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Input-to-state stabilization of linear systems with quantized feedback

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***Abstract*— We consider the problem of achieving input-to-state stability (ISS) with respect to external disturbances for control systems with linear dynamics and quantized state measurements. Quantizers considered in this paper take finitely many values and have an adjustable "zoom" parameter. Building on an approach applied previously to systems with no disturbances, we develop a control methodology that counteracts an unknown disturbance by switching repeatedly between "zooming out" and "zooming in". Two specific control strategies that yield ISS are presented. The first one is implemented in continuous time, while the second one incorporates time sampling. We discover that in the presence of disturbances, time-sampling implementation requires an additional modification which has not been considered in previous work.**

I. Introduction

The subject of this paper is feedback control of linear continuous-time systems with quantized state measurements. Control problems of this kind are motivated by numerous applications where communication between the plant and the controller is limited due to capacity or security constraints. This is a very active and expanding research area [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

The starting point for this paper is the approach developed in [3], [9] (see also [12, Chap. 5]) which we now briefly recall. The quantizer is assumed to take a finite set of values and incorporates an adjustable "zoom" parameter. The control strategy is composed of two stages. The first, "zooming-out" stage consists in increasing the range of the quantizer until the state of the system can be adequately measured; at this stage, the system is open-loop. The second, "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. This results in a hybrid control law, in which discrete transitions are triggered by the values of a suitable Lyapunov function.

The method of [3], [9] was shown to achieve global asymptotic stability (GAS). The focus of the present work is on achieving robustness with respect to disturbances. We characterize the desired robustness by an ISS-like property (see [13]) which involves bounded nonlinear gains from the initial state and the supremum norm of the disturbance to the supremum norm of the state and also from the supremum limit of the disturbance to the supremum limit of the state. In [8], [5], state boundedness in the presence of bounded disturbances is achieved by using the knowledge of a disturbance bound. In [11], mean square stability is obtained by utilizing statistical information about the disturbance (a bound on its appropriate moment). In contrast to these works, here we assume the disturbance to be completely unknown to the controller.

Our first main result (Theorem 1 in Section II) is that the ISS property in the presence of disturbances can be achieved by extending the method of [3], [9]. An extension is necessary because an unknown disturbance may force the state outside the range of the quantizer after it has already been inside. Thus we develop a control strategy that switches multiple times between the zooming-out and zooming-in stages. This strategy is still Lyapunov-based, and its analysis is similar in spirit to that from [9] but is significantly more difficult. When no disturbances are present, the earlier stabilization result is recovered from our new result as a special case.

Next, we turn to the problem of achieving the same robustness property using sampled-data quantized feedback. Time-sampling implementation is important because it guarantees a finite data rate (cf. [7]) and exposes the issue of robustness with respect to time delays. We demonstrate that unless proper care is taken, the straightforward sampled-data adaptation of the continuous-time control strategy in general fails to provide ISS (Section III-B). We then proceed to describe a modified version of the zooming-out procedure which yields ISS in the time-sampling context, obtaining our second main result (Theorem 5 in Section III-C).

The proof of Theorem 5 sharply differs from that of Theorem 1 in that it does not use a Lyapunov function and instead is based entirely on trajectory analysis. Thus another principal contribution of this work is a novel alternative method for analyzing stability and robustness of quantized feedback control schemes (this method can be applied in continuous time as well). In particular, an important component of this time-based analysis consists in recognizing and utilizing a cascade structure[1] of the hybrid closed-loop system. Due to space constraints, the proofs are omitted and can be found in the full on-line version [15].

II. Lyapunov-based continuous-time approach

We consider the linear system

$$\dot{x} = Ax + Bu + Dd \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d \in \mathbb{R}^s$ is a disturbance (u and d are taken to be Lebesgue

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[1]This can be viewed as a special instance of the general small-gain approach to stability analysis of hybrid systems proposed in [14].

measurable and locally bounded). We assume that A is a nonzero, non-Hurwitz matrix. We assume this system is stabilizable, so there exist matrices K and $P = P^T > 0$ such that $A + BK$ is Hurwitz and

$$(A + BK)^T P + P(A + BK) \le -2I\,. \tag{2}$$

Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm, and $\|\cdot\|_J$ denotes the supremum norm of a signal on an interval J. For $x \in \mathbb{R}^n$, $\lceil x \rceil$ is the smallest integer $z \in \mathbb{N}$ such that $z \ge x$. A continuous function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\mathcal{K}_\infty$ if it is zero at zero, strictly increasing, and unbounded.

A *quantizer* is a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. We assume that there exist real numbers $M > \Delta > 0$ such that the following two conditions hold:

$$|z| \le M \Rightarrow |q(z) - z| \le \Delta \tag{3}$$

and

$$|z| > M \Rightarrow |q(z)| > M - \Delta\,. \tag{4}$$

The first condition gives a bound on the quantization error when the quantizer does not saturate, while the second one provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* and the *error* of the quantizer, respectively. We also assume that $q(x) = 0$ on some neighborhood of the origin:

Assumption 1 *There exists a number $\Delta_0 > 0$ such that for all $|z| \le \Delta_0$ we have $q(z) = 0$.*

We will be using the one-parameter family of quantizers

$$q_\mu(x) := \mu q\Big(\frac{x}{\mu}\Big), \qquad \mu > 0\,.$$

Here μ is an adjustable parameter, which can be viewed as a "zoom" variable. At each time t, the quantized measurement $q_{\mu(t)}(x(t))$ will represent the information about $x(t)$ which is available to the controller. This quantity takes on a finite number of values (equal to the cardinality of the set $\mathcal{Q}$). Geometrically, $\mathbb{R}^n$ is divided into a finite number of quantization regions (each corresponding to a fixed value of q) and the controller knows which of these regions contains the state x at every time. The variable μ is an adjustable parameter which we will vary in a discrete fashion in order to extract more information about the state (cf. [3], [9]).

The problem of interest is to design a quantized feedback control law and a scheme for updating μ to achieve the following goal: there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial condition $x_0 = x(t_0)$ and every bounded disturbance d we have

$$|x(t)| \le \gamma_1(|x_0|) + \gamma_2\left(\|d\|_{[t_0,\infty)}\right) \qquad \forall\, t \ge t_0 \tag{5}$$

and

$$\limsup_{t\to\infty} |x(t)| \le \gamma_3\Big(\limsup_{t\to\infty} |d(t)|\Big)\,. \tag{6}$$

We note that the gain functions $\gamma_1, \gamma_2, \gamma_3$ may depend on the choice of the initial value $\mu_0 = \mu(t_0)$ of the zoom variable μ, but do not depend on x_0 or d. Since the closed-loop dynamics will not explicitly depend on time t, all bounds will also be uniform with respect to the initial time t_0.

We know that for continuous systems of the form $\dot{x} = f(x, d)$, the property expressed by the two inequalities (5) and (6) is equivalent to *input-to-state stability* (ISS) with respect to d [13]. In the present case, the closed-loop system will be a hybrid system, as it will contain an additional discrete state μ. With some abuse of terminology, we will refer to the above property as ISS of continuous closed-loop dynamics.

This ISS property also implies that in the disturbance-free case ($d \equiv 0$), the origin is a GAS equilibrium of the continuous closed-loop dynamics (for a fixed μ_0). Thus we recover as a special case the property achieved by the algorithms developed in [3], [9] for the case of no disturbances. In fact, the algorithm presented next is a natural extension of the ones from [3], [9].

The hybrid closed-loop system will contain continuous states (states taking values in a continuum) and discrete states (states taking values in a discrete set). Both continuous and discrete states will be functions of the continuous time $t \in [t_0, \infty)$. The continuous variables will be comprised of the system state x and two auxiliary reset clock variables τ_{out} and τ_{in}, both initialized at 0. They will take values in the intervals $[0, T_{\text{out}}]$ and $[0, T_{\text{in}}]$, respectively, where $T_{\text{out}} \ge T_{\text{in}} > 0$.

The discrete variables will be comprised of the zoom variable μ and an auxiliary logical variable `capt`. The variable μ will be initialized at some $\mu_0 > 0$ and will take values in a discrete subset of $(0, \infty)$ which depends on μ_0. The variable `capt` will take values in the set {"yes", "no"} and will be initialized at "no"; it is needed to distinguish between the "capture" (open-loop) stage and the control (closed-loop) stage. The control law is defined by

$$u(t) = \begin{cases} 0 & \text{if } \texttt{capt} = \text{“no”} \\ Kq_{\mu(t)}(x(t)) & \text{if } \texttt{capt} = \text{“yes”} \end{cases}. \tag{7}$$

The state dynamics describing the evolution of the system variables with respect to time are composed of *continuous evolution* and *discrete events*. During continuous evolution (i.e., while no discrete events occur), μ is held constant, x satisfies (1) with u defined by (7), and the clocks satisfy

$$\dot{\tau}_{\text{in}} = \begin{cases} 1 & \text{if } \tau_{\text{in}} < T_{\text{in}} \\ 0 & \text{if } \tau_{\text{in}} = T_{\text{in}} \end{cases}, \qquad \dot{\tau}_{\text{out}} = \begin{cases} 1 & \text{if } \tau_{\text{out}} < T_{\text{out}} \\ 0 & \text{if } \tau_{\text{out}} = T_{\text{out}} \end{cases}.$$

We now describe the discrete events. Given an arbitrary time t, we will denote by $\mu^-(t)$, or simply by μ^- when the time arguments are omitted, the quantity $\lim_{s\nearrow t} \mu(s)$, and similarly for all other variables. All system variables will be continuous from the right by construction. Let numbers $\Omega_{\text{out}} > 1$, $\Omega_{\text{in}} \in (0, 1)$, $T_{\text{c}} \in (0, T_{\text{out}}/2)$, and $\ell_{\text{out}} > \ell_{\text{in}} > 0$ be given. The discrete events are of three types. They are governed by the following rules, which we write in the form "if <conditions> then <actions>". The conditions are mutually exclusive and are checked continuously in time.

Zoom-out: If

$$\begin{aligned}(\tau_{\text{out}}^- = T_{\text{out}} \text{ and } \texttt{capt}^- = \text{"no"}) \quad \text{or} \\ (|q_{\mu^-}(x)| \geq \ell_{\text{out}}\mu^- \text{ and } \texttt{capt}^- = \text{"yes"})\end{aligned} \tag{8}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and $\tau_{\text{out}} = 0$.

Capture: If

$$|q_{\mu^-}(x)| \leq \ell_{\text{out}}\mu^-,\ \tau_{\text{out}}^- \in [T_c, T_{\text{out}} - T_c] \text{ and } \texttt{capt}^- = \text{"no"} \tag{9}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and $\texttt{capt}$="yes".

Zoom-in: If

$$|q_{\mu^-}(x)| \leq \ell_{\text{in}}\mu^-,\ \tau_{\text{out}}^- = \tau_{\text{in}}^- = T_{\text{in}} \text{ and } \texttt{capt}^- = \text{"yes"} \tag{10}$$

then let $\mu = \Omega_{\text{in}}\mu^-$ and $\tau_{\text{in}} = 0$.

The functioning of the clocks can be understood as follows. While $\texttt{capt}$ = "no", we wait at least T_{out} units of time after a zoom-out before executing another zoom-out. Moreover, we wait at least T_{in} units of time after the last zoom-in or zoom-out before executing another zoom-in. For convenience, the clock τ_{out} is also used to ensure that the capture event is separated in time from the zoom-outs.

For each fixed value of μ, chattering on the boundaries between the quantization regions may occur, and solutions are to be interpreted in the sense of Filippov (this issue does not affect the Lyapunov-based analysis). Solutions of the overall hybrid system are defined as usual, from one discrete event to the next. The only potential issue is the possibility of infinitely many zoom-in/out events in finite time (Zeno behavior), which in principle can happen since a minimal time between zoom-outs is not enforced while $\texttt{capt}$ = "yes". However, when the disturbance is bounded, such behavior is ruled out by the next result, which guarantees that μ remains bounded for all time.

Theorem 1 *Consider the system* (1). *Pick matrices K and $P = P^T > 0$ satisfying* (2). *Let q be a quantizer satisfying the conditions* (3) *and* (4)*, where M and Δ satisfy*

$$M > \left(2 + 2\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|PBK\|\right)\Delta. \tag{11}$$

Let the control be defined by (7) *and let the evolution of μ be as described above, with an arbitrary fixed initial condition $\mu_0 = \mu(t_0) > 0$. Let $\Omega_{\text{in}}, \Omega_{\text{out}}, T_{\text{in}}, T_{\text{out}}, T_c$ be positive numbers satisfying the inequalities $\Omega_{\text{in}} < 1$, $T_c < T_{\text{out}}/2$,*

$$T_{\text{out}} < \log \Omega_{\text{out}}/\|A\|,$$

$$\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - 2\Delta > \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|PBK\|\Delta, \tag{12}$$

$$\Omega_{\text{out}} > \frac{\sqrt{\lambda_{\max}(P)}M}{\sqrt{\lambda_{\min}(P)}(M - 2\Delta)} \tag{13}$$

($T_{\text{in}} > 0$ is arbitrary). Define

$$\ell_{\text{out}} := M - \Delta,\ \ \ell_{\text{in}} := \Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - \Delta. \tag{14}$$

Then there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial state $x_0 = x(t_0)$ and every bounded disturbance d the closed-loop system has the properties that μ remains bounded and the continuous dynamics are ISS in the sense of satisfying (5) *and* (6).

Remark 1 It is straightforward to verify that the inequality (11) ensures the existence of all subsequently defined quantities. The intuitive meaning of this inequality is that the quantizer takes sufficiently many values so that its range M is large enough compared to the error Δ. □

Remark 2 As a corollary, we have that if $d \equiv 0$ then the continuous closed-loop dynamics are GAS. In fact, the rate of convergence of $x(t)$ to 0 is exponential. □

Note that a zoom-out is triggered immediately when the second condition in (8) becomes true. It is clear that this aspect of the above scheme makes it sensitive to time delays and renders it not implementable in the sampled-data framework. Also, in general we cannot rule out Zeno behavior if the disturbance is not bounded. Thus the issue of designing a suitable zooming-out procedure will be central as we turn to the time-sampling scenario in the next section.

III. Trajectory-based sampled-data approach

In this section, we introduce a new sampled-data stabilization scheme which can be regarded as an alternative to the scheme from the previous section. We first discuss the simpler disturbance-free case to illustrate the approach. Then, we study an example of a controller and zooming protocol that do not have robustness in an ISS sense. Finally, in the last part of the section we present a result on ISS of the closed-loop system with respect to disturbances with a modified zooming protocol.

A. Disturbance-free case

We consider the continuous-time linear system (1) and assume that A is a non-zero non-Hurwitz matrix. In this subsection, we assume that $d(\cdot) \equiv 0$. We will control this system with quantized hybrid feedback that is defined next. Let $T > 0$ be a given sampling period and let $t_k := kT$ for $k \in \mathbb{N}$. We define $x(t_k) := x_k$ and a sequence $x_0, \ldots, x_k$ is denoted as $x_{[0,k]}$. Our closed-loop dynamics will be:

$$\text{Plant: } \dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \in \mathbb{R}^n \tag{15}$$

$$\text{Controller: } u(t) = U(\Omega_k, \mu_k, x_k),\ t \in [t_k, t_{k+1}) \tag{16}$$

$$\text{Protocol: } \mu_{k+1} = G(\Omega_k, \mu_k, x_k),\ \mu_0 \in \mathbb{R}_{>0} \tag{17}$$

$$\text{Switching law: } \Omega_k = H(\Omega_{k-1}, \mu_k, x_k),\ \Omega_{-1} = \Omega_{\text{out}} \tag{18}$$

Let $\ell_{\text{out}} > \ell_{\text{in}} > 0$ be strictly positive numbers to be defined below. To simplify the notation, we introduce $q_k := q_{\mu_k}(x_k)$

for arbitrary $k \in \mathbb{N}$, where $q_\mu(\cdot)$ is the one-parameter family of quantizers defined in the previous section. The variable Ω determines the switching rules for the controller and the zooming protocol. It can only take two values Ω_{out} and Ω_{in}, with the initial value $\Omega_{-1} = \Omega_{\text{out}}$. Then, we define the following hysteresis control law and zooming protocol:

$$U(\Omega_k, \mu_k, x_k) := \begin{cases} 0 & \text{if } \Omega_k = \Omega_{\text{out}} \\ K q_k & \text{if } \Omega_k = \Omega_{\text{in}} \end{cases} \tag{19}$$

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\text{out}}\mu_k & \text{if } \Omega_k = \Omega_{\text{out}} \\ \Omega_{\text{in}}\mu_k & \text{if } \Omega_k = \Omega_{\text{in}} \end{cases} \tag{20}$$

$$H(\Omega_{k-1}, \mu_k, x_k) = \begin{cases} \Omega_{\text{out}} & \text{if } q_k > \ell_{\text{out}}\mu_k \\ \Omega_{\text{in}} & \text{if } q_k < \ell_{\text{in}}\mu_k \\ \Omega_{k-1} & \text{if } q_k \in [\ell_{\text{in}}\mu_k, \ell_{\text{out}}\mu_k] \end{cases} \tag{21}$$

where Ω_{in} and Ω_{out} are strictly positive constants to be defined below. We introduce some notation. Note that for all $k \geq 0$ we have that $\Omega_k = \Omega_{\text{out}}$ or $\Omega_k = \Omega_{\text{in}}$. In the former case, we say that the zoom-out condition is triggered at time k and in the latter case we say that the zoom-in condition is triggered at time k. Given an initial condition (and a disturbance), there is a sequence of intervals on which we zoom in or out, i.e., we can introduce $k_j \in \mathbb{N}$ such that

$$\begin{aligned} \Omega_k &= \Omega_{\text{out}} \quad \text{if } k \in [k_{2i}, k_{2i+1} - 1]\,, \\ \Omega_k &= \Omega_{\text{in}} \quad \text{if } k \in [k_{2i+1}, k_{2(i+1)} - 1]\,. \end{aligned}$$

The above system induces the following discrete-time system that is more amenable to analysis:

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) & x(0) &= x_0 & (22) \\ \mu_{k+1} &= G(\Omega_k, \mu_k, x_k) & \mu_0 &\in \mathbb{R}_{>0} & (23) \\ \Omega_{k+1} &= H(\Omega_{k-1}, \mu_k, x_k) & \Omega_{-1} &= \Omega_{\text{out}} & (24) \end{aligned}$$

where

$$\Phi := e^{AT}, \quad \Gamma := \int_0^T e^{As} B ds\,.$$

Note that the switching between the zoom-in and zoom-out stages is determined by the variable $\xi_k := \frac{x_k}{\mu_k}$. Hence, the dynamical equations that describe how ξ_k changes are important for understanding the operation of the system. For instance, during the zoom-out stage we have $\xi_{k+1} = \Phi\xi_k/\Omega_{\text{out}}$. During the zoom-in stage we have

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{\text{in}}}\Gamma K \nu_k \tag{25}$$

where $\nu_k := q(\xi_k) - \xi_k$. We can state the following two standard results.

Lemma 1 *Suppose that $\Phi + \Gamma K$ is Schur. Then, there exists $\Omega^*_{\text{in}} \in (0, 1)$, such that for all $\Omega_{\text{in}} \in [\Omega^*_{\text{in}}, 1)$,*

$$(\Phi + \Gamma K)/\Omega_{\text{in}} \tag{26}$$

is Schur. Moreover, for any such Ω_{in}, there exist strictly positive L_1, λ_1, γ_1 such that the solutions of the system (25) satisfy the following:

$$|\xi_k| \leq L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma_1 \|\nu\| \qquad \forall k \geq 0\,.$$

Note that Lemma 1 imposes a lower bound on Ω_{in}, which is similar to the condition (12) from the previous section.

Corollary 2 *Let Ω_{in} come from Lemma 1. Then, there exist strictly positive M, Δ and Δ_M, with $\Delta_M - \Delta > 0$ such that whenever $|\xi_0| \leq \Delta_M$ and $\|\nu\| \leq \Delta$, we have*

$$q_{\mu_k}(x_k) \leq (M - \Delta)\mu_k \text{ and } |\xi_k| \leq M \qquad \forall k \geq 0\,. \tag{27}$$

Corollary 2 has an appropriate interpretation via Lyapunov functions that links results of this section with the previous section. Indeed, since we assume that $\frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)$ is Schur, there exists a quadratic Lyapunov function $V(\xi) := \xi^T P \xi$ such that for some $a > 0$ the solutions of the system (25) satisfy $|\xi_k| \geq a\,|\nu_k| \Rightarrow V(\xi_{k+1}) < V(\xi_k)$. Suppose that Δ is given. Then, one possible choice of M, Δ_M, Δ is

$$\Delta_M > \max\{1, a\}\Delta \tag{28}$$

and

$$M - 2\Delta > \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)} \cdot \Delta_M\,. \tag{29}$$

A geometrical interpretation of (29) is that the smallest level set of V containing the ball of radius Δ_M is inside the largest level set of V contained in the ball of radius $M - 2\Delta$. If (28) holds, then V decreases for ξ in the annulus between these two level sets as long as $\|\nu\|$ is smaller than Δ. Hence for $\nu_k = q(\xi_k) - \xi_k$ the conditions (27) are satisfied because ξ_k stays within the range of q. The inequalities (28) and (29) basically say that M should be large enough compared to Δ, which is similar to the condition (11).

Theorem 3 *Consider the system (15) and suppose Assumption 1 holds. Suppose that for the given $T > 0$ the pair (Φ, Γ) is stabilizable. Let K be such that $(\Phi + \Gamma K)$ is Schur. Let Ω_{in} be such that (26) is Schur and let $\Omega_{\text{out}} > |\Phi|$. Let the range M of the quantizer be sufficiently larger than the error Δ of the quantizer so that Corollary 2 holds with M, Δ and some Δ_M. Define $\ell_{\text{out}} := M - \Delta$ and $\ell_{\text{in}} := \Delta_M - \Delta$ in (21). Then, μ_k is bounded for all $k \geq 0$ and the system (15), (16), (17), (18), (19), (20), (21) is globally asymptotically stable. More precisely, there exists $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is of class $\mathcal{K}_\infty$ in its first argument for any fixed value of its second argument and such that for all $x_0 \in \mathbb{R}^n$ and any μ_0 we have*

$$|x_k| \leq \varphi(|x_0|, \mu_0) \qquad \forall k \geq 0 \tag{30}$$

and $\lim_{k\to\infty} |x_k| = 0$, exponentially fast.

Remark 3 It is not hard to show that the stability bound valid only at the sampling instants t_k, provided by Theorem 3, can be extended to all $t \geq 0$. The same is true for our ISS results in Section III-C. For similar results, see [16]. □

A more general result for the disturbance case is presented in Section III-C. The control law and protocol (19), (20), (21) are novel in that hysteresis switching is used to switch between the zoom-in and zoom-out stages. This switching strategy simplifies analysis of the time-driven sampling scheme. In particular, the underlying cascaded structure of the system during the zoom-in stage is obtained and used for the first time to establish stability.

While it can be shown that for any fixed $\mu > 0$ we can take $\varphi(\cdot, \mu) \in \mathcal{K}_\infty$ in Theorem 3, we have at the same time that for any fixed $s > 0$ the following holds: $\lim_{\mu \to 0} \varphi(s, \mu) = \infty$. Hence, the overshoot of the x-subsystem is non-uniform in small μ_0. While initializing the system at a particular μ_0 gives a constant overshoot for the x variable and one can prove stability of the x-subsystem, the lack of uniformity of the overshoot leads to an inherent lack of robustness, as the following example illustrates.

B. Example: lack of robustness

Consider the plant

$$x_{k+1} = \Phi x_k + \Gamma u_k + w_k$$

with (16), (17), (19), (20) and suppose that all conditions of Theorem 3 hold for this closed-loop system. Note that we assume that the system is controllable from disturbance in one step to simplify the analysis.

We show that the closed-loop system does not have a finite ISS gain from an additive plant disturbance to x. We do this by showing that for any $C_1 > 0$, any $\varepsilon > 0$, any $x_0 \in R^n$ and any $\mu_0 > 0$ there exists an additive plant disturbance w^ε such that $\|w^\varepsilon\| \leq \varepsilon$ and the following holds: $\limsup_{k \to \infty} |x(k, x_0, \mu_0, w^\varepsilon_{[0,k]})| > C_1$. Let $C_1 > 0$ and $\varepsilon > 0$ be arbitrary. Suppose without loss of generality that there is a positive real eigenvalue λ_m of Φ larger than one and let ζ_m be its corresponding eigenvector with $|\zeta_m| = 1$. Let $\hat{\varepsilon} > 0$ and $\bar{\varepsilon}_1 > 0$ be such that

$$\bar{\varepsilon}_1 \left(|\Phi + \Gamma K| + |\Gamma K| \Delta \right) + \hat{\varepsilon} < \varepsilon \,. \tag{31}$$

Let C_1 and $\hat{\varepsilon}$ generate

$$\overline{T} := \left\lceil \frac{\ln\left(\frac{C_1}{\hat{\varepsilon}}\right)}{\ln(\lambda_m)} \right\rceil \,. \tag{32}$$

Let $\overline{T}$ generate $C_2 > 0$ via

$$C_2 > \max\left\{ \ell_{\text{in}} \cdot \left| \left(\frac{\Omega_{\text{out}}^{\overline{T}}}{\Phi^{\overline{T}}} \right) \right|, \ell_{\text{out}} \right\} \,. \tag{33}$$

Let C_2 and $\hat{\varepsilon}$ generate $\bar{\varepsilon}_2$ as follows:

$$\bar{\varepsilon}_2 := \frac{\hat{\varepsilon}}{\Omega_{\text{in}} C_2} \,. \tag{34}$$

Finally, using $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ define

$$\bar{\varepsilon} := \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\} \,. \tag{35}$$

Note that since the system without disturbance is stable, as shown in the previous section, then for any $x_0 \in \mathbb{R}^n$, $\mu_0 > 0$ there exists $k_0^* > 0$ such that with $w_k \equiv 0$ we have

$$\max\{|x_{k_0^*}|, \mu_{k_0^*}\} \leq \bar{\varepsilon} \text{ and } |\xi_{k_0^*}| \leq M. \tag{36}$$

We now start the construction of the disturbance. Let the disturbance satisfy

$$w_k^\varepsilon = 0 \qquad k \in [0, k_0^* - 1] \tag{37}$$

Hence, (36) holds. Let now

$$w_{k_0^*}^\varepsilon = -(\Phi + \Gamma K) x_{k_0^*} - \Gamma K \mu_{k_0^*} (q_{k_0^*} - \xi_{k_0^*}) + \hat{\varepsilon} \zeta_m \,.$$

This disturbance will yield $x_{k_0^*+1} = \hat{\varepsilon}\zeta_m$. The conditions (31) and (35) guarantee that $|w_{k_0^*}^\varepsilon| \leq \varepsilon$. The conditions (34) and (35) guarantee that

$$\left|\xi_{k_0^*+1}\right| = \left| \frac{x_{k_0^*+1}}{\Omega_{\text{in}} \mu_{k_0^*}} \right| \geq \frac{\hat{\varepsilon}}{\Omega_{\text{in}} \bar{\varepsilon}_2} = C_2 \,, \tag{38}$$

and hence at time $k_0^* + 1$ the zoom-out condition is triggered. Since the ξ dynamics with $w_k \equiv 0$ evolve according to

$$\xi_{k+1} = \Phi \xi_k / \Omega_{\text{out}} \,,$$

there exists an integer k_1^* such that if $w_k^\varepsilon = 0$ for all $k \in [k_0^* + 1, k_1^* - 1]$, then $\left|\xi_{k_1^*}\right| \leq \ell_{\text{in}}$ and the zoom-in condition is triggered at $k = k_1^*$. Moreover, from (32) and (33) we have $k_1^* - k_0^* - 1 \geq \overline{T}$, which implies together with (38) that

$$|x_{k_1^*}| = \left| \lambda_m^{k_1^* - k_0^* - 1} \zeta_m \hat{\varepsilon} \right| \geq \lambda_m^{\overline{T}} \hat{\varepsilon} \geq C_1 \,.$$

Again via stability of the disturbance-free (x, μ) system, there exists k_2^* such that if $w_k^\varepsilon = 0$ for all $k \in [k_1^*, k_2^* - 1]$, then we have

$$\max\{|x_{k_2^*}|, \mu_{k_2^*}\} \leq \bar{\varepsilon} \text{ and } |\xi_{k_2^*}| \leq M \,. \tag{39}$$

Continuing in a similar manner, we construct the disturbance which satisfies $\|w^\varepsilon\| \leq \varepsilon$ and yields

$$|x_{k_{2j+1}^*}| > C_1 \qquad \forall j \in \mathbb{N} \,.$$

The possible non-robustness of the control law in the above example actually holds for a large class of plants, control laws, and zooming protocols. Indeed, the crucial ingredients of closed-loop systems that will exhibit this type of non-robustness are as follows:

1) The closed-loop system has to have the property that in the absence of disturbances, both x and μ converge to zero. Moreover, given any initial conditions x_0 and $\mu_0 > 0$ the zoom-out stage is bounded;
2) The closed-loop system is such that the x component is completely controllable locally around the origin with arbitrarily small disturbances $\|w\| \leq \varepsilon$;
3) For all $k \geq 0$, the zooming protocol takes the form $\mu_{k+1} = \gamma_k(\mu_k)$ where γ_k are continuous, zero at zero, locally invertible and uniformly lower and upper bounded;
4) If the measurement overflows, the controller is switched off.

Hence, a suitable modification in the zooming-out procedure needs to be adopted in order to achieve ISS. We will provide a modification of the zooming-out procedure (see (43) below) and the closed-loop system with the modified scheme will be ISS. In particular, our modification violates the above item 3 and this is sufficient to achieve ISS.

C. Input-to-state stability

Consider the plant with disturbance (1), together with the controller and zooming protocol introduced in Section III-A. The corresponding discrete-time system is

$$x_{k+1} = \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) + w_k,\ x(0) = x_0 \quad (40)$$

$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k),\ \mu_0 > 0 \quad (41)$$

$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k),\ \Omega_{-1} = \Omega_{\text{out}} \quad (42)$$

where U and H are defined in (19), (21) and $w_k := \int_{kT}^{(k+1)T} e^{A((k+1)T-s)} Dd(s)ds$. We use a new protocol:

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\text{out}}[\mu_k + c] & \text{if } \Omega_k = \Omega_{\text{out}} \\ \Omega_{\text{in}}\mu_k & \text{if } \Omega_k = \Omega_{\text{in}} \end{cases} \quad (43)$$

where $c > 0$. The use of this constant c violates item 3 above, and this will be shown to fix the problem.

Next we introduce a discrete-time version of the definition of ISS. This will suffice for our analysis in this section since the discrete-time ISS can be used to prove an appropriate version of continuous-time ISS that takes inter-sample behavior into account (see Remark 3). The system (40), (41) is said to be ISS if there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that the solutions of the system satisfy the following for all $x_0 \in \mathbb{R}^n$ and all w:

$$|x_k| \le \gamma_1(|x_0|) + \gamma_2(\|w\|) \quad \forall k \ge 0\,, \quad (44)$$

$$\limsup_{k\to\infty}|x_k| \le \gamma_3 \left(\limsup_{k\to\infty}|w_k|\right)\,. \quad (45)$$

We note that the functions $\gamma_1, \gamma_2, \gamma_3$ depend on $\mu_0 > 0$ but are independent of x_0 or w.

Again, we consider the dynamics of the variable $\xi_k := \frac{x_k}{\mu_k}$. During the zoom-in stage we have that:

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{\text{in}}}\Gamma K \nu_k + \frac{1}{\Omega_{\text{in}}}\zeta_k\,, \quad (46)$$

where $\nu_k := q(\xi_k) - \xi_k$ and $\zeta_k := \frac{w_k}{\mu_k}$.

Lemma 2 *Suppose that $\frac{1}{\Omega_{\text{in}}}(\Phi+\Gamma K)$ is Schur*[2]*. Then, there exist strictly positive $L_1, \lambda_1, \gamma_1, \gamma_2$ such that the solutions of the system (46) satisfy the following:*

$$|\xi_k| \le L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma_1 \|\nu\| + \gamma_2 \|\zeta\| \qquad \forall k \ge 0\,.$$

Corollary 4 *Let Ω_{in} come from Lemma 2. Then, there exist strictly positive Δ, Δ_M and Δ_w, with $\Delta_M - \Delta > 0$ such that whenever $|\xi_0| \le \Delta_M$, $\|\nu\| \le \Delta$ and $\|\zeta\| \le \Delta_w$, we have*

$$q(\mu_k, x_k) \le (M - \Delta)\mu_k \text{ and } |\xi_k| \le M \qquad \forall k \ge 0\,.$$

[2]In Lemma 1 we showed that we can find an appropriate $\Omega_{\text{in}} \in (0, 1)$ so that this holds whenever $(\Phi + \Gamma K)$ is Schur.

Theorem 5 *Consider the system (40), (41), (42) and suppose that Assumption 1 holds. Suppose that for the given $T > 0$ the pair (Φ, Γ) is stabilizable. Let K be such that $(\Phi+\Gamma K)$ is Schur. Let Ω_{in} be such that (26) is Schur and let $\Omega_{\text{out}} > |\Phi|$. Let the range M of the quantizer be sufficiently larger than the error Δ of the quantizer so that Corollary 4 holds with M, Δ and some Δ_M, Δ_w. Define $\ell_{\text{out}} := M - \Delta$ and $\ell_{\text{in}} := \Delta_M - \Delta$ in (21). Then, μ_k is bounded for all $k \ge 0$ and the system (40), (41), (42) is ISS.*

Remark 4 It is worth noting that the modified zooming protocol of the form (43) can be used in the event-based scheme and it would not change the ISS properties of the system. Actually, this modification would have the added benefits of reducing the number of zoom-outs and providing robustness of the event-based scheme to time delays. □

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