Input-to-State Stabilization of Linear Systems With Quantized State Measurements

Daniel Liberzon, *Senior Member, IEEE*, and Dragan Nešić, *Senior Member, IEEE*

***Abstract*—We consider the problem of achieving input-to-state stability (ISS) with respect to external disturbances for control systems with linear dynamics and quantized state measurements. Quantizers considered in this paper take finitely many values and have an adjustable "zoom" parameter. Building on an approach applied previously to systems with no disturbances, we develop a control methodology that counteracts an unknown disturbance by switching repeatedly between "zooming out" and "zooming in." Two specific control strategies that yield ISS are presented. The first one is implemented in continuous time and analyzed with the help of a Lyapunov function, similarly to earlier work. The second strategy incorporates time sampling, and its analysis is novel in that it is completely trajectory-based and utilizes a cascade structure of the closed-loop hybrid system. We discover that in the presence of disturbances, time-sampling implementation requires an additional modification which has not been considered in previous work.**

***Index Terms*—Disturbances, hybrid control, input-to-state stability (ISS), quantized feedback.**

I. Introduction

THE subject of this paper is feedback control of linear continuous-time systems with quantized state measurements. Control problems of this kind are motivated by numerous applications where communication between the plant and the controller is limited due to capacity or security constraints. This is a very active and expanding research area; see, e.g., [1]–[3], [5], [6], [8], [9], [13], [16], [18], and [19].

The starting point for this paper is the approach developed in [1], [8] (see also [10, Ch. 5]), which we now briefly recall. The quantizer is assumed to take a finite set of values and incorporates an adjustable "zoom" parameter. The control strategy is composed of two stages. The first, "zooming-out" stage consists in increasing the range of the quantizer until the state of the system can be adequately measured; at this stage, the system is open-loop. The second, "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. This results in a hybrid control law, in which the zoom parameter is a discrete variable whose transitions are triggered by the values of a suitable Lyapunov function.

The method of [1], [8] was shown to achieve global asymptotic stability (GAS). The focus of the present work is on achieving robustness with respect to disturbances. We characterize the desired robustness by an ISS-like property (see [17]) which involves bounded nonlinear gains from the initial state and the supremum norm of the disturbance to the supremum norm of the state and also from the supremum limit of the disturbance to the supremum limit of the state. The contributions cited earlier only deal with stability in the absence of external signals, with the notable exceptions of [5], [13], and [18]. In [5] and [18], state boundedness in the presence of bounded disturbances is achieved by using the knowledge of a disturbance bound. In [13], mean square stability in the stochastic setting is obtained by utilizing statistical information about the disturbance (a bound on its appropriate moment). In contrast to these works, here we assume the disturbance to be completely unknown to the controller.

Our first main result (Theorem 1 in Section II) is that the ISS property in the presence of disturbances can be achieved by extending the method of [1], [8]. An extension is necessary because an unknown disturbance may force the state outside the range of the quantizer after it has already been inside. Thus we develop a control strategy that switches multiple times between the zooming-out and zooming-in stages. This strategy is still Lyapunov-based, and its analysis is similar in spirit to that from [8] but is significantly more difficult. When no disturbances are present, the earlier stabilization result is recovered from our new result as a special case.

Next, we turn to the problem of achieving the same robustness property using sampled-data quantized feedback. Time-sampling implementation is important because it guarantees a finite data rate (cf. [6]) and also because it exposes the issue of robustness with respect to time delays. We demonstrate that unless proper care is taken, the straightforward sampled-data adaptation of the continuous-time control strategy in general fails to provide ISS, although it does stabilize the system in the absence of disturbances (see Sections III-A and III-B). We then proceed to describe a modified version of the zooming-out procedure which yields ISS in the time-sampling context, obtaining our second main result (Theorem 2 in Section III-C).

We give a proof of Theorem 2 which sharply differs from that of Theorem 1 in that it does not use a Lyapunov function and instead is based entirely on trajectory analysis. Thus another principal contribution of this work is a novel alternative method for analyzing stability and robustness of quantized feedback control schemes (which can be applied in continuous time as well).

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D. Liberzon is with the Coordinated Science Laboratory, the University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: liberzon@uiuc.edu).

D. Nešić is with the Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, 3052 VIC, Australia (e-mail: d.nesic@ee.unimelb.edu.au).

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In particular, an important component of this time-based analysis consists in recognizing and utilizing a cascade structure of the hybrid closed-loop system. (This can be viewed as a special instance of the general small-gain approach to stability analysis of hybrid systems proposed in [11] and [14].)

In an effort to make the paper accessible to a variety of readers, we organize it in a top-down fashion.[1] The main results are first presented in Sections II and III without proofs; discussions and examples are also provided there to compare and illustrate the results. The basic steps of their proofs are then described in Section IV, and the remaining more technical proofs are collected in the Appendix. Section V offers conclusions and remarks on extending the results to nonlinear dynamics.

II. Lyapunov-Based Continuous-Time Approach

We consider the linear system

$$\dot{x} = Ax + Bu + Dd \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d \in \mathbb{R}^s$ is an unknown disturbance (u and d are taken to be Lebesgue measurable and locally bounded). We assume that A is a nonzero, non-Hurwitz matrix, and that the system (1) is stabilizable, so there exist matrices K and $P = P^T > 0$ such that $A + BK$ is Hurwitz and

$$(A + BK)^T P + P(A + BK) \leq -2I. \tag{2}$$

Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm, and $\|\cdot\|_J$ denotes the supremum norm of a signal on an interval J; sometimes we will omit the subscript J if it is the entire domain of the signal. For $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer z such that $z \geq x$. We use the notation $(x, y) := (x^T y^T)^T$ for arbitrary vectors x, y. A continuous function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}_\infty$ ($\varphi \in \mathcal{K}_\infty$) if it is zero at zero, strictly increasing, and unbounded.

A *quantizer* is a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. As in [8], we assume that there exist real numbers $M > \Delta > 0$ such that the following two conditions hold:

$$|z| \leq M \quad \Rightarrow \quad |q(z) - z| \leq \Delta \tag{3}$$

and

$$|z| > M \quad \Rightarrow \quad |q(z)| > M - \Delta. \tag{4}$$

The first condition gives a bound on the quantization error when the quantizer does not saturate, while the second one provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* and the *error bound* of the quantizer, respectively.

[1]We thank the anonymous referees for suggesting this structure.

We also assume that $q(z) = 0$ on some neighborhood of the origin. This assumption can be stated as follows.

Assumption II.1: There exists a number $\Delta_0 > 0$ such that for all $|z| \leq \Delta_0$ we have $q(z) = 0$.

We will be using the one-parameter family of quantizers

$$q_\mu(x) := \mu q\left(\frac{x}{\mu}\right), \qquad \mu > 0. \tag{5}$$

Here μ is an adjustable parameter, which can be viewed as a "zoom" variable. At each time t, the quantized measurement $q_{\mu(t)}(x(t))$ will represent the information about $x(t)$ that is communicated to the controller. For each fixed μ, the range of the quantizer q_μ is $M\mu$ and the error bound is $\Delta\mu$. Geometrically, at every given time $\mathbb{R}^n$ is divided into a finite number of quantization regions (each corresponding to a fixed value of q_μ) and the controller knows which of these regions contains the state x. The variable μ will be varied in a discrete fashion in order to extract more information about the state. This adjustment policy for μ, or "zooming protocol," will depend only on the quantized measurements of the state; it can be thought of as being implemented synchronously on both ends of the communication channel, starting from some known initial value μ_0. We refer the reader to [1], [8], and [10] for further motivation and discussion.

The problem of interest is to design a quantized feedback control law and a scheme for updating μ to achieve the following property: there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial condition $x_0 = x(t_0)$ and every bounded disturbance d we have

$$|x(t)| \leq \gamma_1(|x_0|) + \gamma_2\left(\|d\|_{[t_0,\infty)}\right) \qquad \forall t \geq t_0 \tag{6}$$

and

$$\limsup_{t\to\infty} |x(t)| \leq \gamma_3\left(\limsup_{t\to\infty} |d(t)|\right). \tag{7}$$

We note that the gain functions $\gamma_1, \gamma_2, \gamma_3$ may depend on the choice of the initial value $\mu_0 = \mu(t_0)$ of the zoom variable μ, but do not depend on x_0 or d. Actually, γ_3 will not depend on μ_0, as we will see from the formula (48) in Section IV-A. Since the closed-loop dynamics will not explicitly depend on time t, all bounds will also be uniform with respect to the initial time t_0.

We know that for continuous systems of the form $\dot{x} = f(x, d)$, the property expressed by the two inequalities (6) and (7) is equivalent to *input-to-state stability* (ISS) with respect to d [17]. In the present case, the closed-loop system will be a hybrid system, because it will contain an additional discrete state μ. With some abuse of terminology, we will refer to the previous property as ISS of the continuous closed-loop dynamics.

This ISS property also implies that in the disturbance-free case ($d \equiv 0$), the origin is a GAS equilibrium of the continuous closed-loop dynamics (for a fixed μ_0). Thus we recover as a special case the property achieved by the algorithms developed in [1], [8] for the case of no disturbances. In fact, the algorithm presented next is a natural extension of the ones from [1], [8].

As we said, the overall closed-loop system will be hybrid: it will contain continuous states (states taking values in a continuum) and discrete states (states taking values in a discrete set). Both continuous and discrete states will be functions of the continuous time $t \in [t_0, \infty)$. The continuous variables will be comprised of the system state x and two auxiliary reset clock variables τ_{out} and τ_{in}, both initialized at 0. They will take values in the intervals $[0, T_{\text{out}}]$ and $[0, T_{\text{in}}]$, respectively, where $T_{\text{out}} \geq T_{\text{in}} > 0$. The discrete variables will be comprised of the zoom variable μ and an auxiliary logical variable `captured`. The variable μ will be initialized at some $\mu_0 > 0$ and will take values in a discrete subset of $(0, \infty)$ which depends on μ_0. The variable `captured` will take values in the set {"yes", "no"} and will be initialized at "no;" it is needed to distinguish between the "capturing" (open-loop) stage and the control (closed-loop) stage. The control law is defined by

$$u(t) = \begin{cases} 0 & \text{if } \texttt{captured} = \text{"no"} \\ Kq_{\mu(t)}(x(t)) & \text{if } \texttt{captured} = \text{"yes"} \end{cases} \tag{8}$$

The state dynamics describing the evolution of the system variables with respect to time are composed of *continuous evolution* and *discrete events*. During continuous evolution (i.e., while no discrete events occur), μ is held constant, x satisfies (1) with u defined by (8), and the clock variables satisfy

$$\dot{\tau}_{\text{in}} = \begin{cases} 1 & \text{if } \tau_{\text{in}} < T_{\text{in}} \\ 0 & \text{if } \tau_{\text{in}} = T_{\text{in}} \end{cases} \quad \dot{\tau}_{\text{out}} = \begin{cases} 1 & \text{if } \tau_{\text{out}} < T_{\text{out}} \\ 0 & \text{if } \tau_{\text{out}} = T_{\text{out}} \end{cases} \tag{9}$$

We now describe the discrete events. Given an arbitrary time t, we will denote by $\mu^-(t)$, or simply by μ^- when the time arguments are omitted, the quantity $\lim_{s \nearrow t} \mu(s)$, and similarly for all other variables. All system variables will be continuous from the right by construction (and of course x is continuous). Let numbers $\Omega_{\text{out}} > 1$, $\Omega_{\text{in}} \in (0,1)$, $T_{\text{c}} \in (0, T_{\text{out}}/2)$, and $\ell_{\text{out}} > \ell_{\text{in}} > 0$ be given. The discrete events are of three types. They are governed by the following rules, which we write in the form "if {conditions} then {actions}." The conditions are mutually exclusive and are assumed to be checked continuously in time. Variables for which no actions are specified remain constant during the corresponding events.

Zoom-out: If

$$\begin{aligned} &\left(\tau_{\text{out}}^- = T_{\text{out}} \text{ and } \texttt{captured}^- = \text{"no"}\right) \text{ or} \\ &\left(\left|q_{\mu^-}(x)\right| \geq \ell_{\text{out}}\mu^- \text{ and } \texttt{captured}^- = \text{"yes"}\right) \end{aligned} \tag{10}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and $\tau_{\text{out}} = 0$.

Capture: If

$$\begin{aligned} &\left|q_{\mu^-}(x)\right| \leq \ell_{\text{out}}\mu^- \text{ and } \tau_{\text{out}}^- \in [T_{\text{c}}, T_{\text{out}} - T_{\text{c}}] \\ &\text{and } \texttt{captured}^- = \text{"no"} \end{aligned} \tag{11}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and `captured` = "yes."

Zoom-in: If

$$\begin{aligned} &\left|q_{\mu^-}(x)\right| \leq \ell_{\text{in}}\mu^- \text{ and } \min\left\{\tau_{\text{out}}^-, \tau_{\text{in}}^-\right\} \geq T_{\text{in}} \\ &\text{and } \texttt{captured}^- = \text{"yes"} \end{aligned} \tag{12}$$

then let $\mu = \Omega_{\text{in}}\mu^-$ and $\tau_{\text{in}} = 0$.

Noting the saturation in (9) and recalling that $T_{\text{out}} \geq T_{\text{in}}$, the functioning of the clocks can be understood as follows. While `captured` = "no," we wait at least T_{out} units of time after a zoom-out before executing another zoom-out. Moreover, we wait at least T_{in} units of time after the last zoom-in or zoom-out before executing another zoom-in. For convenience, the clock τ_{out} is also used to ensure that the capture event is separated in time from the zoom-outs.

For each fixed value of μ, chattering on the boundaries between the quantization regions may occur, and solutions are to be interpreted in the sense of [4] (this issue does not affect the Lyapunov-based analysis that follows). Solutions of the overall hybrid system are defined as usual, from one discrete event to the next. The only potential issue is the possibility of infinitely many zoom-in/out events in finite time (Zeno behavior), which in principle can happen since a minimal time between zoom-outs is not enforced while `captured` = "yes". However, such behavior is ruled out by Theorem 1 given next, which guarantees that μ remains bounded for all time when the disturbance is bounded. Indeed, first note that the variable `captured` cannot change its value more than once and hence can be ignored. Now suppose that on a finite interval $[t_0, t_1]$ we have Zeno behavior. Since d is locally bounded, it is bounded on $[t_0, t_1]$. By causality, $\mu(t_1)$ would not change if we reset $d(t)$ for $t > t_1$, say, to 0. Therefore, $\mu(t_1)$ must be bounded. We have $\mu(t_1) = \Omega_{\text{in}}^{k_1}\Omega_{\text{out}}^{k_2}\mu(t_0)$, where k_1 and k_2 denote the (possibly infinite) number of zoom-ins and zoom-outs on the interval $[t_0, t_1]$, respectively. Our algorithm enforces that k_1 is finite, since $[t_0, t_1]$ is bounded and each zoom-in is preceded by a time interval of length at least T_{in}. Hence, only k_2 can be infinite. But this would contradict the boundedness of $\mu(t_1)$ since $\Omega_{\text{out}} > 1$ and $\mu(t_0) > 0$.

We now state the main result of this section; see Section IV-A for the proof.

Theorem 1: Consider the system (1). Pick matrices K and $P = P^T > 0$ satisfying (2). Let q be a quantizer fulfilling Assumption II.1 and the conditions (3) and (4), where M and Δ satisfy

$$M > \left(2 + 2\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|PBK\|\right)\Delta. \tag{13}$$

Let Ω_{in}, Ω_{out}, T_{in}, T_{out}, T_{c} be positive numbers satisfying the inequalities $\Omega_{\text{in}} < 1$, $T_{\text{out}} \geq T_{\text{in}}$, $T_{\text{c}} < T_{\text{out}}/2$,

$$\begin{aligned} &\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - 2\Delta \\ &\quad > \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|PBK\|\Delta \end{aligned} \tag{14}$$

$$\Omega_{\text{out}} > \frac{\sqrt{\lambda_{\max}(P)}M}{\sqrt{\lambda_{\min}(P)}(M - 2\Delta)} \tag{15}$$

$$T_{\text{out}} < \log\Omega_{\text{out}}/\|A\| \tag{16}$$

($T_{\text{in}} > 0$ is arbitrary). Define

$$\ell_{\text{out}} := M - \Delta \quad \ell_{\text{in}} := \Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - \Delta. \tag{17}$$

Let the control be defined by (8) and let the evolution of μ be as described earlier, with an arbitrary fixed initial condition $\mu_0 = \mu(t_0) > 0$. Then there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial state $x_0 = x(t_0)$ and every bounded disturbance d the closed-loop system has the properties that μ

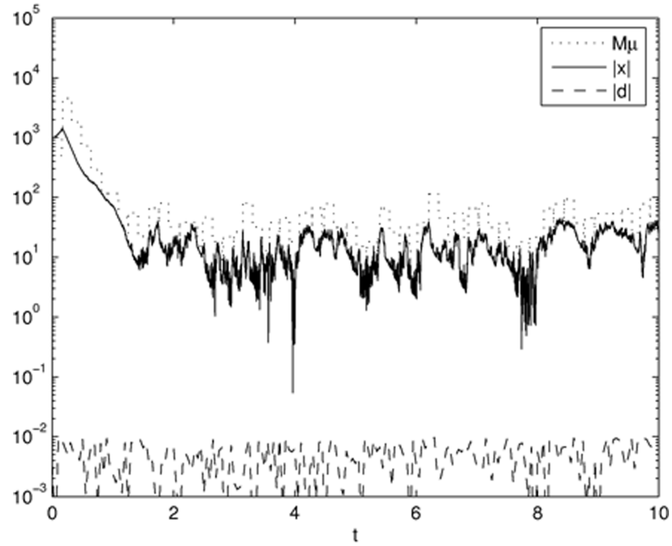

Fig. 1. Simulation results for Example II.2.

remains bounded and the continuous dynamics are ISS in the sense of satisfying (6) and (7).

It is straightforward to verify that the inequality (13) ensures the existence of all subsequently defined quantities in the theorem statement. The intuitive meaning of this inequality is that the quantizer takes sufficiently many values so that its range M is large enough compared to the error bound Δ.

As a corollary of Theorem 1, we have that if $d \equiv 0$ then the continuous closed-loop dynamics are GAS. In fact, the rate of convergence of $x(t)$ to 0 is exponential. This can be deduced from the proof of the theorem, but also follows from the fact that the convergence provided by the present event-based strategy is no slower than the one obtained in [8] via dwell-time switching.

Example II.2: To illustrate Theorem 1, we simulated the previous control algorithm with system parameters

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad D = \begin{pmatrix} -1.2 \\ 2 \end{pmatrix} \cdot 10^5$$

uniform quantizer with $M = 50$ and $\Delta = \sqrt{2}/10$, and control design parameters $K = (-1.1 \ \ -2.6)$, $P = I$, $\Omega_{\rm in} = 0.4$, $\Omega_{\rm out} = 3$, $T_{\rm in} = T_{\rm out} = 0.16$, $T_{\rm c} = 0.01$. The behavior (on a log scale) of the state $x(t)$ and the quantizer's range $M\mu(t)$ for the initial conditions $x(0) = \binom{1000}{0}$ and $\mu(0) = 10$ in response to random disturbance $d(t)$ uniformly distributed on the interval $[0, 0.01]$ is shown in Fig. 1. Large values for D and $x(0)$ were chosen simply to achieve separation between the different plots in the figure. As expected, after an initial overshoot the state settles below a bound which, actually, is several orders of magnitude lower than that provided by the formulas (7) given earlier and (48) given in the proof of Theorem 1. □

It is important to note that in the control strategy just described, a zoom-out is triggered immediately whenever the last two conditions in (10) are true. This property is crucial for Theorem 1 to hold (it enables the last claim of Lemma IV.1 in Section IV). In Fig. 1, we indeed see rapid changes of μ in response to rapidly varying d. This aspect of the present scheme makes it sensitive to time delays and renders it not implementable in the sampled-data framework. Thus the issue of designing a suitable zooming-out procedure will be central as we turn to the time-sampling scenario in the next section.

III. Trajectory-Based Sampled-Data Approach

In this section, we introduce a new sampled-data stabilization scheme, which can be regarded as an alternative to the scheme from the previous section. We first discuss the simpler disturbance-free case to illustrate the new technique. Then, we study an example of a controller and zooming protocol that do not have robustness in an ISS sense. Finally, we present a result on ISS of the closed-loop system with respect to disturbances with a modified zooming protocol.

A. Disturbance-Free Case

We consider again the continuous-time linear system (1) with A a nonzero, non-Hurwitz matrix. In this subsection, we assume that $d(\cdot) \equiv 0$. We will control this system with quantized hybrid feedback that is defined next. Let $T > 0$ be a given sampling period and let $t_k := kT$ for $k \in \mathbb{N}$. We define $x_k := x(t_k)$, and similarly for other variables. Our closed-loop dynamics will consist of the plant, controller, and zooming protocol described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(0) = x_0 \in \mathbb{R}^n \tag{18}$$
$$u(t) = U(\Omega_k, \mu_k, x_k) \quad t \in [t_k, t_{k+1}) \tag{19}$$
$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k) \quad \mu_0 \in \mathbb{R}_{>0} \tag{20}$$
$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k) \quad \Omega_{-1} = \Omega_{\rm out}. \tag{21}$$

Let $\ell_{\rm out} > \ell_{\rm in}$ be strictly positive numbers to be defined. To simplify the notation, we introduce $q_k := q_{\mu_k}(x_k)$ for arbitrary $k \in \mathbb{N}$, where $q_\mu(\cdot)$ is the one-parameter family of quantizers defined in (5) and satisfying (3) and (4). The variable Ω determines the switching rules for the controller and the zooming protocol. This variable can only take two values $\Omega_{\rm out}$ and $\Omega_{\rm in}$, with the initial value $\Omega_{-1} = \Omega_{\rm out}$. Then, we define the following hysteresis control law and zooming protocol:

$$U(\Omega_k, \mu_k, x_k) := \begin{cases} 0 & \text{if } \Omega_k = \Omega_{\rm out} \\ Kq_k & \text{if } \Omega_k = \Omega_{\rm in} \end{cases} \tag{22}$$
$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\rm out}\mu_k & \text{if } \Omega_k = \Omega_{\rm out} \\ \Omega_{\rm in}\mu_k & \text{if } \Omega_k = \Omega_{\rm in} \end{cases} \tag{23}$$
$$H(\Omega_{k-1}, \mu_k, x_k) := \begin{cases} \Omega_{\rm out} & \text{if } |q_k| > \ell_{\rm out}\mu_k \\ \Omega_{\rm in} & \text{if } |q_k| < \ell_{\rm in}\mu_k \\ \Omega_{k-1} & \text{if } |q_k| \in [\ell_{\rm in}\mu_k, \ell_{\rm out}\mu_k] \end{cases} \tag{24}$$

where $\Omega_{\rm in}$ and $\Omega_{\rm out}$ are strictly positive constants to be defined.

Remark III.1: The control law and protocol (22), (23), (24) are novel in that hysteresis switching is used to switch between the zooming-in and zooming-out stages, and there is a notable difference compared to the control law and protocol of the previous section. For instance, the controller (8) runs in the open-loop mode (i.e., $u(t) = 0$) only during the first zoom-out interval until the state is "captured" and then it behaves as the certainty equivalence controller of the form $u(t) = Kq_{\mu(t)}(x(t))$ for all future time. On the other hand, the controller (22) switches to the open-loop mode (i.e., $u_k = 0$) whenever the quantized measurement is saturated and remains zero until we have $\Omega_k = \Omega_{\text{in}}$. This yields a cascade structure of the closed-loop system during the zooming-in stage, which greatly simplifies the proofs (see the formula (49) in Section IV-B). One may expect that the controller (22) might yield larger overshoots than the controller (8) since it runs more in the open-loop mode. Indeed, this will be confirmed later by simulations.

We emphasize that it is not necessary to use (22), (23), (24) in this section to prove our results, and this choice is used for convenience. Indeed, instead of (22), one could use a sampled-data version of the control law from the previous section and our results could still be proved. We do not pursue this option for space reasons; in fact, this difference serves to illustrate the flexibility in the design that the two approaches offer. □

We introduce some notation. Note that for each $k \geq 0$ we have $\Omega_k = \Omega_{\text{out}}$ or $\Omega_k = \Omega_{\text{in}}$. In the former case, we say that the zoom-out condition is triggered at time k and in the latter case we say that the zoom-in condition is triggered at time k. Given an initial condition (and a disturbance), there is a sequence of intervals on which we zoom in or out, i.e., we can introduce $k_j \in \mathbb{N}$ such that

$$\begin{aligned} \Omega_k &= \Omega_{\text{out}}, \quad \text{if } k \in [k_{2i}, k_{2i+1} - 1] \\ \Omega_k &= \Omega_{\text{in}}, \quad \text{if } k \in \left[k_{2i+1}, k_{2(i+1)} - 1\right] \end{aligned}$$

where $i = 0, 1, \ldots, N$, with either finite $N \in \mathbb{N}$ or $N = \infty$ (we may have either infinitely many zoom-in/out switchings or finitely many). For notational purposes we will always let $k_0 = 0$ and if we actually have that the zoom-in condition is triggered at $k_0 = k = 0$, then we let $k_1 = k_0$ and we have that the first zoom-out interval is $[k_0, k_1 - 1] = [0, -1] = \emptyset$. In this way, all the proofs will start with a zoom-out interval knowing that this interval may actually be empty. This convention simplifies the presentation.

The dynamics described earlier induce the following discrete-time system, which is more amenable to analysis

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) \quad x_0 \in \mathbb{R}^n \\ \mu_{k+1} &= G(\Omega_k, \mu_k, x_k) \quad \mu_0 \in \mathbb{R}_{>0} \\ \Omega_k &= H(\Omega_{k-1}, \mu_k, x_k) \quad \Omega_{-1} = \Omega_{\text{out}} \end{aligned}$$

where

$$\Phi := e^{AT} \quad \Gamma := \int_0^T e^{As} B ds. \tag{25}$$

Note that the switching between the zooming-in and zooming-out stages is determined by the variable

$$\xi_k := \frac{x_k}{\mu_k}. \tag{26}$$

Hence, the dynamical equations that describe how ξ_k changes are important for understanding the operation of the system. For instance, during the zooming-out stage we have for all $k \in [k_{2i}, k_{2i+1} - 1]$ that

$$\xi_{k+1} = \frac{\Phi}{\Omega_{\text{out}}} \xi_k. \tag{27}$$

During the zooming-in stage we have for all $k \in [k_{2i+1}, k_{2(i+1)} - 1]$ that

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}} (\Phi + \Gamma K) \xi_k + \frac{1}{\Omega_{\text{in}}} \Gamma K \nu_k \tag{28}$$

where $\nu_k := q(\xi_k) - \xi_k$. We can state the following two standard results whose proofs are omitted. The first result follows directly from [7, Ex. 3.4].

Lemma III.2: Suppose that $\Phi + \Gamma K$ is Schur. Then, there exists an $\Omega_{\text{in}}^* \in (0, 1)$ such that for all $\Omega_{\text{in}} \in [\Omega_{\text{in}}^*, 1)$,

$$\frac{1}{\Omega_{\text{in}}} (\Phi + \Gamma K) \tag{29}$$

is Schur. Moreover, for any such Ω_{in}, there exist strictly positive L_1, λ_1, γ_1 such that the solutions of the system (28) satisfy the following:

$$|\xi_k| \leq L_1 \exp(-\lambda_1 k) |\xi_0| + \gamma_1 \|\nu\| \qquad \forall k \geq 0.$$

In particular, let $\kappa > 0$ and $\sigma \in (0, 1)$ be such that[2] $\|(1/\Omega_{\text{in}}^k)(\Phi + \Gamma K)^k\| \leq \kappa \sigma^k$ for all $k \geq 0$. Then, we can let

$$L_1 = \kappa \quad \lambda_1 = -\ln(\sigma) \quad \gamma_1 = \frac{\kappa \|\Gamma K\|}{\Omega_{\text{in}}(1 - \sigma)}. \tag{30}$$

Corollary III.3: Let $\Omega_{\text{in}}, L_1, \gamma_1$ come from Lemma III.2 and let strictly positive M and Δ be such that the following holds:

$$M > (2 + L_1 + \gamma_1) \Delta. \tag{31}$$

Then, there exists a $\Delta_M > 0$, with $\Delta_M - \Delta > 0$, such that whenever $|\xi_0| \leq \Delta_M$ and $\|\nu\| \leq \Delta$, we have

$$|q_{\mu_k}(x_k)| \leq (M - \Delta) \mu_k \text{ and } |\xi_k| \leq M \qquad \forall k \geq 0. \tag{32}$$

We have the following result for the disturbance-free case; a sketch of its proof is given in Section IV-B. It can be viewed as a sampled-data counterpart of the stabilization results from [1], [8]. This is a corollary of a more general result (Theorem 2) for the disturbance case that will be presented in Section III-C.

Proposition III.4: Consider the system (18) and let q be a quantizer fulfilling Assumption II.1 and the conditions (3) and (4). Suppose that for the given $T > 0$ the pair (Φ, Γ) is stabilizable. Let K be such that $(\Phi + \Gamma K)$ is Schur. Let Ω_{in} be such that (29) is Schur and let $\Omega_{\text{out}} > \|\Phi\|$. Let the range M of q be sufficiently larger than the error bound Δ so that Corollary

[2]These numbers always exist since $(1/\Omega_{\text{in}})(\Phi + \Gamma K)$ is Schur.

III.3 holds with M, Δ and some Δ_M. Define $\ell_{\rm out} := M - \Delta$ and $\ell_{\rm in} := \Delta_M - \Delta$. Then, μ_k is bounded for all $k \geq 0$ and the system (18), (19), (20), (21), (22), (23), (24) is globally asymptotically stable. More precisely, there exists a $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is of class $\mathcal{K}_\infty$ in its first argument for any fixed value of its second argument and such that for all $x_0 \in \mathbb{R}^n$ and any μ_0 we have

$$|x_k| \leq \varphi(|x_0|, \mu_0) \qquad \forall k \geq 0 \tag{33}$$

and $|x_k| \to 0$ as $k \to \infty$, exponentially fast.

Remark III.5: It is not hard to show that the stability bound valid only at the sampling instants t_k, which is provided by Proposition III.4, can be extended to all $t \geq 0$. The same is true for our ISS results in Section III-C. For similar results, see [15]. □

Corollary III.3 has an appropriate interpretation via Lyapunov functions, which links the results of this section with those of the previous section (whose proofs are Lyapunov-based). Indeed, since we assume that $(1/\Omega_{\rm in})(\Phi + \Gamma K)$ is Schur, there exists a quadratic Lyapunov function $V(\xi) := \xi^T P \xi$ such that for some $a > 0$ the solutions of the system (28) satisfy

$$|\xi_k| \geq a|\nu_k| \quad \Rightarrow \quad V(\xi_{k+1}) < V(\xi_k).$$

Suppose that Δ is given. Then, one possible choice of M, Δ_M, Δ is given by

$$\Delta_M > \max\{1, a\}\Delta \tag{34}$$

and

$$M - 2\Delta > \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\Delta_M. \tag{35}$$

A geometrical interpretation of (35) is that the smallest level set of V containing the ball of radius Δ_M is inside the largest level set of V contained in the ball of radius $M - 2\Delta$. If (34) holds, then V decreases for ξ in the annulus between these two level sets as long as $\|\nu\|$ is smaller than Δ. Hence for $\nu_k = q(\xi_k) - \xi_k$ the conditions (32) are satisfied because ξ_k stays within the range of q.

Lemma III.2 imposes a lower bound on $\Omega_{\rm in}$, while the inequalities (34) and (35) basically say that M should be large enough compared to Δ. In this sense, these conditions are similar to the conditions (13) and (14) from Section II. However, it is important to note the following difference. In Theorem 1, the inequality (13) involves only the system and quantizer parameters, and the subsequent conditions impose constraints on the controller parameters. In Proposition III.4, on the other hand, the controller parameters $\Omega_{\rm in}$ and $\Omega_{\rm out}$ are selected on the basis of the system parameters only, and the choice of $\Omega_{\rm in}$ affects the quantizer parameters.

While it can be shown that for any fixed $\mu > 0$ we can take $\varphi(\cdot, \mu)$ to be of class $\mathcal{K}_\infty$, we have at the same time that for any fixed $s > 0$ the following holds: $\lim_{\mu \to 0} \varphi(s, \mu) = \infty$. Hence, the overshoot of the x-subsystem is non-uniform in small μ_0. While it is true that initializing the system at a particular μ_0 gives a constant overshoot for the x variable and one can prove stability of the x-subsystem, the lack of uniformity of the overshoot leads to an inherent lack of robustness of this scheme, as the following example illustrates.

B. Lack of Robustness

Our next result shows that when the plant dynamics in the closed-loop system satisfying all conditions of Proposition III.4 are perturbed with a disturbance, the system is not ISS in general, although we showed in Proposition III.4 its stability in the absence of disturbances.

Proposition III.6: Consider the closed-loop system consisting of the plant[3]

$$x_{k+1} = \Phi x_k + \Gamma u_k + w_k$$

and the controller with protocol (19)–(24). Suppose that Φ has a real eigenvalue $\lambda_m > 1$ and that all conditions of Proposition III.4 hold. Then, for any $x_0 \in \mathbb{R}^n$, any $\mu_0 > 0$ and any positive C_1 and ε there exists a disturbance w^ε with $\|w^\varepsilon\| \leq \varepsilon$ which gives

$$\limsup_{k \to \infty} |x_k| > C_1.$$

Next, we explain the intuition behind the construction of a disturbance that illustrates that the ISS gain is not finite, i.e., the system is not ISS. A detailed construction is presented in the proof of Proposition III.6 in Section IV-C. First, the disturbance is set to zero and, using the proof of Proposition III.4, we can show that if we wait long enough with the zero disturbance, both x and μ will become arbitrarily small and we will be in the zoom-in mode of operation. Then, when both x and μ are sufficiently small and since the plant is one-step completely controllable from the disturbance, we can find a disturbance w of arbitrarily small norm that makes the ratio of x to μ arbitrarily large in one step. We apply such a disturbance and then set it to zero again. Consequently, large $\xi = x/\mu$ forces the switching logic into the zoom-out mode of operation. More importantly, since prior to the action of the disturbance we had that the system was in the zoom-in mode (i.e., $|\xi| \leq M$) and since the norm of ξ can be made arbitrarily large with the action of the (arbitrarily small) disturbance, it may take arbitrarily long time before the system switches again to the zoom-in mode. As a result, the state x exhibits arbitrarily large overshoots during the zoom-out since the open-loop plant is unstable. Finally, we repeat this construction *ad infinitum* in order to force $\limsup_{k \to \infty} |x_k|$ to be arbitrarily large.

Example III.7: To illustrate Proposition III.6, we applied the control algorithm of Section III-A to the same system as in Example II.2, taking the sampling period to be $T = 0.16$ and retaining the initial conditions and all relevant quantizer and controller parameters listed in Example II.2. We constructed a

[3]Note that since $x, w \in \mathbb{R}^n$, the plant is one-step completely controllable from disturbance w. This is not necessary as our simulation example will illustrate.

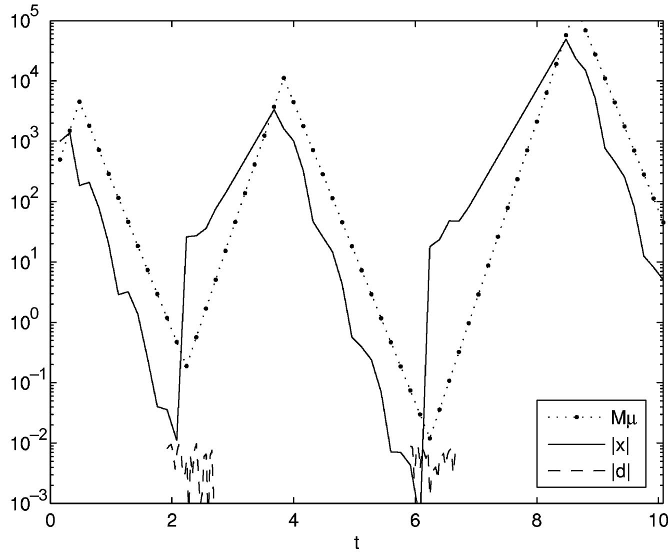

Fig. 2. Simulation results for Example III.7.

disturbance $d(t)$ in the spirit of the previous discussion, keeping it zero most of the time and turning it on for a short period of time when the state becomes small. Note that the system does not satisfy the assumption of being controllable in one step, and that we did not follow exactly the disturbance construction given in the proof of Proposition III.6. Nevertheless, the simulation results shown in Fig. 2 confirm that the system may exhibit arbitrarily large overshoots in response to a small disturbance, because the quantizer's range (whose values at sampling times are indicated by larger dots) takes a long time to catch up with the state. In fact, if we keep increasing the intervals on which the disturbance is zero to let the state drop to progressively smaller values, we will obtain $\limsup_{k\to\infty}|x_k| = \infty$. □

The possible non-robustness of the control law in Proposition III.6 actually holds for a large class of plants, control laws, and zooming protocols. Indeed, the crucial ingredients of closed-loop systems that will exhibit this type of non-robustness are as follows.

1) The closed-loop system has to have the property that in the absence of disturbances, both x and μ converge to zero. Moreover, given any initial conditions x_0 and $\mu_0 > 0$, the zooming-out stage is bounded.
2) The closed-loop system is such that the x-component is completely controllable locally around the origin with arbitrarily small disturbances $\|w\| \leq \varepsilon$.
3) For all $k \geq 0$, the zooming protocol takes the form $\mu_{k+1} = \gamma_k(\mu_k)$, where γ_k are continuous, zero at zero, locally invertible, and uniformly bounded from below and from above.
4) When the measurement overflows, the controller is switched off.

Hence, a suitable modification in the zooming-out procedure needs to be adopted in order to achieve ISS. In what follows, we provide a modification of the zooming-out procedure (see (39) in the next subsection) and subsequently prove that the closed-loop system with the modified scheme is ISS. In particular, our modification violates item 3) and we will show that this is sufficient to guarantee ISS.

We remark that the same observation applies to an even larger class of stabilizing quantized feedback control strategies, including those with a moving quantization center (cf. [1], [5], [9], [16], and [18]). Since asymptotic stabilization using such strategies relies on the convergence of the quantization center to zero, the previous argument can be slightly extended to reveal a lack of robustness with respect to disturbances.

C. Input-to-State Stability

Consider the plant with disturbance (1), together with the controller and zooming protocol introduced in Section III-A. The corresponding discrete-time system is

$$x_{k+1} = \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) + w_k \quad x_0 \in \mathbb{R}^n \tag{36}$$

$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k) \quad \mu_0 > 0 \tag{37}$$

$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k) \quad \Omega_{-1} = \Omega_{\text{out}} \tag{38}$$

where Φ and Γ are defined in (25), U and H are defined in (22) and (24), and $w_k := \int_{kT}^{(k+1)T} e^{A((k+1)T-s)}Dd(s)ds$. We use here a new zooming protocol

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\text{out}}(\mu_k + c) & \text{if } \Omega_k = \Omega_{\text{out}} \\ \Omega_{\text{in}}\mu_k & \text{if } \Omega_k = \Omega_{\text{in}} \end{cases} \tag{39}$$

where $c > 0$. For simplicity, we assume in the sequel that $c = 1$. The use of this constant c violates item 3) given in Section III-B, and this will be shown to fix the problem identified there. The best value of c in general cannot be determined without having some information about the disturbance. We do not pursue this interesting issue further here; we just note that as c is reduced, the ISS gain will increase and in the limit as $c \to 0$, we lose ISS as shown in the previous section.

Next, we introduce a discrete-time version of the definition of ISS. This will suffice for our analysis in this subsection since the discrete-time ISS can be used to prove an appropriate version of continuous-time ISS that takes inter-sample behavior into account (see Remark III.5). The system (36), (37) is said to be ISS if there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that the solutions of the system satisfy the following for all $x_0 \in \mathbb{R}^n$ and all w:

$$|x_k| \leq \gamma_1(|x_0|) + \gamma_2(\|w\|) \qquad \forall k \geq 0 \tag{40}$$

and

$$\limsup_{k\to\infty} |x_k| \leq \gamma_3\left(\limsup_{k\to\infty} |w_k|\right). \tag{41}$$

As in Section II, the functions γ_1, γ_2 will depend on $\mu_0 > 0$ (but not on x_0 or w) while γ_3 will be independent of μ_0, as we will see at the end of the proof of Theorem 2 in Section IV-D.

Again, we consider the dynamics of the variable ξ_k defined in (26). During the zooming-in stage we have

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{\text{in}}}\Gamma K \nu_k + \frac{1}{\Omega_{\text{in}}}\zeta_k \tag{42}$$

where $\nu_k := q(\xi_k) - \xi_k$ and $\zeta_k := w_k/\mu_k$. We can state the following results; the first one follows directly from [7, Ex. 3.4].

Lemma III.8: Suppose that $(1/\Omega_{\rm in})(\Phi + \Gamma K)$ is Schur.[4] Then, there exist strictly positive L_1, λ_1, γ_1, γ_2 such that the solutions of the system (42) satisfy the following:

$$|\xi_k| \le L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma_1\|\nu\| + \gamma_2\|\zeta\| \qquad \forall k \ge 0.$$

In particular, let $\kappa > 0$ and $\sigma \in (0,1)$ be such that $\|(1/\Omega_{\rm in}^k)(\Phi + \Gamma K)^k\| \le \kappa\sigma^k$ for all $k \ge 0$ (see footnote 2 in Lemma III.2). Then, L_1, λ_1, γ_1 are given by (30) and we can let $\gamma_2 = \kappa/(\Omega_{\rm in}(1-\sigma))$.

Corollary III.9: Let $\Omega_{\rm in}$, L_1, γ_1 come from Lemma III.8 and let strictly positive M and Δ be such that (31) holds. Then, there exist strictly positive Δ_M and Δ_w, with $\Delta_M - \Delta > 0$, such that whenever $|\xi_0| \le \Delta_M$, $\|\nu\| \le \Delta$, and $\|\zeta\| \le \Delta_w$, we have

$$|q_{\mu_k}(x_k)| \le (M-\Delta)\mu_k \quad \text{and} \quad |\xi_k| \le M \qquad \forall k \ge 0.$$

We can now state the main result of this section; its proof is given in Section IV-D.

Theorem 2: Consider the system (36)–(38) and let q be a quantizer fulfilling Assumption II.1 and the conditions (3) and (4). Suppose that for the given $T > 0$ the pair (Φ, Γ) is stabilizable. Let K be such that $(\Phi + \Gamma K)$ is Schur. Let $\Omega_{\rm in}$ be such that (29) is Schur and let $\Omega_{\rm out} > \|\Phi\|$. Let the range M of q be sufficiently larger than the error bound Δ so that Corollary III.9 holds with M, Δ and some Δ_M, Δ_w. Define $\ell_{\rm out} := M - \Delta$ and $\ell_{\rm in} := \Delta_M - \Delta$. Then, μ_k is bounded for all $k \ge 0$ and the system (22), (24), (36), (37), (38), (39) is ISS.

The numbers L_1, γ_1, γ_2 in Lemma III.8 can be computed using a Lyapunov function. Moreover, since (31) is a strict inequality, there exist two strictly positive numbers ε_1, ε_2 such that we have $\Delta(L_1(1+\varepsilon_1) + \gamma_1 + 2) + \varepsilon_2 = M$. Then, it is not hard to show that we can use in Corollary III.9 the following: $\Delta_M = (1+\varepsilon_1)\Delta$ and $\Delta_w = \varepsilon_2/\gamma_2$. Alternatively, we can directly obtain M, Δ, Δ_M, Δ_w in Corollary III.9 from a Lyapunov function (see the discussion for the disturbance-free case following Remark III.5).

Example III.10: This is a continuation of Example III.7. The system is the same as in that example, and the disturbance behaves in the same way, but here we adopt the modified control scheme with $c = 1$ in (39). Fig. 3 confirms that the problem observed in Fig. 2 is overcome, and the system performance is comparable with that shown earlier in Fig. 1 for the continuous-time case. We note, however, slightly larger overshoots here compared to Fig. 1, which are due to the hysteresis switching logic (see Remark III.1). □

A careful inspection of the proof of Theorem 2 reveals that the gain functions γ_1, γ_2, γ_3 grow faster than any linear function both for small and for large values of their arguments. It turns out that this is not an artifact of our control design, but rather a consequence of a recent result by Nuno Martins who showed, using techniques from information theory, that it is impossible to achieve ISS with linear gain for any linear system with finite data rate feedback [12]. Thus, in the presence of state quantization it is indeed necessary to formulate the disturbance attenuation problem in terms of nonlinear ISS gains (despite the fact that the given open-loop system is linear), and our control strategy complements the findings of [12] by providing a constructive solution to this problem.

[4]In view of Lemma III.2, we can find an appropriate $\Omega_{\rm in} \in (0,1)$ so that this holds whenever $(\Phi + \Gamma K)$ is Schur.

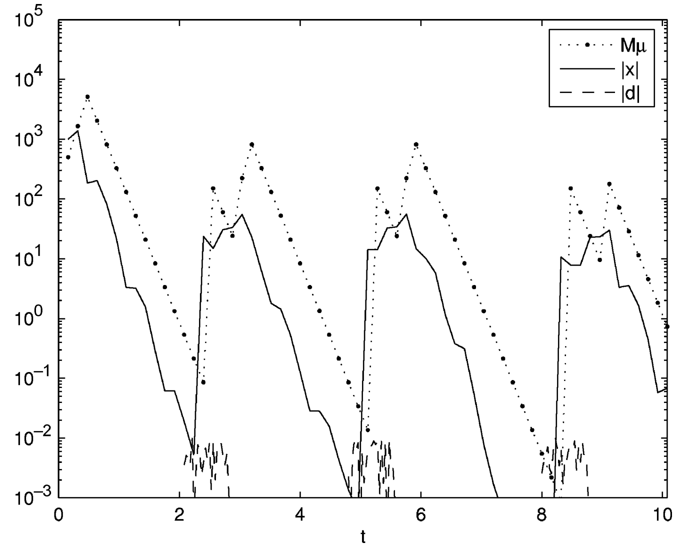

Fig. 3. Simulation results for Example III.10.

It is worth noting that the modified zooming protocol of the form (39) can be used in the event-based scheme and it would not change the ISS properties of the system. Actually, this modification would have the added benefits of reducing the number of zoom-outs and providing robustness of the event-based scheme with respect to time delays.

IV. Proofs of the Main Results

A. Proof of Theorem 1

The proof of Theorem 1 will rely on a series of lemmas, whose proofs can be found in the Appendix. We assume throughout that $\mu_0 > 0$ is fixed, d is bounded, and all hypotheses of Theorem 1 hold.

Lemma IV.1: There exist a time $t_1 \ge t_0$ and functions $\rho_x, \rho_\mu : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$\|x\|_{[t_0,t_1]} \le \rho_x\left(|x_0| + \|D\|\,\|d\|_{[t_0,\infty)}\right) \tag{43}$$

$$\mu(t_1) \le \rho_\mu\left(|x_0| + \|D\|\,\|d\|_{[t_0,\infty)}\right) \tag{44}$$

and for all $t \ge t_1$ we have $\texttt{captured}(t) =$ "yes" and $|x(t)| \le M\mu(t)$.

Lemma IV.1 establishes the existence of a time t_1 such that from this time onward, the continuous state x always remains within the range of the quantizer q_μ. In other words, the "capturing" stage has a finite duration and does not need to be repeated. The lemma also provides bounds on the overshoots of the system states x and μ during the capturing stage. By time-invariance of the dynamics, $|t_1 - t_0|$ is independent of t_0. On the other hand, $|t_1 - t_0|$, ρ_x, and ρ_μ are all affected by the choice of μ_0 (see the proof of Lemma IV.1).

Lemma IV.2: Define $V(x) := (1/2)x^TPx$. Then for $t \ge t_1$ we have

$$|x| > \|PBK\|\Delta\mu + \|PD\||d| \quad \Rightarrow \quad \dot V < 0 \tag{45}$$

along the continuous dynamics (i.e., on every subinterval of $[t_1, \infty)$ on which μ remains constant).

Lemma IV.2 says that after the capturing stage is completed, and away from the discrete events (zoom-ins and zoom-outs), V serves as a Lyapunov function for the closed-loop system as long as x remains outside a ball around the origin whose size is determined by μ, $|d|$, and system parameters. This implies, in particular, that every sublevel set of V which contains this ball (and is contained in the range of q_μ) is invariant with respect to the continuous dynamics.

Lemma IV.3: Consider some $t \geq t_1$ such that $x(t) \in \mathcal{R}_1(\mu(t))$, where

$$\mathcal{R}_1(\mu) := \left\{x : V(x) < \lambda_{\min}(P)(M - 2\Delta)^2\mu^2\right\}.$$

Suppose that $\mu(t)$ satisfies

$$\sqrt{\lambda_{\min}(P)}(M - 2\Delta)\mu(t) > \sqrt{\lambda_{\max}(P)}\left(\|PBK\|\Delta\mu(t) + \|PD\|\,\|d\|_{[t,\infty)}\right). \quad (46)$$

Then the next discrete event can only be a zoom-in. Moreover, if $\mu(t)$ satisfies

$$\sqrt{\lambda_{\min}(P)}\left(\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - 2\Delta\right)\mu(t) > \sqrt{\lambda_{\max}(P)}\left(\|PBK\|\Delta\mu(t) + \|PD\|\,\|d\|_{[t,\infty)}\right) \quad (47)$$

then this zoom-in will happen in finite time.

The first claim of Lemma IV.3 provides a specific instance of the general statement immediately preceding Lemma IV.3: $\mathcal{R}_1(\mu)$ is a suitable sublevel set of V, and the condition (46) guarantees that it contains the appropriate ball, hence a zoom-out cannot occur. The meaning of the second claim is that a zoom-in will eventually be triggered unless μ is already small enough relative to the disturbance.

Lemma IV.4: For every $\varepsilon > 0$ there exists a $\delta > 0$ with the property that if $|x_0| \leq \delta$ and $\|d\|_{[t_0,\infty)} \leq \delta$ then there exists a time $t_2 \geq t_1$ such that:

1) $\mathcal{R}_1(\mu(t_2)) \subset \{x : |x| \leq \varepsilon\}$;
2) $x(t) \in \mathcal{R}_1(\mu(t_2))$ for all $t \in [t_0, t_2]$;
3) inequality (46) holds with $t = t_2$.

In the absence of the disturbance, Lemma IV.4 gives stability of the origin in the sense of Lyapunov. Its proof is an extension of the proof of Lyapunov stability given in [8, p. 1547], and relies on Assumption II.1.

Proof of Theorem 1: Define

$$\hat{\mu} := \frac{\sqrt{\lambda_{\max}(P)}\|PD\|\,\|d\|_{[t_0,\infty)}}{\sqrt{\lambda_{\min}(P)}(M - 2\Delta) - \sqrt{\lambda_{\max}(P)}\|PBK\|\Delta}.$$

It is straightforward to check that (46) holds whenever $\mu(t) > \hat{\mu}$.

Claim 1: For all $t \geq t_1$ we have $\mu(t) \leq \Omega_{\text{out}} \max\{\mu(t_1), \hat{\mu}\}$.

If the claim is not true, then a zoom-out must have occurred after t_1 with $\mu^- > \max\{\mu(t_1), \hat{\mu}\}$. This in turn implies that the discrete event prior to that was either a zoom-out or a zoom-in which also occurred after t_1 and resulted in $\mu > \max\{\mu(t_1), \hat{\mu}\}$. By Lemma IV.1 we have $|x(t)| \leq M\mu(t)$ for $t \geq t_1$. It is easy to see from (15) and (17) that after a zoom-out or a zoom-in with $|x| \leq M\mu^-$ we necessarily have $|x| \in \mathcal{R}_1(\mu)$. Therefore, Lemma IV.3 tells us that the next discrete event could not be a zoom-out, and the resulting contradiction proves the claim.

Combining Claim 1 and the definition of $\hat{\mu}$ with the bounds (43) and (44) from Lemma IV.1, we see that the estimate (6) holds with some functions γ_1 and γ_2 which can be made continuous and increasing, but not necessarily 0 at 0. Moreover, for every $\varepsilon > 0$ we can apply Lemma IV.4 to find a $\delta > 0$ with the three properties stated in that lemma. Lemma IV.3 then implies that the first discrete event after t_2 (if one occurs) is a zoom-in. It follows that $\mu(t) \leq \mu(t_2)$ for all $t \geq t_2$, because if μ returns to the value $\mu(t_2)$ then Lemma IV.3 again applies. This means that for $|x_0|$ and $\|d\|_{[t_0,\infty)}$ sufficiently small, Lemmas IV.1 and IV.4 yield an arbitrarily small bound for $|x(t)|$ for all time. Therefore, we can modify the functions γ_1 and γ_2 to make them 0 at 0, hence class $\mathcal{K}_\infty$, and the first ISS estimate (6) is established.

Next, pick an arbitrary $\tilde{\varepsilon} > 0$ and define

$$\tilde{\mu} := \frac{\lambda_{\max}(P)\|PD\|}{a}\left(\limsup_{t\to\infty}|d(t)| + \tilde{\varepsilon}\right)$$

where $a := \Omega_{\text{in}}\lambda_{\min}(P)(M - 2\Delta) - \lambda_{\max}(P)\|PBK\|\Delta - \sqrt{\lambda_{\min}(P)\lambda_{\max}(P)}2\Delta$. There exists a time $t_{\tilde{\varepsilon}} \geq t_1$ such that $|d(t)| \leq \limsup_{t\to\infty}|d(t)| + \tilde{\varepsilon}$ for all $t \geq t_{\tilde{\varepsilon}}$. It is straightforward to check that (47) holds whenever $t \geq t_{\tilde{\varepsilon}}$ and $\mu(t) > \tilde{\mu}$.

Claim 2: There exists a time $\tilde{t}_{\tilde{\varepsilon}} \geq t_{\tilde{\varepsilon}}$ such that $\mu(t) \leq \Omega_{\text{out}}\tilde{\mu}$ for all $t \geq \tilde{t}_{\tilde{\varepsilon}}$.

If $\mu(t) \leq \tilde{\mu}$ for all $t \geq t_{\tilde{\varepsilon}}$, then the claim is trivially true. Otherwise, pick some $t \geq t_{\tilde{\varepsilon}}$ such that $\mu(t) > \tilde{\mu}$. If $x \notin \mathcal{R}_1(\mu(t))$, then Lemma IV.2 guarantees that either x will enter $\mathcal{R}_1(\mu(t))$ before the next discrete event occurs or a zoom-out will occur and we will have $x \in \mathcal{R}_1(\mu)$ for the new value of μ, i.e., $\Omega_{\text{out}}\mu(t)$. After that, Lemma IV.3 ensures that as long as $\mu > \tilde{\mu}$, zoom-ins will keep occurring. Therefore, we will eventually have $\mu \leq \tilde{\mu}$. This proves the claim, because if μ returns to a value in $(\tilde{\mu}, \Omega_{\text{out}}\tilde{\mu}]$, then the same argument again applies and a further zoom-out is not possible.

In view of Claim 2, the definition of $\tilde{\mu}$, the bound $|x(t)| \leq M\mu(t)$ for $t \geq t_1$ provided by Lemma IV.1, and the fact that $\tilde{\varepsilon} > 0$ was arbitrary, the second ISS estimate (7) is also established, with the linear gain function

$$\gamma_3(r) := \frac{M\Omega_{\text{out}}\lambda_{\max}(P)\|PD\|}{a}r. \quad (48)$$

This completes the proof of the theorem. □

B. Proof of Proposition III.4 (Sketch)

If the initial conditions are such that a zoom-in is triggered initially, then the zoom-in condition is triggered for all future times and the system dynamics evolve according to the following equations:

$$x_{k+1} = (\Phi + \Gamma K)x_k + \Gamma K \mu_k \left(q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k} \right)$$
$$\mu_{k+1} = \Omega_{\text{in}}\mu_k. \tag{49}$$

Since $|q(\xi_k) - \xi_k| \le \Delta$, this system is a cascade of the GAS μ-subsystem and the ISS x-subsystem and hence there exist $\overline{K}, \lambda > 0$ such that for all $k \ge 0$ we have

$$|(x_k, \mu_k)| \le \overline{K}\exp(-\lambda k)\,|(x_0, \mu_0)|.$$

On the other hand, if a zoom-out is triggered initially, then for any x_0 and μ_0 there exists a $k^* := k^*(|x_0|, \mu_0)$ such that $|(x_{k^*}/\mu_{k^*})| \le \ell_{\text{in}}$ and hence the zoom-in condition is triggered. Moreover, for all $k \ge k^*$ the zoom-in condition is triggered and for all $k \ge 0$ we have

$$|(x_k, \mu_k)| \le \overline{K}\exp\left(-\lambda(k - k^*)\right)|(x_{k^*}, \mu_{k^*})|. \tag{50}$$

For all $k \in [0, k^*]$ we have

$$|x_k| \le \|\Phi\|^{k^*(|x_0|,\mu_0)}|x_0| =: \rho_1\left(|x_0|, \mu_0\right)$$
$$\mu_k \le \Omega_{\text{out}}^{k^*(|x_0|,\mu_0)}|\mu_0| =: \rho_2\left(|x_0|, \mu_0\right). \tag{51}$$

Combining the bounds (50) and (51), we can write

$$|x_k| \le \exp(-\lambda k)\cdot \overline{K}\exp\left(\lambda k^*\left(|x_0|, \mu_0\right)\right)\cdot\sqrt{\rho_1^2 + \rho_2^2}$$

which shows that x_k converges to zero exponentially (note that we suppressed the arguments of ρ_i). The proof would be over if we had $\overline{K}\exp(\lambda k^*(0,\mu_0))\cdot\sqrt{\rho_1^2(0,\mu_0) + \rho_2^2(0,\mu_0)} = 0$ but this is not true since $\rho_2(0,\mu) \ne 0$ for any $\mu > 0$.

In order to prove stability, we use Assumption II.1 to prove that there exists continuous and bounded $\varphi : \mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\varphi(0,\mu) = 0$ so that (33) holds. With these properties, there is no loss of generality in taking $\varphi(\cdot,\mu) \in \mathcal{K}_\infty$ for any fixed $\mu > 0$ (just bound the original function with a class $\mathcal{K}_\infty$ one). Let an arbitrary $\rho > 0$ be given and introduce

$$T^* := \max\left\{\left\lceil \ln\left(\frac{\rho|x_{k^*}|}{M}\right)(\ln(\Omega_{\text{in}}))^{-1}\right\rceil, 0\right\}.$$

Then we have for all $k \ge k^* + T^*$ that

$$|x_k| \le M\mu_k = M\Omega_{\text{in}}^{(k-k^*)}\mu_{k^*} \le M\Omega_{\text{in}}^{T^*}\mu_{k^*}$$
$$\le \rho\mu_{k^*}|x_{k^*}| =: \chi_1\left(|x_{k^*}|, \mu_{k^*}\right).$$

Note that Assumption II.1 guarantees that there exists an $L_q > 0$ such that $q(z) \le L_q|z|$ for all z. Hence, for $k \in [k^*, k^* + T^*]$ we can write

$$|x_k| \le \left(\|\Phi\| + \|\Gamma K\|L_q\right)^{T^*}|x_{k^*}| =: \chi_2\left(|x_{k^*}|\right).$$

Since $\|\Phi\| > 1$ and $\Omega_{\text{in}} < 1$, $\chi_2(0) = 0$ and $\chi_2(s)$ is bounded for all $s \ge 0$. Hence, we can bound it by a $\chi_3 \in \mathcal{K}_\infty$. Finally, we define

$$\tilde{\varphi}\left(|x|, \mu\right) := \max\left\{\chi_3\left(|x|\right), \chi_1\left(|x|, \mu\right)\right\}$$

and it is clear that $\tilde{\varphi}(0,\mu) = 0$ and this function is increasing in both its arguments. Hence, we can write that for all $k \ge k^*$,

$$|x_k| \le \tilde{\varphi}\left(|x_{k^*}|, \mu_{k^*}\right) \le \tilde{\varphi}\left(\rho_1\left(|x_0|, \mu_0\right), \rho_2\left(|x_0|, \mu_0\right)\right)$$
$$=: \bar{\varphi}\left(|x_0|, \mu_0\right).$$

Note that $\bar{\varphi}(0,\mu) = 0$ for any $\mu > 0$. Finally, the conclusion in (33) follows by noting that there exists a φ with the right properties such that $\varphi(s,\mu) \ge \max\{\bar{\varphi}(s,\mu), \rho_1(s,\mu)\}\ \forall \mu, s.$ □

C. Proof of Proposition III.6

Let $C_1 > 0$ and $\varepsilon > 0$ be arbitrary. Since we assumed that there exists a positive real eigenvalue $\lambda_m > 1$ of Φ, let ζ_m be its corresponding eigenvector with $|\zeta_m| = 1$. Let $\hat{\varepsilon} > 0$ and $\bar{\varepsilon}_1 > 0$ be such that

$$\bar{\varepsilon}_1\left(\|\Phi + \Gamma K\| + \|\Gamma K\|\Delta\right) + \hat{\varepsilon} < \varepsilon. \tag{52}$$

Let C_1 and $\hat{\varepsilon}$ generate

$$\overline{T} := \left\lceil \ln\left(\frac{C_1}{\hat{\varepsilon}}\right)(\ln(\lambda_m))^{-1}\right\rceil. \tag{53}$$

Let $\overline{T}$ generate $C_2 > 0$ via

$$C_2 > \max\left\{\ell_{\text{in}}(\Omega_{\text{out}})^{\overline{T}}\|\Phi\|^{-\overline{T}}, \ell_{\text{out}}\right\}. \tag{54}$$

Let C_2 and $\hat{\varepsilon}$ generate $\bar{\varepsilon}_2$ as follows:

$$\bar{\varepsilon}_2 := \hat{\varepsilon}(\Omega_{\text{in}}C_2)^{-1}. \tag{55}$$

Finally, using $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$, define

$$\bar{\varepsilon} := \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}. \tag{56}$$

Note that since the system without disturbance is stable, as shown by Proposition III.4, for any $x_0 \in \mathbb{R}^n$, $\mu_0 > 0$ there exists a $k_0^* > 0$ such that with $w_k \equiv 0$ we have

$$\max\left\{\left|x_{k_0^*}\right|, \mu_{k_0^*}\right\} \le \bar{\varepsilon} \quad \text{and} \quad \left|\xi_{k_0^*}\right| \le M. \tag{57}$$

We now start the construction of the disturbance. Let the disturbance satisfy $w_k^\varepsilon = 0\ \forall k \in [0, k_0^* - 1]$. Hence, (57) holds. Let now

$$w_{k_0^*}^\varepsilon = -(\Phi + \Gamma K)x_{k_0^*} - \Gamma K\mu_{k_0^*}\left(q_{k_0^*} - \xi_{k_0^*}\right) + \hat{\varepsilon}\zeta_m.$$

This disturbance will yield $x_{k_0^*+1} = \hat{\varepsilon}\zeta_m$. The conditions (52) and (56) guarantee that $|w_{k_0^*}^\varepsilon| \le \varepsilon$. The conditions (55) and (56) guarantee that

$$\left|\xi_{k_0^*+1}\right| = \left|\frac{x_{k_0^*+1}}{\Omega_{\text{in}}\mu_{k_0^*}}\right| \ge \frac{\hat{\varepsilon}}{\Omega_{\text{in}}\bar{\varepsilon}_2} = C_2 \tag{58}$$

and hence at time $k_0^* + 1$ the zoom-out condition is triggered. Since the ξ-dynamics with $w_k \equiv 0$ evolve according to (27), there exists an integer k_1^* such that if the disturbance satisfies

$w_k^\varepsilon = 0\,\forall k \in [k_0^*+1, k_1^*-1]$, then $|\xi_{k_1^*}| \leq \ell_{\rm in}$ and the zoom-in condition is triggered at $k = k_1^*$. Moreover, from (53) and (54) we have $k_1^* - k_0^* - 1 \geq \overline{T}$, which implies together with (58) that

$$\left|x_{k_1^*}\right| = \left|\lambda_m^{k_1^*-k_0^*-1}\zeta_m\hat{\varepsilon}\right| \geq \lambda_m^{\overline{T}}\hat{\varepsilon} \geq C_1.$$

Again via stability of the disturbance-free (x,μ)-system, there exists a k_2^* such that if $w_k^\varepsilon = 0\ \forall k \in [k_1^*, k_2^*-1]$, then we have

$$\max\left\{\left|x_{k_2^*}\right|, \mu_{k_2^*}\right\} \leq \bar{\varepsilon} \quad \text{and} \quad \left|\xi_{k_2^*}\right| \leq M.$$

In a similar manner, we construct the disturbance so that for all $k \neq k_{2j}^*$, $j \in \mathbb{N}$ we have $w_k^\varepsilon = 0$ and for all $k = k_{2j}^*$, $j \in \mathbb{N}$ we have

$$w_{k_{2j}^*}^\varepsilon = -(\Phi + \Gamma K)x_{k_{2j}^*} - \Gamma K \mu_{k_{2j}^*}\left(q_{k_{2j}^*} - \xi_{k_{2j}^*}\right) + \hat{\varepsilon}\zeta_m.$$

This disturbance by construction satisfies $\|w^\varepsilon\| \leq \varepsilon$ and yields $|x_{k_{2j+1}^*}| > C_1\ \forall j \in \mathbb{N}$, which completes the argument. □

D. *Proof of Theorem 2*

The proof of Theorem 2 is carried out using several lemmas, whose proofs are given in the Appendix.

Lemma IV.5: Suppose that all conditions of Theorem 2 hold. Then, there exist $\rho_1, \rho_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that for any $i \in \mathbb{N}$, $x_{k_{2i}} \in \mathbb{R}^n$, $\mu_{k_{2i}} > 0$, and w we have

$$k_{2i+1} - k_{2i} \leq 1 + \varphi_1\left(|x_{k_{2i}}|\right) + \varphi_2\left(\|w\|_{[k_{2i}, k_{2i+1}-1]}\right) \quad (59)$$

and, moreover, for all $k \in [k_{2i}, k_{2i+1}]$,

$$|x_k| \leq \rho_1\left(|x_{k_{2i}}|\right) + \rho_2\left(\|w\|_{[k_{2i}, k_{2i+1}-1]}\right). \quad (60)$$

Lemma IV.5 implies that the zoom-out condition can be only triggered for finitely many time steps. Hence, if N is finite, then $k_{2N+2} = \infty$. In other words, there exists a $k_{2N+1} \in \mathbb{N}$ such that the zoom-in condition is triggered on the interval $[k_{2N+1}, \infty)$. Moreover, Lemma IV.5 establishes a bound on the state x during the zoom-out intervals. We remark that the functions ρ_i and φ_i are independent of μ.

Lemma IV.6: There exists a continuous bounded function $\rho_\mu^{\rm out}$ such that for any $\mu > 0$ we have $\rho_\mu^{\rm out}(\mu, 0, 0) > 0$ and the following is true for all $i \in \{0, 1, \ldots, N\}$ and all $\mu_{k_{2i}} > 0$, $x_{k_{2i}} \in \mathbb{R}^n$, $w \in l_\infty$:

$$\mu_{k_{2i+1}} \leq \rho_\mu^{\rm out}\left(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w\|_{[k_{2i}, k_{2i+1}-1]}\right).$$

Lemma IV.6 establishes a bound on μ at the end of each zoom-out interval in terms of the values of μ and x at the beginning of that interval and the infinity norm of the disturbance during that interval.

Lemma IV.7: There exist positive $\overline{K}$, λ, γ such that for any $s, t \in [k_{2i+1}, k_{2i+2}]$ with $s \geq t$, any x_s, μ_s, and $w \in l_\infty$,

$$|x_s| \leq \overline{K}\exp\left(-\lambda(s-t)\right)\left(|x_t| + \mu_t\right) + \gamma\|w\|_{[t, s-1]}. \quad (61)$$

In particular, we have from (61) that for all $k \in [k_{2i+1}, k_{2i+2}]$ the following holds:

$$|x_k| \leq \overline{K}\exp\left(-\lambda(k - k_{2i+1})\right)\left(|x_{2i+1}| + \mu_{2i+1}\right) + \gamma\|w\|_{[k_{2i+1}, k-1]}.$$

Lemma IV.7 establishes an appropriate bound on the state x during the zoom-in intervals. This bound is a direct consequence of the fact that during the zoom-ins the system behaves as a cascade of x- and μ-subsystems. The x-subsystem is ISS when μ is regarded as an input, and the μ-subsystem is globally exponentially stable. Note that γ is a fixed constant, independent of μ.

Lemma IV.8: There exists a continuous function $\rho_x^{\rm in} : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\rho_x^{\rm in}(\mu, 0, 0) = 0$ for all $\mu > 0$, and such that for any $s \geq 0$, $\rho_x^{\rm in}(\cdot, \cdot, s)$ is nondecreasing in its first two arguments and for any $i \in \{0, 1, \ldots, N\}$ the following holds for all $\mu_{k_{2i+1}}, x_{k_{2i+1}}, w$ and all $k \in [k_{2i+1}, k_{2i+2}]$:

$$|x_k| \leq \rho_x^{\rm in}\left(\mu_{k_{2i+1}}, \left|x_{k_{2i+1}}\right|, \|w\|_{[k_{2i+1}, k_{2(i+1)}-1]}\right)$$

Lemma IV.8 establishes a different bound during zoom-in intervals for the state x than the bound given in Lemma IV.7. Indeed, note that $\rho_x^{\rm in}(\mu, 0, 0) = 0$ can not be directly concluded from Lemma IV.7. As with Lemma IV.4, we need to use Assumption II.1 in the proof of Lemma IV.8.

Lemma IV.9: Consider an arbitrary $i \in \{0, 1, 2, \ldots, N\}$. If $k_{2i+2} < \infty$, then $i < N-1$ and there exists a $\tilde{\gamma} > 0$ such that

$$\max\left\{\left|x_{k_{2i+2}}\right|, \mu_{k_{2i+2}}\right\} \leq \tilde{\gamma}\|w\|_{[k_{2i+1}, k_{2i+2}-1]}. \quad (62)$$

Lemma IV.9 establishes that if a zoom-in interval is bounded (i.e., is followed by a zoom-out interval) then at the end of this zoom-in interval we have that x and μ are bounded by a function of disturbance only, i.e., the initial conditions are "forgotten". Note that $\tilde{\gamma}$ is a fixed constant.

Proof of Theorem 2: First, we prove that there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that (40) holds. The proof is carried out using the previous lemmas and induction.

Step $i = 0$: Let $k_0 = 0$. Without loss of generality we suppose that $\Omega_0 = \Omega_{\rm out}$ and we zoom out on the non-empty interval $[k_0, k_1 - 1]$. We have from Lemma IV.5 that

$$|x_k| \leq \rho_1\left(|x_0|\right) + \rho_2\left(\|w\|_{[k_0, k_1-1]}\right) \qquad \forall k \in [0, k_1].$$

Then, using the fact that $\rho_x^{\rm in}$ is nondecreasing in its first two arguments, Lemmas IV.6 and IV.8, and (59), we can write that for all $k \in [k_1, k_2]$,

$$\begin{aligned}|x_k| &\leq \rho_x^{\rm in}\left(\mu_{k_1}, |x_{k_1}|, \|w\|_{[k_1, k-1]}\right)\\ &\leq \rho_x^{\rm in}\left(\rho_\mu^{\rm out}, \rho_1 + \rho_2, \|w\|_{[k_1, k-1]}\right)\\ &\leq \overline{\gamma}_1\left(\mu_0, |x_0|\right) + \overline{\gamma}_2\left(\mu_0, \|w\|_{[0, k-1]}\right)\end{aligned}$$

where $\overline{\gamma}_i$ are nondecreasing in μ and for each fixed $\mu > 0$ we have that $\overline{\gamma}_1(\mu,\cdot), \overline{\gamma}_2(\mu,\cdot) \in \mathcal{K}_\infty$.

We either have $k_2 = \infty$ or $k_2 < \infty$. If the former is true, the proof is complete. If the latter is true, then we have from Lemma IV.9 that (62) holds, i.e.,

$$\max\{|x_{k_2}|, \mu_{k_2}\} \le \tilde{\gamma}\|w\|_{[k_1,k_2-1]}. \tag{63}$$

Step $i = 1$: Using Lemma IV.5 and (63), it follows that for all $k \in [k_2, k_3]$,

$$\begin{aligned} |x_k| &\le \rho_1\left(\tilde{\gamma}\|w\|_{[k_1,k_2-1]}\right) + \rho_2\left(\|w\|_{[k_2,k_3-1]}\right) \\ &\le \overline{\gamma}\left(\|w\|_{[k_1,k-1]}\right) \end{aligned}$$

where $\overline{\gamma}(s) := \rho_1(\tilde{\gamma}s) + \rho_2(s)$ is independent of μ since ρ_i and $\tilde{\gamma}$ are independent of μ. Moreover, using Lemma IV.8, (59) and (63), we can write for all $k \in [k_3, k_4]$ that

$$\begin{aligned} |x_k| &\le \rho_x^{\text{in}}\left(\mu_{k_3}, |x_{k_3}|, \|w\|_{[k_3,k-1]}\right) \\ &\le \rho_x^{\text{in}}\left(\rho_\mu^{\text{out}}, \rho_1 + \rho_2, \|w\|_{[k_3,k-1]}\right) \\ &\le \overline{\gamma}_1\left(\mu_{k_2}, |x_{k_2}|\right) + \overline{\gamma}_2\left(\mu_{k_2}, \|w\|_{[k_2,k-1]}\right) \\ &\le \overline{\gamma}_1\left(\tilde{\gamma}\|w\|_{[k_1,k_2-1]}, \tilde{\gamma}\|w\|_{[k_1,k_2-1]}\right) \\ &\quad + \overline{\gamma}_2\left(\tilde{\gamma}\|w\|_{[k_1,k_2-1]}, \|w\|_{[k_2,k-1]}\right) \\ &\le \widehat{\gamma}\left(\|w\|_{[k_1,k-1]}\right) \end{aligned}$$

where $\widehat{\gamma}(s) := \overline{\gamma}_1(\tilde{\gamma}s, \tilde{\gamma}s) + \overline{\gamma}_2(\tilde{\gamma}s, s)$ is independent of μ since $\tilde{\gamma}$ is. Either $k_4 = \infty$, in which case we have completed the proof, or $k_4 < \infty$, in which case

$$\max\{|x_{k_4}|, \mu_{k_4}\} \le \tilde{\gamma}\|w\|_{[k_3,k_4-1]}$$

and hence we can repeat the argument.

Step $i \ge 1$: Repeating the previous argument, it follows that for any $i \in \{1, 2, \ldots, N\}$ the following holds:

$$\begin{aligned} |x_k| &\le \overline{\gamma}\left(\|w\|_{[k_{2i-1},k-1]}\right) \qquad k \in [k_{2i}, k_{2i+1}] \\ |x_k| &\le \widehat{\gamma}\left(\|w\|_{[k_{2i-1},k-1]}\right) \qquad k \in [k_{2i+1}, k_{2i+2}]. \end{aligned}$$

The proof follows by induction. Indeed, we have that (40) holds with $\gamma_1(\mu, s) := \max\{\rho_1(s), \overline{\gamma}_1(\mu, s)\}$ and $\gamma_2(\mu, s) := \max\{\rho_2(s), \overline{\gamma}_2(\mu, s), \overline{\gamma}(s), \widehat{\gamma}(s)\}$.

The proof of (41) is completed in a similar fashion. In particular, if N is finite, then the last stage is zooming-in and Lemma IV.7 guarantees that $\limsup_{k\to\infty}|x_k| \le \gamma \limsup_{k\to\infty}|w_k|$. If $N = \infty$, then we have already proved that for $k \ge k_2$ we have $|x_k| \le \overline{\gamma}(\|w\|_{[k_{2i-1},k-1]})$ for $k \in [k_{2i}, k_{2i+1}]$ and $|x_k| \le \widehat{\gamma}(\|w\|_{[k_{2i-1},k-1]})$ for $k \in [k_{2i+1}, k_{2i+2}]$. Hence, we can take $\gamma_3(s) := \max\{\gamma s, \overline{\gamma}(s), \widehat{\gamma}(s)\}$. Note that γ_3 is independent of μ_0 by construction since we showed that $\gamma, \overline{\gamma}(s)$ and $\widehat{\gamma}(s)$ are independent of μ_0. □

V. Conclusion

This paper is the first investigation of the problem of achieving ISS with respect to completely unknown disturbances for control systems with quantized state measurements. We proposed a new quantized control design methodology, which relies on multiple switchings between the zooming-out and zooming-in stages. We described two specific control strategies that achieve ISS. The first strategy was implemented in continuous time, and its Lyapunov-based analysis was an extension of the one from [8]. We highlighted the difficulties that arise in implementing a similar strategy in the time-sampling context. We then presented the second strategy which takes time sampling into account, and analyzed it using a novel method which is trajectory-based and utilizes a cascade structure of the closed-loop hybrid system.

Although the results in this paper are limited to linear dynamics, it is possible to extend them to nonlinear dynamics. The ingredients in this extension are similar to the ones used in [8], and are as follows. First, we need to assume forward completeness of the uncontrolled system, in order to have upper bounds on the state expansion during the zooming-out stage. Second, we need to assume that the state feedback law renders the continuous dynamics ISS with respect to both the quantization error and the disturbance (in the linear case this is true for every stabilizing feedback). In the sampled-data scenario, we would also need to have an exact discrete-time model of the system. These assumptions are quite restrictive, and the algorithm becomes less constructive, but conceptually the generalization is relatively straightforward. Other topics for future work include: obtaining similar results for "coarse" quantizers not satisfying conditions such as (13); achieving other robustness properties besides ISS, such as L_p stability; and explicitly addressing robustness with respect to time delays.

Appendix
Proofs of the Technical Lemmas

Proof of Lemma IV.1: We have `captured`$(t_0) =$ "no". As long as `captured` $=$ "no", the continuous dynamics are given by $\dot{x} = Ax + Dd$. Thus, we have

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}Dd(s)ds.$$

A (very crude) upper bound for this is

$$|x(t)| \le e^{\|A\|(t-t_0)}\left(|x_0| + \|D\|\,\|d\|_{[t_0,t]}\right). \tag{64}$$

In the meantime, zoom-outs occur every T_{out} units of time, hence we have $\mu(t_0+kT_{\text{out}}) = \Omega_{\text{out}}^k\mu_0, k = 0, 1, \ldots$. In view of (16), (64), and the boundedness of d, the values $\mu(t_0 + kT_{\text{out}})$ grow faster than the largest values $|x|$ can attain on the intervals $[t_0 + kT_{\text{out}}, t_0 + (k+1)T_{\text{out}})$. However, if `captured` remains equal to "no," then by (11) we have an infinite sequence of times $kT_{\text{out}} + T_{\text{c}}, k = 0, 1, \ldots$ at which $|q_\mu(x)| > \ell_{\text{out}}\mu = (M - \Delta)\mu$, hence $|x| > (M - 2\Delta)\mu$ by (3). We reach a contradiction, hence there exists a t_1 at which the value of `captured` is switched to "yes." The existence of functions ρ_x and ρ_μ with the indicated properties follows from the previous calculations: every value of $|x_0| + \|D\|\|d\|_{[t_0,\infty)}$ gives an upper bound for t_1, which in turn gives upper bounds for $\|x\|_{[t_0,t_1]}$ and $\mu(t_1)$.

There is no discrete event at which the value of `captured` would switch back to "no," thus `captured`$(t) =$ "yes" for all

$t \geq t_1$. At $t = t_1$ we must have $|q_{\mu^-}(x)| \leq \ell_{\text{out}}\mu^- = (M - \Delta)\mu^-$, hence (4) implies that $|x(t)| \leq M\mu^-(t_1) < M\mu(t_1)$. The inequality $|x| \leq M\mu$ cannot become violated as a result of a zoom-out, because zoom-outs increase the value of μ (here and later we are using continuity of x). It also cannot become violated as a result of a zoom-in in view of (3), (4), (12), and the definition of ℓ_{in} in (17). Finally, the inequality $|x| \leq M\mu$ cannot become violated along the continuous dynamics, because if at some $t \geq t_1$ we have $|x(t)| = M\mu^-(t)$, then (3) implies that $|q_{\mu^-(t)}(x(t))| \geq (M - \Delta)\mu^-(t) = \ell_{\text{out}}\mu^-(t)$, hence by (10) and the definition of ℓ_{out} in (17) a zoom-out occurs and we have $|x(t)| < M\mu(t)$. □

Proof of Lemma IV.2: For $t \geq t_1$, we have $\texttt{captured}(t) =$ "yes" by Lemma IV.1, hence by (8) the closed-loop system is

$$\begin{aligned}\dot{x} &= Ax + BKq_\mu(x) + Dd\\ &= (A + BK)x + BK\mu\left(q\left(\frac{x}{\mu}\right) - \frac{x}{\mu}\right) + Dd.\end{aligned}$$

In view of (2), the derivative of V along solutions satisfies

$$\dot{V} \leq -|x|^2 + x^T PBK\mu\left(q\left(\frac{x}{\mu}\right) - \frac{x}{\mu}\right) + x^T PDd.$$

Using the last claim of Lemma IV.1 and (3), we obtain

$$\begin{aligned}\dot{V} \leq & -|x|^2 + |x|\|PBK\|\Delta\mu + |x|\|PD\||d|\\ = & -|x|\left(|x| - \|PBK\|\Delta\mu - \|PD\||d|\right)\end{aligned}$$

from which the statement follows. □

Proof of Lemma IV.3: The open ellipsoid $\mathcal{R}_1(\mu(t))$ is a strict sublevel set of V, and (46) ensures that it contains the ball

$$\left\{x : |x| \leq \|PBK\|\Delta\mu(t) + \|PD\|\,\|d\|_{[t,\infty)}\right\}. \tag{65}$$

Therefore, by Lemma IV.2 this ellipsoid is invariant with respect to the continuous dynamics as long as μ remains constant. Moreover, $\mathcal{R}_1(\mu(t))$ is contained in the ball $\{x : |x| < (M - 2\Delta)\mu(t)\}$, in which a zoom-out cannot occur because of (3) and the definition of ℓ_{out} in (17). Thus the first claim of the lemma is established.

To prove the second claim, define

$$\mathcal{R}_2(\mu) := \left\{x : V(x) < \lambda_{\min}(P) \times \left(\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \cdot (M - 2\Delta) - 2\Delta\right)^2 \mu^2\right\}.$$

The ellipsoid $\mathcal{R}_2(\mu(t))$ is contained in $\mathcal{R}_1(\mu(t))$ and contains the ball (65) if (47) holds. Applying Lemma IV.2, we know that there exists a time $\bar{t} \geq t$ at which $x(\bar{t}) \in \mathcal{R}_2(\mu(t))$, unless a zoom-in occurs earlier. Since $\mathcal{R}_2(\mu(t))$ is also invariant and contained inside the ball

$$\left\{x : |x| \leq \left(\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - 2\Delta\right)\mu(t)\right\}$$

we conclude from (3), (12), and the definition of ℓ_{in} in (17) that a zoom-in must happen prior to time $\bar{t} + T_{\text{in}}$. □

Proof of Lemma IV.4: Let $\varepsilon > 0$ be given. Find a positive integer k such that $\bar{\mu} := \Omega_{\text{in}}^k\Omega_{\text{out}}\mu_0$ satisfies $\mathcal{R}_1(\bar{\mu}) \subset \{x : |x| \leq \varepsilon\}$. Since $\sqrt{\lambda_{\min}(P)}(M - 2\Delta) > \sqrt{\lambda_{\max}(P)}\|PBK\|\Delta$ by virtue of (13), there exists a $\delta_d > 0$ such that $\sqrt{\lambda_{\min}(P)}(M - 2\Delta)\bar{\mu} > \sqrt{\lambda_{\max}(P)}(\|PBK\|\Delta\bar{\mu} + \|PD\|\delta_d)$. Next, take a $\delta_x > 0$ such that $\bar{x} := e^{\|A\|(T_{\text{c}}+kT_{\text{in}})}(\delta_x + \|D\|\delta_d)$ satisfies $|\bar{x}| \leq \Omega_{\text{in}}^{k-1}\mu_0\Delta_0$ and $\sqrt{\lambda_{\max}(P)}|\bar{x}| < \sqrt{\lambda_{\min}(P)}(M - 2\Delta)\bar{\mu}$. Then the statement of the lemma holds with $\delta := \min\{\delta_x, \delta_d\}$ and $t_2 := t_0 + T_{\text{c}} + kT_{\text{in}}$. Indeed, the previous inequalities guarantee the occurrence of the capture event at time $t_0 + T_{\text{c}}$ (i.e., $t_1 = t_0 + T_{\text{c}}$) followed by k zoom-ins at times $t_0 + T_{\text{c}} + T_{\text{in}}, \ldots, t_0 + T_{\text{c}} + kT_{\text{in}}$, while x satisfies $\dot{x} = Ax + Dd$ (because $q_\mu(x) = 0$ thanks to Assumption II.1) and hence remains in $\mathcal{R}_1(\bar{\mu}) = \mathcal{R}_1(\mu(t_2))$ for $t \in [t_0, t_2]$ due to (64). □

Proof of Lemma IV.5: We consider two cases. If $k_{2i+1} - k_{2i} = 1$, then

$$|x_k| \leq \|\Phi\|\,|x_{k_{2i}}| + |w_{k_{2i}}| \qquad k \in [k_{2i}, k_{2i+1}] \tag{66}$$

since $\|\Phi\| > 1$. Suppose now that $k_{2i+1} - k_{2i} > 1$. In this case we can write that with ξ_k given by (26),

$$|\xi_{k_{2i}+1}| = \left|\frac{\Phi x_{k_{2i}} + w_{k_{2i}}}{\Omega_{\text{out}}\left(\mu_{k_{2i}} + 1\right)}\right| \leq \|\Phi\|\,|x_{k_{2i}}| + |w_{k_{2i}}|. \tag{67}$$

Moreover, we can also write for all $k \in [k_{2i} + 1, k_{2i+1} - 1]$ that the following holds:

$$|\xi_{k+1}| \leq \left|\frac{\Phi x_k + w_k}{\Omega_{\text{out}}(\mu_k + 1)}\right| \leq \frac{\|\Phi\|}{\Omega_{\text{out}}}|\xi_k| + \frac{|w_k|}{\mu_k}.$$

This implies that for all $k \in [k_{2i} + 1, k_{2i+1}]$ we have

$$|\xi_k| \leq \left(\frac{\|\Phi\|}{\Omega_{\text{out}}}\right)^{k-k_{2i}-1}|\xi_{k_{2i}+1}| + \sum_{j=k_{2i}+1}^{k-1}\left(\frac{\|\Phi\|}{\Omega_{\text{out}}}\right)^{k-1-j}|v_j|$$

where $v_j := w_j/\mu_j$ and, since $\mu_j \geq \sum_{s=1}^{j-k_{2i}}\Omega_{\text{out}}^s$, we can write

$$|v_j| := \frac{|w_j|}{\mu_j} \leq \frac{|w_j|}{\sum_{s=1}^{j-k_{2i}}\Omega_{\text{out}}^s}.$$

Using (67), the fact that $\|\Phi\|/\Omega_{\text{out}} < 1$ and that as $j \to \infty$ we have $v_j \to 0$, we conclude that eventually we must have $|\xi_k| \leq \ell_{\text{in}}$ and, hence, $k_{2i+1} - k_{2i} - 1$ is bounded. Moreover, we can write for some continuous, nondecreasing, and bounded function $\tilde{\varphi}$ that

$$\begin{aligned}k_{2i+1} - k_{2i} - 1 &\leq \tilde{\varphi}\left(|\xi_{k_{2i}+1}|, \|w\|_{[k_{2i}+1, k_{2i+1}-1]}\right)\\ &\leq \tilde{\varphi}\left(\|\Phi\|\,|x_{k_{2i}}| + |w_{k_{2i}}|, \|w\|_{[k_{2i}+1, k_{2i+1}-1]}\right)\\ &\leq \varphi\left(|x_{k_{2i}}|, \|w\|_{[k_{2i}, k_{2i+1}-1]}\right).\end{aligned}$$

Note that we can let $\varphi(0, 0) = 0$ since if $x_{k_{2i}+1} = 0$, then $k_{2i+1} - k_{2i} = 1$. Hence, we can find $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ so that (59) holds. Note also that there exist $\tilde{\rho}_1, \tilde{\rho}_2 \in \mathcal{K}_\infty$ such that for all $k \in [k_{2i}+1, k_{2i+1}]$ we have

$$\begin{aligned} |x_k| \le & \|\Phi\|^{k-k_{2i}-1}|x_{k_{2i}+1}| + \sum_{j=k_{2i}+1}^{k-1-i} \|\Phi\|^{k-1-j}|w_j| \\ \le & \|\Phi\|^{k_{2i+1}-k_{2i}-1}\left|x_{k_{2i}+1}\right| \\ & + \frac{\|\Phi\|^{k_{2i+1}-k_{2i}-2}-1}{\|\Phi\|-1}\|w\|_{[k_{2i}+1,k_{2i+1}-1]} \\ \le & \|\Phi\|^{\varphi_1(|x_{k_{2i}+1}|)+\varphi_2(\|w\|_{[k_{2i}+1,k_{2i+1}-1]})}\left|x_{k_{2i}+1}\right| \\ & + \frac{\|\Phi\|^{\varphi_1(|x_{k_{2i}+1}|)+\varphi_2(\|w\|_{[k_{2i}+1,k_{2i+1}-1]})-2}-1}{\|\Phi\|-1} \\ & \times \|w\|_{[k_{2i}+1,k_{2i+1}-1]} \\ \le & \tilde{\rho}_1\left(|x_{k_{2i}+1}|\right) + \tilde{\rho}_2\left(\|w\|_{[k_{2i}+1,k_{2i+1}-1]}\right). \end{aligned} \tag{68}$$

Finally, using (66) and (68) we have that (60) holds. □

Proof of Lemma IV.6: Recall that $k_0 = 0$ and $\Omega_{-1} = \Omega_{\text{out}}$ is used to initialize the system. Hence, we have

$$\begin{aligned} \mu_{k_{2i+1}} &= \Omega_{\text{out}}^{k_{2i+1}-k_{2i}}\mu_{k_{2i}} + \sum_{j=1}^{k_{2i+1}-k_{2i}} \Omega_{\text{out}}^{j} \\ &\le \Omega_{\text{out}}^{1+\varphi_1+\varphi_2}\mu_{k_{2i}} + \frac{\Omega_{\text{out}}^{\varphi_1+\varphi_2+1}-1}{\Omega_{\text{out}}-1} \\ &=: \rho_\mu^{\text{out}}\left(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w\|_{[k_{2i},k_{2i+1}-1]}\right) \end{aligned}$$

where we used (59). □

Proof of Lemma IV.7: Note that for all $k \in [k_{2i+1}, k_{2i+2}]$ we have by construction

$$|x_k| \le M\mu_k \quad \left|q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k}\right| \le \Delta$$

and the system evolves according to

$$\begin{aligned} x_{k+1} &= (\Phi + \Gamma K)x_k + \Gamma K\mu_k\left(q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k}\right) + w_k \\ \mu_{k+1} &= \Omega_{\text{in}}\mu_k. \end{aligned}$$

This is a cascade of an ISS system and a GAS system and, hence, the conclusion follows immediately. □

Proof of Lemma IV.8: In order to obtain the desired bound, we consider two cases.

- Case 1) $|x_{k_{2i+1}}| \ge \|w\|_{[k_{2i+1},k_{2i+2}-1]}$.
- Case 2) $|x_{k_{2i+1}}| \le \|w\|_{[k_{2i+1},k_{2i+2}-1]}$.

Case 1: Let $\rho_x > 0$ be arbitrary and introduce

$$T_x^* := \max\left\{\left\lceil \ln\left(\frac{\rho_x}{M}\left|x_{k_{2i+1}}\right|\right)(\ln(\Omega_{\text{in}}))^{-1}\right\rceil, 0\right\}.$$

Hence, for all $k \ge k_{2i+1} + T_x^*$ we have

$$\begin{aligned} |x_k| &\le M\mu_k \le M\Omega_{\text{in}}^{k-k_{2i+1}}\mu_{k_{2i+1}} \le M\Omega_{\text{in}}^{T_x^*}\mu_{k_{2i+1}} \\ &\le \rho_x\mu_{k_{2i+1}}\left|x_{k_{2i+1}}\right| =: \chi_1^x\left(\mu_{k_{2i+1}}, \left|x_{k_{2i+1}}\right|\right). \end{aligned}$$

On the other hand, using Assumption II.1 we have that there exists an L_q such that $|q(z)| \le L_q|z|$ for all z. Hence, for $k \in [k_{2i+1}, k_{2i+1} + T_x^*]$ we have that $|x_{k+1}| \le H|x_k| + |w_k|$ with $H := (\|\Phi\| + \|\Gamma K\|L_q)$, which implies that for all $k \in [k_{2i+1}, k_{2i+1} + T_x^*]$,

$$|x_k| \le \frac{H^{T_x^*}-1}{H-1}\left|x_{k_{2i+1}}\right| =: \chi_2^x\left(\left|x_{k_{2i+1}}\right|\right)$$

since $|x_{k_{2i+1}}| \ge \|w\|_{[k_{2i+1},k_{2i+2}-1]}$. Using arguments similar to those in the proof of Proposition III.4, we conclude that

$$|x_k| \le \chi^x\left(\mu_{k_{2i+1}}, \left|x_{k_{2i+1}}\right|\right) \qquad k \in [k_{2i+1}, k_{2i+2}]$$

where $\chi^x(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Case 2: The proof of this case follows exactly the same steps as the proof of Case 1 with the following modification. We let $\rho_w > 0$ be arbitrary and introduce

$$T_w^* := \max\left\{\left\lceil \ln\left(\frac{\rho_w}{M}\|w\|_{[k_{2i+1},k_{2i+2}-1]}\right)(\ln\Omega_{\text{in}})^{-1}\right\rceil, 0\right\}.$$

With this change, we can conclude that

$$|x_k| \le \chi^w\left(\mu_{k_{2i+1}}, \|w\|_{[k_{2i+1},k_{2i+2}-1]}\right) \qquad k \in [k_{2i+1}, k_{2i+2}]$$

where $\chi^w(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$. The conclusion of the lemma follows by defining $\rho_x^{\text{in}}(\mu, s, p) := \chi^x(\mu, s) + \chi^w(\mu, p)$ and noting that χ^x and χ^w are nondecreasing in μ. □

Proof of Lemma IV.9: The inequality $i < N - 1$ follows by definition of N. Note that by construction (see Corollary III.9), a zoom-out can occur after a zoom-in only if there exists a $k^* \in [k_{2i+1}, k_{2i+2} - 1]$ such that $\Delta_w^{-1}|w_{k^*}| \ge \mu_{k^*}$. Indeed, if $\Delta_w^{-1}|w_k| \le \mu_k$ for all k during a zoom-in, then we have from Corollary III.9 that $|\zeta_k| = |w_k/\mu_k| \le \Delta_w$ and hence $|x_k| \le M\mu_k$ for all k. Moreover, during a zoom-in we must have $|x_{k^*}| \le M\mu_{k^*}$, and also $\Delta_w^{-1}M|w_{k^*}| \ge |x_{k^*}|$. Using (61) with $s = k_{2i+2}$, $t = k^*$, we can obtain (62) with $\tilde{\gamma} = \overline{K}M\Delta_w^{-1} + \gamma + \Delta_w^{-1}$. □

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Daniel Liberzon (M'98–SM'04) was born in the former Soviet Union in 1973. He was a student in the Department of Mechanics and Mathematics at Moscow State University from 1989 to 1993 and received the Ph.D. degree in mathematics from Brandeis University, Waltham, MA, in 1998 (under the supervision of Prof. Roger W. Brockett of Harvard University).

Following a postdoctoral position in the Department of Electrical Engineering at Yale University, New Haven, CT, from 1998 to 2000, he joined the University of Illinois at Urbana-Champaign, where he is now an Associate Professor in the Electrical and Computer Engineering Department and an Associate Research Professor in the Coordinated Science Laboratory. His research interests include nonlinear control theory, analysis and synthesis of switched systems, control with limited information, and uncertain and stochastic systems. He is the author of the book *Switching in Systems and Control* (Boston, MA: Birkhäuser, 2003).

Dr. Liberzon served as an Associate Editor on the IEEE Control Systems Society Conference Editorial Board in 1999–2000. He received the IFAC Young Author Prize and the NSF CAREER Award, both in 2002.

Dragan Nešić (M'01–SM'02) received the B.E. degree from The University of Belgrade, Belgrade, Yugoslavia, in 1990, and the Ph.D. degree from Systems Engineering, RSISE, Australian National University, Canberra, Australia, in 1997.

He is currently a Professor in the Department of Electrical and Electronic Engineering (DEEE) at The University of Melbourne, Australia. In the period of 1997–1999 he held postdoctoral positions at DEEE, The University of Melbourne, Australia; CESAME, Université Catholique de Louvain, Louvain la Neuve, Belgium; and ECE, University of California, Santa Barbara, CA, USA. Since February 1999 he has been with The University of Melbourne. His research interests include networked control systems, discrete-time, sampled-data and continuous-time nonlinear control systems, input-to-state stability, extremum seeking control, applications of symbolic computation in control theory, and so on.

In 2003, Dr. Nešić was awarded a Humboldt Research Fellowship funded by the Alexander von Humboldt Foundation, Germany. Currently, he is an Australian Professorial Fellow (2004–2009), which is a research position funded by the Australian Research Council. He is a Fellow of IEAust. He is an Associate Editor for the journals *Automatica*, the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and *Systems and Control Letters*.