding solution satisfies the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]})$$

as long as it exists. In particular, a system without inputs given by

$$\begin{aligned}\dot{x} &= f(x)\\ y &= h(x)\end{aligned}$$

will be called *0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ the corresponding solution satisfies the following inequality as long as it exists:

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]}). \tag{11}$$

These concepts were studied in (Sontag and Wang, 1997) under the names of *input/output-to-state stability* and *output-to-state stability*, respectively. In this paper we use the term "0-detectability" as a shorthand.

Let us call the system (10) *uniformly 0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ and every input u the inequality (11) holds along the corresponding solution. As the name suggests, uniform 0-detectability amounts to 0-detectability that is uniform with respect to inputs. This property was called *uniform output-to-state stability* in (Krichman *et al.*, 1999) and *strong detectability* in (Hespanha and Morse, 1999).

When working with output derivatives, it is helpful to introduce an auxiliary system whose output contains derivatives of the output of the original system (restricting admissible inputs if necessary to ensure that these derivatives are well defined). Consider the system (1) with $m = l = 1$ (the SISO case). Take a nonnegative integer k. Restricting the input u to be in C^{k-1}, we can consider the *k-output extension* of (1):

$$\begin{aligned}\dot{x} &= f(x,u)\\ \mathbf{y}^k &= \mathbf{h}_k(x, u, \dots, u^{(k-1)})\end{aligned} \tag{12}$$

where $\mathbf{h}_k(x, u, \dots, u^{(k-1)}) := \big(H_0(x); \dots; H_k(x, u, \dots, u^{(k-1)})\big)$ is the new output map (here we are using the notation of Section 3). That is, we redefine the output of the system to be $\mathbf{y}^k$. Of course, for $k = 0$ we recover the original system. Note that the differentiability assumption on u can be relaxed if the function H_k is known to be independent of $u_0, \dots, u_{k-1}$ for all k between 0 and some positive integer. Let us call the system (1) *weakly 0-detectable of order* k if its k-output extension (12) is 0-detectable. Also, let us call the system (1) *weakly uniformly 0-detectable of order* k if its k-output extension (12) is uniformly 0-detectable. The extension to MIMO systems is straightforward.

5. OUTPUT-INPUT STABILITY

We first study the SISO case, represented by the system (1) with $m = l = 1$. Our main result is the following characterization of output-input stability for SISO systems.

Theorem 1.
(1) Suppose that the system (1) has a relative degree r in the sense of Definition 2 and is weakly uniformly 0-detectable of order k, for some k. Then (1) is output-input stable in the sense of Definition 1, with $N = \max\{r, k\}$.
(2) Suppose that the system (1) is output-input stable in the sense of Definition 1. Then it is weakly uniformly 0-detectable of order N.
(3) Suppose that the system (1) is output-input stable in the sense of Definition 1, that the function $f(x, \cdot)$ is real analytic in u for each fixed x, and that $f(0,0) = 0$ and $h(0) = 0$. Then (1) has a relative degree $r \le N$ in the sense of Definition 2.

The theorem implies that for systems with relative degree, output-input stability is equivalent to weak uniform 0-detectability of order k for some $k \le N$, and that for systems satisfying the additional assumptions on f and h stated in part (3), output-input stability is equivalent to the existence of a relative degree $r \le N$ plus weak uniform 0-detectability of order k for some $k \le N$.

Let us call the system (1) *strongly minimum-phase* if it has a relative degree r and is weakly uniformly 0-detectable of order $r - 1$. Note that $\mathbf{y}^{r-1}$ is a function of the state x only: $\mathbf{y}^{r-1} = \mathbf{h}_{r-1}(x)$; no differentiability is required from u. By part (1) of Theorem 1, every strongly minimum-phase system is output-input stable.

Example 2. Consider an affine system in global normal form

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ &\dots\\ \dot{\xi}_r &= b(\xi,\eta) + a(\xi,\eta)u\\ \dot{\eta} &= q(\xi,\eta)\\ y &= \xi_1\end{aligned} \tag{13}$$

where $\xi := (\xi_1; \dots; \xi_r)$, $b(0,0) = 0$, and $a(\xi,\eta) \ne 0 \ \forall \xi, \eta$ (so that r is the uniform relative degree by Proposition 2). This system is usually called minimum-phase if the *zero dynamics* $\dot{\eta} = q(0,\eta)$ have an asymptotically stable equilibrium at $\eta = 0$ (Byrnes and Isidori, 1988). Since $\mathbf{y}^{r-1} = \xi$, the strong minimum-phase property in this case demands that the equation for η in (13), which represents the internal dynamics, be input-to-state stable (ISS) with respect to ξ. □

The results of (Krichman *et al.*, 1999; Sontag and Wang, 1997) imply that the system (1) is weakly uniformly 0-detectable of order $r-1$ if there exists a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$\nabla V(x) f(x,u) \leq -\alpha(|x|) + \chi(|\mathbf{y}^{r-1}|) \quad \forall x, u \quad (14)$$

for some functions $\alpha, \chi \in \mathcal{K}_\infty$. This Lyapunov-like dissipation inequality can be used to check the strong minimum-phase property, once the relative degree of the system is known. In fact, the inequality (14) provides a necessary and sufficient condition for weak uniform 0-detectability of order $r-1$ if controls take values in a compact set (Krichman *et al.*, 1999). But this condition is only sufficient if the control set is unbounded, as seen from the simple example $\dot{x} = u$, $y = x$.

Example 3. We now give an example of an output-input stable system that is not strongly minimum-phase. Consider the system

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_2 + u^2 \\ y &= x_1 \end{aligned} \qquad (15)$$

It has relative degree 1. From the equation $\dot{x}_2 = -x_2 + \dot{y}^2$, which is ISS with respect to $\dot{y}$, we deduce that the system (15) is output-input stable (with $N = 1$). Now let us show that this system is not strongly minimum-phase. It is enough to find a solution trajectory along which x_1 converges to zero while x_2 does not converge to zero. Take the initial state to be 0, and apply the following input: $u(t) \equiv 1$ for $0 \leq t < 1$, $u(t) \equiv -1$ for $1 \leq t < 2$, $u(t) \equiv 1$ for $2 \leq t < 2\frac{1}{2}$, $u(t) \equiv -1$ for $2\frac{1}{2} \leq t < 3$, $u(t) \equiv 1$ for $3 \leq t < 3\frac{1}{4}$, and so on. Then $x_1 \to 0$, but $\dot{x}_2 = -x_2 + 1$ so $x_2 \to 1$. □

For MIMO systems, the existence of a relative degree is quite a restrictive assumption. Fortunately, Definition 1 does not have the shortcoming of applying only to systems with relative degree, as illustrated by the following example.

Example 4. The system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 + u_1^2 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= -x_4 + x_1^3 \\ y &= (x_1; x_2) \end{aligned}$$

does not have a relative degree. This system is output-input stable, as can be seen from the formulas $|u_2| = |\ddot{y}_2 - 2\dot{y}_1\ddot{y}_1| \leq |\ddot{y}_2| + \dot{y}_1^2 + \ddot{y}_1^2$, $|x_3| \leq |\dot{y}_2| + \dot{y}_1^2$, and the fact that the equation for x_4 is ISS with respect to x_1. □

The above example is to be contrasted with:

Example 5. The system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 + x_2 u_2 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= -x_4 + x_1^3 \\ y &= (x_1; x_2) \end{aligned}$$

does not have a relative degree. The zero dynamics $\dot{x}_4 = -x_4$ are asymptotically stable. However, it can be shown that arbitrarily large u_2 can lead to arbitrarily small x, y, and derivatives of y. Therefore, this system is not output-input stable. □

An important feature of Definition 1 is that for linear MIMO systems it is equivalent to the classical definition of the minimum-phase property:

Theorem 2. A left-invertible linear control system is output-input stable if and only if all its transmission zeros have negative real parts.

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