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A small-gain approach to stability analysis of hybrid systems

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Abstract— **We propose to use ISS small-gain theorems to analyze stability of hybrid systems. We demonstrate that the small-gain analysis framework is very naturally and generally applicable in the context of hybrid systems, and thus has a potential to be useful in many applications. The main idea is illustrated on specific problems in the context of control with limited information, where it is shown to provide novel interpretations, powerful extensions, and a more unified treatment of several previously available results. The reader does not need to be familiar with ISS or small-gain theorems to be able to follow the paper.**

I. Introduction

The small-gain theorem is a classical tool for analyzing input-output stability of feedback systems; see, e.g., [1]. More recently, small-gain tools have been used extensively to study feedback interconnections of nonlinear state-space systems in the presence of disturbances; see, e.g., [2].

Hybrid systems can be naturally viewed as feedback interconnections of simpler subsystems. For example, every hybrid system can be regarded as a feedback interconnection of its continuous and discrete dynamics. This makes small-gain theorems a very natural tool to use for studying internal and external stability of hybrid systems. However, we are not aware of any systematic application of this idea in the hybrid systems literature.

The purpose of this paper is to bring the small-gain analysis method to the attention of the hybrid systems community. We review, in a leisurely tutorial fashion, the concept of input-to-state stability (ISS) introduced by Sontag [3] and a general nonlinear small-gain theorem based on this concept. We then illustrate the power of this approach by treating several specific problems from the area of hybrid control with communication constraints. We demonstrate how the small-gain analysis provides insightful interpretations of existing results, immediately leads to generalizations, and allows a unified treatment of problems that so far have been studied separately.

The case studies presented below are rooted in the previous work by the authors [4], [5], [6]. Due to the pervasive nature of hybrid systems in applications, we expect that the main idea proposed in this paper will be useful in many other areas as well.

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A. Hybrid system model

We begin by describing the model of a hybrid system to which our subsequent results will apply. This model easily fits into standard modeling frameworks for hybrid systems (see, e.g., [7], [8]), and the reader can consult these references for background and further technical details.

We label the hybrid system to be defined below as $\mathcal{H}$. The *state variables* of $\mathcal{H}$ are divided into continuous variables $x \in \mathbb{R}^n$, discrete variables $\mu \in \mathbb{R}^r$, and additional variables $\tau \in \mathbb{R}^l$. We note that μ takes discrete values, which we embed in $\mathbb{R}^r$ just for simplicity. The variables τ represent auxiliary states thought of primarily as continuous clocks. The *time* is continuous: $t \in [t_0, \infty)$. We also consider *external variables* $w \in \mathbb{R}^s$, viewed as disturbances.

The *state dynamics* describing the evolution of these variables with respect to time are composed of *continuous evolution* and *discrete events*. During continuous evolution (i.e., while no discrete events occur), μ is held constant, x satisfies the ordinary differential equation $\dot{x} = f(x, \mu, w)$ with $f : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ locally Lipschitz, and τ satisfies $\dot{\tau}_i = 1$, $i = 1, \dots, l$. We now describe the discrete events. Given an arbitrary time t, we will denote by $x^-(t)$, or simply by x^- when the time arguments are omitted, the quantity $x(t^-) = \lim_{s \nearrow t} x(s)$, and similarly for the other state variables. Consider a *guard map* $G : \mathbb{R}^{n+r+l} \to \mathbb{R}^p$ (where p is a positive integer) and a *reset map* $R : \mathbb{R}^{n+r+l} \to \mathbb{R}^{n+r+l}$. The discrete events are defined as follows: whenever

$$G\big(x^-, \mu^-, \tau^-\big) \geq 0 \tag{1}$$

(component-wise), we let

$$\begin{pmatrix} x \\ \mu \\ \tau \end{pmatrix} = R(x^-, \mu^-, \tau^-) = \begin{pmatrix} R_x(x^-, \mu^-, \tau^-) \\ R_\mu(x^-, \mu^-, \tau^-) \\ R_\tau(x^-, \mu^-, \tau^-) \end{pmatrix}.$$

By construction, all signals are right-continuous.

Some remarks on the above relations are in order. In many situations, $R_x(x, \mu, \tau) \equiv x$, i.e., the continuous state does not jump at the event times. We want inequality rather than equality in (1) because for a discrete event to occur, we might need several conditions which do not become valid simultaneously (e.g., some relation between x and μ holds *and* a clock has reached a certain value). Of course, equality conditions are easily described by pairs of inequalities. The continuous time t does not explicitly appear in the dynamics. If desired, it could be incorporated either in x (with equation $\dot{t} = 1$) or in τ, and in either case it is not reset at event times. For simplicity, we assume that the disturbances w affect only the continuous evolution and the auxiliary variables τ affect only the discrete events, because

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this will be the case in the examples studied below. Discrete events can in general occur completely asynchronously.

Well-posedness (existence and uniqueness of solutions) of the hybrid system $\mathcal{H}$ is an issue; see, e.g., [7]. At the general level of the present discussion, we are going to assume it. For example, we can assume that the use of auxiliary variables (clocks) in the reset condition (1) ensures that a bounded number of discrete events occurs in any bounded time interval. Then, to obtain a solution (in the sense of Carathéodory), we simply flow the continuous dynamics until either the end of their domain is reached (finite escape) or a discrete event occurs; in the latter case, repeat from the new state, and so on. This construction will apply in all the examples treated below.

B. Feedback interconnection structure

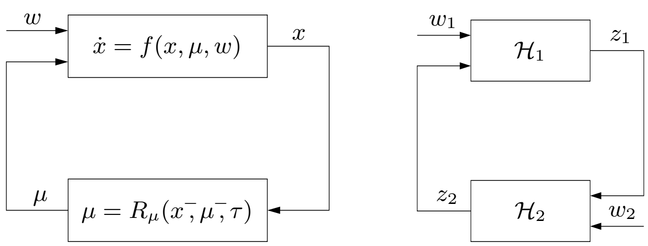

Fig. 1. Hybrid system viewed as feedback interconnection: (a) simple decomposition, (b) general decomposition

The starting point for our results is the observation that we can view the hybrid system $\mathcal{H}$ as a feedback interconnection of its continuous and discrete parts, as shown in Figure 1(a). The auxiliary variables τ are available to the discrete subsystem (because they are needed to determine the event times and execute resets) and possibly also available to the continuous subsystem. We do not display their dynamics explicitly in the picture because we will not be concerned about their behavior.

It is clear that the above is just one possible way to split the hybrid system $\mathcal{H}$ into a feedback interconnection of two subsystems. There may be many other ways to do it. Each subsystem in the decomposition can be continuous, discrete, or hybrid, and may be affected by the disturbances. This more general situation is illustrated in Figure 1(b). Here, the state variables and the external signals of $\mathcal{H}$ are split as $x = (x_1, x_2)$, $\mu = (\mu_1, \mu_2)$, $w = (w_1, w_2)$, the first subsystem $\mathcal{H}_1$ has states $z_1 = (x_1, \mu_1)$ and inputs $v_1 = (z_2, w_1)$, and the second subsystem $\mathcal{H}_2$ has states $z_2 = (x_2, \mu_2)$ and inputs $v_2 = (z_1, w_2)$. As before, we can actually omit some variables that are not of immediate interest from the states of the feedback interconnection (these variables would still be used in establishing the desired properties of the two components). Generalization to the case of partial measurements (outputs) is also straightforward.

Coming up with an appropriate decomposition of the above kind is the first step in the analysis of a given hybrid system $\mathcal{H}$. As we pointed out, such a decomposition always exists. It can also happen that the hybrid system model is given from the beginning as an interconnection of several hybrid systems. Thus the structure we consider is very general and not at all restrictive.

C. Stability definitions

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ (which we write as $\alpha \in \mathcal{K}$) if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of *class* $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$). A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to zero as $t \to \infty$ for each fixed $r \geq 0$.

We now define the stability notions of interest. Consider a hybrid system with state z and input v (as a special case, it can have only continuous dynamics or only discrete events). Following [3], we say that this system is *input-to-state stable* (ISS) with respect to v if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $z(t_0)$ and every input $v(\cdot)$ the corresponding solution satisfies

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma(\|v\|_{[t_0,t]}) \qquad (2)$$

for all $t \geq t_0$, where $\|v\|_{[t_0,t]} := \sup\{|v(s)| : s \in [t_0, t]\}$ (except possibly on a set of measure 0). We will refer to γ as an *ISS gain function*, or just a *gain* if clear from the context. For time-invariant systems, we can take the initial time to be 0 without loss of generality. In the case of no inputs ($v \equiv 0$), the above inequality reduces to $|z(t)| \leq \beta(|z(t_0)|, t)$ $\forall\, t \geq t_0$ which corresponds to the standard notion[1] of *global asymptotic stability* (GAS). In the presence of inputs, ISS captures the property that bounded inputs and inputs converging to 0 produce states that are also bounded and converging to 0, respectively. If the inputs are split as $v = (v_1, v_2)$, then (2) is equivalent to

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma_1(\|v_1\|_{[t_0,t]}) + \gamma_2(\|v_2\|_{[t_0,t]})$$

for some functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. In this case, we will call γ_1 the ISS gain from v_1 to z, and so on.

We note that asymptotic stability of a *linear* system (continuous or sampled-data) can always be characterized by a class $\mathcal{KL}$ function of the form $\beta(r, t) = cre^{-\lambda t}$, $c, \lambda > 0$. Moreover, an asymptotically stable linear system is automatically ISS with respect to external inputs, with a linear ISS gain function: $\gamma(r) = ar$, $a > 0$.

II. ISS SMALL-GAIN THEOREM

Consider the hybrid system $\mathcal{H}$ defined in Section I-A, and suppose that it has been represented as a feedback interconnection of two subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ in the way described earlier and shown in Figure 1(b). The small-gain theorem stated next reduces the problem of verifying ISS of $\mathcal{H}$ to that of verifying ISS of $\mathcal{H}_1$ and $\mathcal{H}_2$ and checking a condition that relates their respective ISS gains. The result we give is a special case of the small-gain theorem

[1] This can be equivalently restated in the more classical ε–δ style [10].

from [2]. That paper treats continuous systems, but since the statement and the proof given there involve only properties of system signals, the fact that the dynamics are hybrid in our case does not change the validity of the argument. We note that the result presented in [2] is much more general in that it treats partial measurements (input-to-output-stability, in conjunction with detectability) and deals with practical stability notions. Many other versions are also possible, e.g., we can replace the sup norm used in (2) by an L_p norm [6].

Theorem 1 *Suppose that:*
1. $\mathcal{H}_1$ is ISS with respect to $v_1 = (z_2, w_1)$, with gain γ_1 from z_2 to z_1.
2. $\mathcal{H}_2$ is ISS with respect to $v_2 = (z_1, w_2)$, with gain γ_2 from z_1 to z_2.
3. There exists a function $\rho \in \mathcal{K}_\infty$ such that[2]

$$(id + \rho) \circ \gamma_1 \circ (id + \rho) \circ \gamma_2(r) \le r \qquad \forall\, r \ge 0. \tag{3}$$

Then $\mathcal{H}$ is ISS with respect to the input $w = (w_1, w_2)$.

Three special cases are worth mentioning explicitly, because they will arise in what follows. First, in the case of no external signals ($w_1 = w_2 \equiv 0$), we conclude that $\mathcal{H}$ is GAS. Second, when the two ISS gain functions are linear: $\gamma_i(r) = c_i r$, $i = 1, 2$, the small-gain condition (3) reduces to the simple one $c_1 c_2 < 1$. Third, the theorem covers the case of a cascade connection, where one of the gains is 0 and hence the small-gain condition (3) automatically holds.

As already mentioned, we will sometimes concentrate only on some states of the overall system, excluding the other states from the feedback interconnection. Typically, these "hidden" states have very simple dynamics and remain bounded for all time. Theorem 1 is still valid if we let z_1 and z_2 include only the states of interest for each subsystem.[3]

Small-gain theorems have been widely used for analysis of continuous-time as well as discrete-time systems with feedback interconnection structure. The discussion of Section I-B suggests that it is also very natural to use this idea to analyze (internal or external) stability of hybrid systems. Of course, we will need to be able to prove that the subsystems in a given feedback decomposition satisfy suitable ISS properties, and calculate the ISS gains in order to check (3). There exist efficient tools for doing this, and the purpose of the next section is to discuss examples of application of these tools to a variety of hybrid systems arising in the context of control with limited information.

III. Examples of applications

A. Quantized feedback

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{4}$$

with A not Hurwitz. We assume that it is stabilizable, so that for some matrices $P = P^T > 0$ and K we have

$$(A + BK)^T P + P(A + BK) \le -2I. \tag{5}$$

We denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a symmetric matrix, respectively. A *quantizer* is a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$, for which there exist positive numbers M (the *range* of q, possibly ∞) and Δ (the *quantization error*) satisfying

$$|z| \le M \Rightarrow |q(z) - z| \le \Delta. \tag{6}$$

It is well known that quantization errors in general destroy asymptotic stability, in the sense that the quantized feedback law $u = Kq(x)$ is no longer stabilizing. To overcome this problem, we will use quantized measurements of the form

$$q_\mu(x) := \mu q(x/\mu), \qquad \mu > 0. \tag{7}$$

The quantizer q_μ has range $M\mu$ and quantization error $\Delta\mu$. The "zoom" variable μ will be the discrete variable of the hybrid closed-loop system, initialized at some fixed value. The idea behind achieving asymptotic stability is to "zoom in", i.e., decrease μ to 0 in a suitable discrete fashion, while applying the feedback law $u = Kq_\mu(x)$. To simplify the exposition, we will assume that the condition $|x| \le M\mu$ always holds, i.e., x always remains within the range of q_μ. This is automatically true if M is infinite. For finite M, this can be achieved by incorporating an initial "zooming-out" scheme and subsequently ensuring that the condition is never violated (see [4] for details).

We view the closed-loop system as the feedback interconnection of Figure 1(a), with $w \equiv 0$ for the time being. Its continuous dynamics are

$$\dot{x} = Ax + BKq_\mu(x) = (A+BK)x + BK\mu\big(q(x/\mu) - x/\mu\big).$$

In view of (6) and the fact that $A+BK$ is Hurwitz, we have ISS with respect to μ. Let us use the Lyapunov function $V(x) := \frac{1}{2}x^T P x$, with P from (5), to compute the ISS gain. Its derivative along solutions satisfies $\dot{V} \le -|x|^2 + |x|\|PBK\|\Delta\mu$, where $\|\cdot\|$ stands for matrix induced norm. Simple square completion shows that for each $\varepsilon > 0$,

$$|x|^2 \ge (1+\varepsilon)^2 \|PBK\|^2 \Delta^2 \mu^2 \quad \Rightarrow \quad \dot{V} \le -\varepsilon |x|^2/(1+\varepsilon).$$

The condition (16) of Lemma 1 from the Appendix holds with $\rho_x(r) := \lambda_{\max}(P)(1+\varepsilon)^2\|PBK\|^2\Delta^2 r^2$, and hence the x-subsystem is ISS with respect to μ, with gain

$$\gamma_x(r) := \sqrt{\lambda_{\max}(P)}(1+\varepsilon)\|PBK\|\Delta r/\sqrt{\lambda_{\min}(P)}. \tag{8}$$

We now need to describe a scheme for updating μ, which we refer to as a *quantization protocol*. The goal is to guarantee ISS of the μ-subsystem with respect to x, with the ISS gain γ_μ such that the small-gain condition of Theorem 1 holds. Pick a number C satisfying

$$C > \sqrt{\lambda_{\max}(P)}\|PBK\|\Delta/\sqrt{\lambda_{\min}(P)}. \tag{9}$$

[2] Here id is the identity function and $\circ$ denotes function composition. If one replaces $\beta + \gamma$ with $\max\{\beta, \gamma\}$ in the definition (2) of ISS, then (3) can be simplified to $\gamma_1 \circ \gamma_2(r) < r$ for all $r > 0$.

[3] This amounts to replacing ISS with an input-to-output stability notion (cf. [2], [11]) and requiring that the ISS gain from the hidden states in each subsystem to the states of interest in the other subsystem be 0.

Define the guard map by $G(x,\mu,\tau) := ((C+\Delta)\mu - |q_\mu(x)|,\, \tau-\delta)^T$ for some $\delta > 0$, where the auxiliary clock variable τ is scalar-valued. Define the reset map by $R(x,\mu,\tau) := (x,\, \Omega_i\mu,\, 0)^T$ for some $\Omega_i \in (0,1)$. In other words, when $|q_{\mu^-}(x^-)| \leq (C+\Delta)\mu^-$ and $\tau^- \geq \delta$, we set $\mu = \Omega_i\mu^-$ ("zoom in") and $\tau = 0$ (reset the clock). In view of (6), it is easy to see that the condition (17) of Lemma 2 from the Appendix is satisfied with $W(\mu) := \mu^2$ and $\rho_\mu(r) := r^2/C^2$. The other hypotheses of that lemma are also satisfied by construction. Therefore, the μ-subsystem is ISS with respect to x, with gain $\gamma_\mu(r) := r/C$. We see that both gain functions are linear gains, and to apply Theorem 1 we need their product to be smaller than 1. Since ε in (8) can be arbitrarily small, the small-gain condition is exactly (9).

This quantization protocol has a clear geometric interpretation. We zoom in if the quantized measurements show that $|x| \leq (C+2\Delta)\mu$, which is guaranteed to happen whenever $|x| \leq C\mu$. The condition (9) means that for each μ, the ball of radius $C\mu$ around the origin contains the level set of V superscribed around the ball of radius $\|PBK\|\Delta\mu$, outside of which V is known to decay (thus ensuring that the zoom-in will be triggered). Similar ideas were used in [12], [4], but previous analyses did not employ the small-gain argument and were arguably less transparent.

We remark that the choice of precise values for C and Ω_i is also dictated by the need to keep x within the range of q_μ. Since these considerations seem to be decoupled from the small-gain argument, we refer the reader to [4] for details. Several variations of the above scheme are also possible (cf. [12], [13]), and can be analyzed similarly. Nonlinear quantized control systems can be treated along the same lines, under the assumption of ISS of the continuous closed-loop dynamics with respect to measurement errors. In particular, the quantization protocol for nonlinear systems proposed in [4] lends itself to an analogous small-gain interpretation, except that the ISS gains are nonlinear and so the general small-condition (3) must be used.

A very important advantage of the small-gain viewpoint is that it allows us to establish robustness (in the ISS sense) with respect to external disturbances. This aspect has not been addressed in earlier work and seems to be much more difficult to handle with previously used tools. In the case of infinite quantizer range M, the small-gain approach makes the extension immediate. Namely, if we augment the system with an external disturbance w to have $\dot{x} = Ax + Bu + Dw$, then Theorem 1 yields ISS of the closed-loop system described above with respect to w. If the quantizer range M is finite, then the situation is much more complicated because it is necessary to "zoom out" to keep x within the range of q_μ, and in the presence of the disturbance we will keep switching between the zooming-in and zooming-out stages. A solution to this problem is described in [13].

B. Encoded sampled-data feedback

Consider again the stabilizable linear system (4). As before, we are interested in the problem of designing a controller that asymptotically stabilizes this system using limited information about its state x. Here we specify what we mean by limited information as follows: (i) the measurements are to be received by the controller at discrete times δ, 2δ, $\ldots$, where $\delta > 0$ is a fixed sampling period; and (ii) at each of these sampling times, the measurement received by the controller must be a number in the set $\{0,1,2,\ldots,N\}$, where N is a fixed positive integer. For notational simplicity, we assume that $\sqrt[n]{N}$ is an integer. It will be convenient to use the norm $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$ on $\mathbb{R}^n$ and the induced matrix norm $\|A\|_\infty := \max\{\sum_{j=1}^n |A_{ij}| : 1 \leq i \leq n\}$. We define $\Lambda := \max_{0\leq t\leq\tau} \|e^{At}\|_\infty \geq 1$ and assume that

$$\Lambda < \sqrt[n]{N}. \tag{10}$$

To represent the above information constraint, we consider a quantizer q that partitions the cubic box centered at the origin in $\mathbb{R}^n$ with edges 2, i.e., $\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$, into N equal cubic quantization regions ($\sqrt[n]{N}$ in each dimension), and assigns numbers between 1 and N to these regions in a one-to-one fashion. The "overflow" symbol 0 is assigned to the complement of the overall box. Thus the quantized measurement $q(x)$ uniquely determines the quantization region containing x. We will use quantized measurements of the form $q_\mu(x-\hat{x})$, where q_μ is defined by (7), μ will be a discrete "zoom" variable similar to the one used in Section III-A, and $\hat{x}$ will be an estimate of x generated using previously received quantized measurements. The control law will be $u = K\hat{x}$, where K is a stabilizing feedback gain as in Section III-A.

The continuous states of the closed-loop system are x and $\hat{x}$. During continuous evolution, x satisfies $\dot{x} = Ax + BK\hat{x}$ and $\hat{x}$ is a solution of a "copy" of the same system: $\dot{\hat{x}} = A\hat{x} + BK\hat{x}$. Initially, we set $\hat{x}(0) := 0$. We also use a scalar auxiliary clock variable τ, initialized at 0 as well.

We now describe a "quantization protocol" that defines discrete events at which both μ and $\hat{x}$ (as well as τ) will be reset. We simplify matters by assuming that an upper bound on the size of the initial state is known: $\|x(0)\|_\infty \leq E_0$ for some $E_0 > 0$. Such a bound may be given to us in advance or may be obtained on the basis of quantized measurements by performing an initial "zoom-out"; see [5] for details. We set $\mu(0) := \Lambda E_0$. The guard map is simply $G(x,\hat{x},\mu,\tau) := \tau - \delta$, and the reset map is defined by $R(x,\hat{x},\mu,\tau) := (x,\, R(x,\hat{x},\mu),\, (\Lambda/\sqrt[n]{N})\mu,\, 0)^T$, where $R(x,\hat{x},\mu)$ is the center of the quantization region of $q_\mu(x-\hat{x})$ which, according to the quantized measurement, contains x. Thus discrete events occur at every sampling time: the values of μ and $\hat{x}$ determine the quantizer, the new value of $\hat{x}$ is computed based on the quantized measurement, and a "zoom-in" is executed (by virtue of (10), the value of μ is decreased at discrete events).

To analyze the resulting hybrid system, let us introduce the *estimation error*

$$e := x - \hat{x}. \tag{11}$$

Initially, we have $\|e(0)\|_\infty \leq E_0$. During continuous evolution, e satisfies $\dot{e} = Ae$. In view of the definitions of Λ and $\mu(0)$, we see that at the first event time $t = \delta$ we have $q_{\mu^-}(x^- - \hat{x}^-) \neq 0$. From the definition of the reset map for $\hat{x}$, it then follows that $\|e(\delta)\|_\infty \leq \Lambda E_0/\sqrt[n]{N}$. Repeating the analysis for subsequent event times, we deduce that e converges (exponentially) to 0. The final step is that in the (x, e) coordinates we can rewrite the closed-loop system as

$$\begin{aligned} \dot{x} &= (A + BK)x - BKe \\ \dot{e} &= Ae, \qquad\qquad\qquad t \neq k\delta,\ k = 1, 2, \ldots \\ e &= R_e(x^-, e^-, \mu^-), \qquad t = k\delta,\ k = 1, 2, \ldots \end{aligned}$$

This is a feedback connection of two subsystems: the e-subsystem, which in view of the above analysis is GAS (i.e., ISS with respect to x with 0 gain, uniformly over μ), and the x-subsystem, which is ISS with respect to e because $A+BK$ is Hurwitz. Therefore, GAS of the overall hybrid system follows from Theorem 1. The above analysis is essentially equivalent to the one given in [5], but the present hybrid model was not used there and the small-gain interpretation was not explicit.

A nonlinear extension of the control strategy from [5] was presented in [14]. For a nonlinear system $\dot{x} = f(x, u)$, we define the estimator equation $\dot{\hat{x}} = f(\hat{x}, u)$, which with the same definition of the estimation error via (11) gives $\dot{e} = f(x, u) - f(\hat{x}, u)$ during continuous evolution. Denoting by L the Lipschitz constant for the function f on some region $D \subset \mathbb{R}^n \times \mathbb{R}^m$, we obtain for solutions with (x, u) and $(\hat{x}, u)$ remaining in D the bound $\|e(t)\|_\infty \leq \xi(t)$, where ξ is the solution of the system

$$\begin{aligned} \dot{\xi} &= L\xi, \qquad\qquad t \neq k\delta,\ k = 1, 2, \ldots \\ \xi &= \xi^-/\sqrt[n]{N}, \qquad t = k\delta,\ k = 1, 2, \ldots \end{aligned}$$

with initial condition $\xi(0) = E_0$. Assume that $e^{L\tau} < \sqrt[n]{N}$. Then ξ converges to 0, hence so does e. Applying a control law $u = k(\hat{x})$, we obtain the x-subsystem $\dot{x} = f(x, k(\hat{x})) = f(x, k(x - e))$. Assume that the control law k renders this system ISS with respect to e. Then it is possible to define the region D in such a way that the above estimates are valid for all $t \geq 0$, and consequently Theorem 1 guarantees GAS of the overall hybrid system just as before (see [14]).

It is clear that many generalizations are possible within the small-gain framework. First, the ISS gain from x to e need not be 0, as long as the small-gain condition is satisfied. Second, external disturbances entering the continuous dynamics can be naturally incorporated. In this case, however, quantizer saturation ($q_\mu(x - \hat{x}) = 0$) can occur, and so a more sophisticated quantization protocol involving "zoom-outs" is required (cf. [13]).

C. Networked control systems

In this section we explain how results from [6] fit into a similar framework where the small gain theorem is an instrumental tool in proving the results. We first present a very simplified version of the problem considered in [6] to make a relationship with results of previous sections more apparent. Then, we discuss its various extensions.

Consider again the linear system (4) that satisfies (5). In the networked control system, control signals and state measurements need to be transmitted via a serial communication channel. The channel operates as follows: (i) at each transmission time δk, $k = 1, 2, \ldots$, where $\delta > 0$, a part of the control and/or state vector is transmitted; (ii) if some signal is not transmitted, then its value is kept constant at the latest value sent via the communication channel; and (iii) all values of state and control are kept constant between transmission instants. Let us denote the most recently transmitted values of the state and control respectively as $\hat{x}$ and $\hat{u}$. In order to analyze the system, it is useful to introduce the error $e := (x - \hat{x}, u - \hat{u})^T = (e_1, \ldots, e_\ell)^T$, where ℓ is referred to as the number of "nodes". We assume for the time being that whenever a node j gets transmitted at time $k\delta$, the corresponding part of the error vector is reset to zero, i.e., $e_j(k\delta) = 0$.

The closed loop system can be written as

$$\left.\begin{aligned} \dot{x} &= Ax + B\hat{u} \\ u &= K\hat{x} \\ \dot{\hat{u}} &= 0 \\ \dot{\hat{x}} &= 0 \end{aligned}\right\} \quad t \neq k\delta \tag{12}$$

$$\left.\begin{aligned} \hat{x} &= x^- + h_x(k, e^-) \\ \hat{u} &= u^- + h_u(k, e^-) \end{aligned}\right\} \quad t = k\delta \tag{13}$$

where h_x and h_u describe the "protocol" which is an algorithm that determines time scheduling of access to the network for different nodes. Explicit expressions for h_x and h_u are given in [6] for several examples of most commonly used protocols, such as token ring.

This model is more amenable to analysis if we represent it in (x, e) coordinates:

$$\left.\begin{aligned} \dot{x} &= A_{11}x + A_{12}e \\ \dot{e} &= A_{21}x + A_{22}e \end{aligned}\right\} \quad t \neq k\delta \tag{14}$$

$$e = h(k, e^-) \qquad t = k\delta \tag{15}$$

where the A_{ij} matrices can easily be computed from the above equations and $h(k, e) := (h_x(k, e), h_u(k, e))$. Hence, for this system we can define the guard map as $G(k, x, e, \tau) := \tau - \delta$ and the reset map as $R(k, x, e, \tau) := (k + 1,\, x,\, h(k, e),\, 0)^T$, where τ satisfies $\dot{\tau} = 1$ as before. Here the explicit dependence on the time step k is needed to cover protocols of round robin type, and usually we can replace k in the above formulas by k mod ℓ.

Similarly to Section III-B, we decompose the overall system into the continuous x-subsystem and the hybrid e-subsystem, and choose the design parameters so that the small-gain condition can be used to prove stability of the overall system. It is easy to show that $A_{11} = A+BK$ and by assumption this matrix is Hurwitz. Hence, the x-subsystem is ISS from e to x with some linear gain γ_x. Suppose now that there exist a function $W : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}$ and constants $L, c, a_1, a_2 > 0$ and $\lambda \in [0, 1)$ with the following properties:

(i) $a_1|e| \leq W(k,e) \leq a_2|e|$, $W(k+1, h(k,e)) \leq \lambda W(k,e)$ for all k and e;
(ii) $\langle \frac{\partial W}{\partial e}, A_{21}x + A_{22}e \rangle \leq LW + c|x|$;
(iii) $\delta \leq \frac{1}{L} \ln \frac{1}{\lambda}$.

Then, the e subsystem is ISS from x to e, with gain $\gamma_e := c(e^{L\delta}-1)/(La_1(1-\lambda e^{L\delta}))$. More importantly, the gain γ_e can be made arbitrarily small by reducing δ. Hence, under the above conditions there exists a value $\delta^* > 0$ such that for all $\delta \in (0, \delta^*)$ the small gain condition holds: $\gamma_x \gamma_e < 1$, and the networked control system is stable.

Condition (i) is a property of the protocol itself and protocols with this property were referred to as *uniformly globally exponentially stable* (UGES) protocols in [6]. It was shown there that token ring and the so-called try-once discard (TOD) protocols are UGES, and appropriate Lyapunov functions W were constructed for these two important cases. Moreover, it was shown for linear systems and for each of these two protocols that (ii) also holds, hence the networked control system is stable for values of the transmission interval δ small enough to fulfill (iii).

The results presented in [6] are more general in several different directions. Nonlinear plants and controllers were considered in [6] and controllers themselves were allowed to be dynamical systems. Moreover, systems with exogenous disturbances were considered and results were proved for $\mathcal{L}_p$ stability. In all of these generalizations, the small-gain approach was instrumental in proving the results.

The small-gain approach can be used to deal with even more general problems, where besides time scheduling one also uses coding schemes for transmitting information via the communication channel. Components of the error vector will then no longer be reset to zero, but rather will be divided by a certain number, much like in Section III-B. The reasoning sketched above, based on the UGES property of the protocol and an application of the small-gain theorem, is readily applicable to such scenarios as well. This clarifies the very close relationship between the quantization/coding literature and the work on networked control systems, and opens the door for a unified treatment of these subjects, which will be presented in a forthcoming paper.

Appendix

Here we give the technical lemmas needed for proving ISS in Section III-A. All lowercase Greek letters denote $\mathcal{K}_\infty$ functions. Consider the hybrid system $\mathcal{H}$ defined in Section I-A and shown in Figure 1(a). We limit ourselves to the case of no disturbances, and write the continuous dynamics as $\dot{x} = f(x,u)$. The two lemmas stated below provide Lyapunov-based sufficient conditions for ISS of the continuous and discrete dynamics, respectively. The first result is well known [3]; the second one is [15, Theorem 4].

Lemma 1 *Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $\alpha_{1,x}(|x|) \leq V(x) \leq \alpha_{2,x}(|x|)$ and*

$$V(x) \geq \rho_x(|\mu|) \Rightarrow \nabla V(x) f(x,\mu) \leq -\alpha_{3,x}(V(x)). \quad (16)$$

Then the x-subsystem is ISS with respect to μ, with gain $\gamma_x := \alpha_{1,x}^{-1} \circ \rho_x$.

Lemma 2 *Suppose there exists a C^1 function $W : \mathbb{R}^r \to \mathbb{R}$ satisfying $\alpha_{1,\mu}(|\mu|) \leq W(\mu) \leq \alpha_{2,\mu}(|\mu|)$ such that*

$$W(\mu) \geq \rho_\mu(|x|) \quad \Rightarrow \quad W(R_\mu(x,\mu,\tau)) - W(\mu) \leq -\alpha_{3,\mu}(W(\mu)) \quad (17)$$

and $W(\mu) \leq \rho_\mu(r)$, $|x| \leq r \Rightarrow W(R_\mu(x,\mu,\tau)) \leq \rho_\mu(r)$. Suppose also that there exist positive numbers δ_a and N_0 with the following property: for each $T > t_0$ such that $W(\mu(t)) \geq \rho_\mu(\|x\|_{[t_0,t]})$ for all $t \in [t_0, T)$, the number $N(T, t_0)$ of discrete events in the interval $[t_0, T]$ satisfies

$$N(T, t_0) \geq -N_0 + (T - t_0)/\delta_a. \quad (18)$$

Then the μ-subsystem is ISS with respect to x, with gain $\gamma_\mu := \alpha_{1,\mu}^{-1} \circ \rho_\mu$.

Remark 1 If the second inequality in (17) holds always, i.e., the hypotheses are satisfied with $\rho_\mu = 0$, then the proof shows that the μ-subsystem is GAS ($\gamma_\mu = 0$). □

References

[1] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems, part 1: conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Trans. Automat. Control*, vol. 11, pp. 228–238, 1966.
[2] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math. Control Signals Systems*, vol. 7, pp. 95–120, 1994.
[3] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, pp. 435–443, 1989.
[4] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.
[5] ——, "On stabilization of linear systems with limited information," *IEEE Trans. Automat. Control*, vol. 48, pp. 304–307, 2003.
[6] D. Nešić and A. R. Teel, "Input-output stability properties of networked control systems," *IEEE Trans. Automat. Control*, vol. 49, pp. 1650–1667, 2004.
[7] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. London: Springer, 2000.
[8] N. Lynch, R. Segala, and F. Vaandrager, "Hybrid I/O automata," *Information and Computation*, vol. 185, pp. 105–157, 2003.
[9] R. Goebel, J. P. Hespanha, A. R. Teel, C. Cai, and R. Sanfelice, "Hybrid systems: generalized solutions and robust stability," in *Proc. 6th IFAC Symp. on Nonlinear Control Systems (NOLCOS)*, 2004.
[10] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey: Prentice Hall, 2002.
[11] E. D. Sontag and Y. Wang, "Notions of input to output stability," *Systems Control Lett.*, vol. 38, pp. 235–248, 1999.
[12] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Control*, vol. 45, pp. 1279–1289, 2000.
[13] D. Liberzon and D. Nešić, "Input-to-state stabilization of linear systems with quantized feedback," in *Proc. 44th IEEE Conf. on Decision and Control*, 2005, submitted. [Online]. Available: http://decision.csl.uiuc.edu/~liberzon/publications.html
[14] D. Liberzon and J. P. Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Trans. Automat. Control*, vol. 50, pp. 910–915, 2005.
[15] J. P. Hespanha, D. Liberzon, and A. R. Teel, "On input-to-state stability of impulsive systems," in *Proc. 44th IEEE Conf. on Decision and Control*, 2005, submitted. [Online]. Available: http://decision.csl.uiuc.edu/~liberzon/publications.html
[16] Z. P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.