

A Lie-algebraic condition for stability of switched nonlinear systems

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Abstract—We present a stability criterion for switched nonlinear systems which involves Lie brackets of the individual vector fields but does not require that these vector fields commute. A special case of the main result says that a switched system generated by a pair of globally asymptotically stable nonlinear vector fields whose third-order Lie brackets vanish is globally uniformly asymptotically stable under arbitrary switching. This generalizes a known fact for switched linear systems and provides a partial solution to the open problem posed in [1]. To prove the result, we consider an optimal control problem which consists in finding the “most unstable” trajectory for an associated control system, and show that there exists an optimal solution which is bang-bang with a bound on the total number of switches. By construction, our criterion also automatically applies to the corresponding relaxed differential inclusion.

Index Terms—Switched nonlinear system, global asymptotic stability, Lie bracket, optimal control, maximum principle, differential inclusion.

I. INTRODUCTION

A finite family of smooth vector fields f_0, f_1, \dots, f_m on \mathbb{R}^n gives rise to the *switched system*

$$\dot{x} = f_\sigma(x), \quad x \in \mathbb{R}^n. \quad (1)$$

Here $\sigma : [0, \infty) \rightarrow \{0, 1, \dots, m\}$ is a piecewise constant function of time, called a *switching signal*. A frequently discussed example, which we will occasionally use for illustration, is when the given vector fields are linear: $f_i(x) = A_i x$, where A_i is an $n \times n$ matrix, $i = 0, 1, \dots, m$. This yields the *switched linear system*

$$\dot{x} = A_\sigma x, \quad x \in \mathbb{R}^n. \quad (2)$$

Switched systems have numerous applications and represent a subject of extensive ongoing research (see, e.g., [2] and the many references therein).

By a solution of (1) we understand an absolutely continuous \mathbb{R}^n -valued function $x(\cdot)$ which satisfies the differential equation for almost all t in its domain. In the context of switched systems, one usually assumes for simplicity that the switching signal σ has a finite number of discontinuities, or *switches*, on every bounded time interval. Our results will in fact automatically apply to measurable switching signals

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and also to solutions of the relaxed differential inclusion associated with (1). This is the differential inclusion

$$\dot{x} \in \text{co}\{f_0(x), f_1(x), \dots, f_m(x)\} \quad (3)$$

where co denotes the convex hull. Its solutions are absolutely continuous functions $x(\cdot)$ satisfying (3) for almost all t , and this includes all solutions of the switched system (1) with piecewise constant σ (see, e.g., [3] for more information on differential inclusions).

When we refer to stability of the switched system (1), the precise property of interest will be global asymptotic stability which is uniform over the set of all switching signals, formally defined as follows. The switched system (1) is called *globally uniformly asymptotically stable* (GUAS) if there exists a class \mathcal{KL} function¹ β such that for every switching signal σ and every initial condition $x(0)$ the corresponding solution of (1) satisfies

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}(0)|, t) \quad \forall t \geq 0 \quad (4)$$

(here and below, $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^n).² If the function β in (4) takes the form $\beta(r, t) = a r e^{-bt}$ for some $a, b > 0$, then the switched system (1) is called *globally uniformly exponentially stable* (GUES). The above definitions reduce to the usual notions of global asymptotic and exponential stability in the case of a single system. If (4) holds along all solutions of the differential inclusion (3), then this differential inclusion is called *globally asymptotically stable*. This property implies GUAS of the switched system (1) because solutions of (1) are contained in those of (3).

It is well known and easy to demonstrate that global asymptotic stability of the individual subsystems $\dot{x} = f_i(x)$, $i = 0, 1, \dots, m$ is necessary but not sufficient for global asymptotic stability of the switched system (1) for every possible switching signal, let alone GUAS. In this paper, we are concerned with the problem of identifying conditions on the individual subsystems—besides the obviously necessary requirement of their global asymptotic stability—which guarantee GUAS of (1). This problem has received considerable attention in the literature; see [2, Chapter 2] for some available results.

¹Recall that a function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to be of class \mathcal{K} if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K} for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \rightarrow \infty$ for each fixed $s \geq 0$.

²An equivalent definition can be given in the ϵ - δ style (cf. [4]).

We now review two directions of attack on the GUAS problem. The difficulty in analyzing the stability of (1) is that the switched system admits an infinite number of solutions for each initial condition. A natural idea is to try to characterize the “worst-case” (that is, the “most unstable”) switching law, and then analyze the behavior of the unique trajectory produced by this law. Pyatnitskiy and Rapoport [5], [6] developed a variational approach to describe the “worst-case” switching law, and the corresponding trajectory, for the switched linear system (2). Applying the Maximum Principle, they derived an implicit characterization of this switching law in terms of a two-point boundary value problem. More recently, the same variational problem was addressed using a dynamic programming approach [7]. For the particular case of second-order switched linear systems, this approach yields an easily verifiable necessary and sufficient condition for GUAS [8] (see also [9] as well as the related work [10]).

Another particularly promising research avenue is to explore the role of commutation relations among the subsystems being switched. We now briefly recall available results, starting with the case of the switched linear system (2). The commutator, or *Lie bracket*, is defined as $[A_i, A_j] := A_i A_j - A_j A_i$. First, suppose that the matrices commute: $[A_i, A_j] = 0$ for all i, j . Then it is well known that the system (2) is GUES provided that A_1, \dots, A_m are all Hurwitz. (This is not hard to show by a direct analysis of the transition matrix, or by constructing a quadratic common Lyapunov function via the iterative procedure proposed in [11].) It was shown in [12] that for $m = 1$ the above statement remains true if the Lie bracket condition is relaxed to $[A_0, [A_0, A_1]] = [A_1, [A_0, A_1]] = 0$. This result was extended in [13] to all nilpotent and solvable matrix Lie algebras for arbitrary m by using simultaneous triangularization (Lie’s Theorem), and later in [14] to a more general class of matrix Lie algebras; the latter paper also established that no further generalization based on the Lie bracket relations alone is possible.

Let us now return to the nonlinear setting described by the switched system (1), which is much less thoroughly understood. Linearizing the individual subsystems and applying the previous results together with Lyapunov’s indirect method, it is not difficult to obtain conditions for local asymptotic stability of (1); this was done in [13], [14]. To tackle the global stability question, one can try to inspect commutation relations between the original nonlinear vector fields $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_m$. The *Lie bracket* is now defined as

$$[\mathbf{f}_i, \mathbf{f}_j](\mathbf{x}) := \frac{\partial \mathbf{f}_j(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_i(\mathbf{x}) - \frac{\partial \mathbf{f}_i(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_j(\mathbf{x}).$$

It turns out that if the two vector fields commute, i.e., if $[\mathbf{f}_i, \mathbf{f}_j](\mathbf{x}) \equiv \mathbf{0}$ for all i, j , then global asymptotic stability of the individual subsystems still implies GUAS of the switched system (1). Similarly to the linear case, this can be proved either by direct analysis of the flow (see [15]) or by an iterative construction of a common

Lyapunov function in the spirit of [11] (see [16] and [17]). All attempts to formulate global asymptotic stability criteria valid beyond the commuting nonlinear case have so far been unsuccessful, as the methods employed to obtain the corresponding results for switched linear systems do not seem to apply. These issues are explained in [1], where this is proposed as an open problem for which an altogether different approach seems to be required.

Such a different approach is presented in this paper. It is based on merging the two directions outlined above; namely, it combines the variational approach with a Lie-algebraic analysis of the “worst-case” switching law. It is a well-known fact that Lie brackets play an essential role in the Maximum Principle of optimal control (see, e.g., [18]). In the present context, it turns out that the “worst-case” switching law for the switched nonlinear system (1) is governed by the signs of suitable functions of time φ_{ij} , $1 \leq i, j \leq m$. The derivatives of these functions are defined by Lie brackets of $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_m$. We impose conditions on these Lie brackets which guarantee that each φ_{ij} is a polynomial. This leads to a bound on the total number of switches for the “worst-case” switching law, and GUAS of the switched system can be deduced from global asymptotic stability of the individual subsystems.

We now formally state our main result. We denote by $(\text{ad } \mathbf{f})^r$, $r = 0, 1, \dots$ the operators defined by $(\text{ad } \mathbf{f})^0(\mathbf{g}) := \mathbf{g}$ and $(\text{ad } \mathbf{f})^r(\mathbf{g}) := [\mathbf{f}, (\text{ad } \mathbf{f})^{r-1}(\mathbf{g})]$ for $r \geq 1$, where \mathbf{f} and \mathbf{g} are smooth vector fields on \mathbb{R}^n .

Theorem 1 *Suppose that $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_m$ are analytic vector fields on \mathbb{R}^n such that the systems $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$, $i = 0, 1, \dots, m$ are globally asymptotically stable and backward complete. Suppose that there exist a positive integer r and an index $l \in \{0, 1, \dots, m\}$ such that for all $\mathbf{x} \in \mathbb{R}^n$ and all $i \neq l$ we have*

$$(\text{ad } \mathbf{f}_l)^r(\mathbf{f}_i)(\mathbf{x}) = \mathbf{0}, \quad (5)$$

and for all $\mathbf{x} \in \mathbb{R}^n$, $s \in \{0, 1, \dots, r-1\}$, and all $i, k \neq l$ we have

$$[\mathbf{f}_k - \mathbf{f}_l, (\text{ad } \mathbf{f}_l)^s(\mathbf{f}_i - \mathbf{f}_l)](\mathbf{x}) = \mathbf{0}. \quad (6)$$

Then the differential inclusion (3) is globally asymptotically stable, and in particular the switched system (1) is GUAS.

For example, it is easy to check that for $r = 1$, the conditions (5) and (6) are equivalent to $[\mathbf{f}_i, \mathbf{f}_j](\mathbf{x}) \equiv \mathbf{0}$ for all i, j , and we recover the known result for the commuting case from [15]. The first interesting situation is when $r = 2$. It is not difficult to verify that the conditions (5) and (6) then reduce to

$$[\mathbf{f}_i, [\mathbf{f}_j, \mathbf{f}_k]](\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall i, j, k \in \{0, 1, \dots, m\}$$

and

$$[\mathbf{f}_k - \mathbf{f}_l, \mathbf{f}_i - \mathbf{f}_l](\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall i, k \neq l.$$

Note that in the particular case when $m = 1$ (i.e., the case of two subsystems) the last condition is trivially satisfied because we must have $i = k$, and we arrive at the following nonlinear extension of the result proved in [12].³

Corollary 1 *Suppose that $\mathbf{f}_0, \mathbf{f}_1$ are analytic vector fields on \mathbb{R}^n such that the systems $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$, $i = 0, 1$ are globally asymptotically stable and backward complete, and suppose that*

$$[\mathbf{f}_0, [\mathbf{f}_0, \mathbf{f}_1]](\mathbf{x}) = [\mathbf{f}_1, [\mathbf{f}_0, \mathbf{f}_1]](\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (7)$$

Then the differential inclusion (3) is globally asymptotically stable, and in particular the switched system (1) is GUAS.

The remainder of the paper is devoted to proving Theorem 1. Due to space limitations, certain details are not given here. They can be found in an extended version of this paper [19].

II. OPTIMAL CONTROL APPROACH

Permuting the vector fields $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_m$ if necessary, we henceforth assume for notational convenience that the value of l in (5) and (6) is 0. Our starting point is to replace the differential inclusion (3) by the control system with drift

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{k=1}^m \mathbf{g}_k(\mathbf{x})u_k, \quad \mathbf{u} = (u_1, \dots, u_m)^T \in \mathcal{U} \quad (8)$$

where

$$\mathbf{g}_k(\mathbf{x}) := \mathbf{f}_k(\mathbf{x}) - \mathbf{f}_0(\mathbf{x}), \quad i = 1, \dots, m \quad (9)$$

and \mathcal{U} is the standard simplex

$$\mathcal{U} := \left\{ \mathbf{u} \in \mathbb{R}^m : u_k \geq 0, \sum_{k=1}^m u_k \leq 1 \right\}.$$

Throughout the rest of the paper, admissible control inputs for (8) will be measurable functions of time taking values in \mathcal{U} (i.e., the control $\mathbf{u}(t) = (u_1(t), \dots, u_m(t))^T \in \mathcal{U}$ for all t). The equivalence between the control system (8) and the differential inclusion (3) is revealed by the result stated next, which is well known as Filippov's Selection Lemma (see, e.g., [3, Section 14, Corollary 1] or [20, Theorem 2.3.13]).

Lemma 1 *The set of solutions of the control system (8) coincides with the set of solutions of the differential inclusion (3).*

Note that trajectories of the original switched system (1) correspond to piecewise constant controls taking values in the set of $m + 1$ vertices of \mathcal{U} . In particular, $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x})$ results by setting $\mathbf{u} \equiv \mathbf{0}$ in (8), while each $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$, $i = 1, \dots, m$ results by setting $u_i \equiv 1$ and $u_j \equiv 0$, $j \neq i$.

³Leonid Gurvits has pointed out to us that he recently also obtained a generalization of his original result to nonlinear systems, using a technique different from ours.

i. We also remark that the switched linear system (2) is associated in this way with the bilinear control system $\dot{\mathbf{x}} = A_0\mathbf{x} + \sum_{k=1}^m (A_k\mathbf{x} - A_0\mathbf{x})u_k$.

Fix an arbitrary point $\mathbf{p} \in \mathbb{R}^n$. Since the right-hand side of (8) is bounded on every bounded ball in \mathbb{R}^n , we can find a time T (which depends on $|\mathbf{p}|$) such that all solutions of (8) starting at \mathbf{p} are bounded by, say, $2|\mathbf{p}|$ for all $t \in [0, T]$. In particular, these solutions are defined at least up to time T . Thus there exists the largest time $T_{\max}(|\mathbf{p}|) \leq \infty$ such that all solutions of (8) starting at \mathbf{p} are defined for $t \in [0, T_{\max}]$.

Let $\mathbf{x}(\cdot; \mathbf{p}, \mathbf{u})$ denote the solution of the system (8) with initial condition $\mathbf{x}(0) = \mathbf{p}$ corresponding to an admissible control \mathbf{u} . Picking a positive final time $t_f < T_{\max}(|\mathbf{p}|)$, we define

$$J(t_f, \mathbf{p}, \mathbf{u}) := |\mathbf{x}(t_f; \mathbf{p}, \mathbf{u})|^2.$$

We pose the following optimal control problem in terms of the cost functional J .

Problem 1 *Find a control \mathbf{u} that maximizes J along the solutions of (8).*

It follows from [21, §7, Theorem 3] that this problem is well posed, i.e., an optimal control does exist.

Lemma 2 *Problem 1 admits a solution for every pair (\mathbf{p}, t_f) such that $t_f < T_{\max}(|\mathbf{p}|)$.*

The intuitive interpretation of Problem 1 is clear: find a control that “pushes” the state as far away from the origin as possible (from a given initial condition in a given amount of time). If we can show that the resulting closed-loop system is stable, then the same property should hold for all other controls, and stability of the original switched system—as well as the differential inclusion—will be established. This program will be formally carried out in what follows.

Consider an optimal control $\tilde{\mathbf{u}} = (\tilde{u}_1, \dots, \tilde{u}_m)^T$ and the corresponding optimal trajectory $\tilde{\mathbf{x}}$. We will study properties of $\tilde{\mathbf{u}}$ using the Maximum Principle (MP) (see, e.g., [22] for background on the MP of optimal control, in a geometric framework suitable for our subsequent developments). The Hamiltonian for Problem 1 is

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) := \boldsymbol{\lambda}^T \mathbf{f}_0(\mathbf{x}) + \sum_{k=1}^m \boldsymbol{\lambda}^T \mathbf{g}_k(\mathbf{x})u_k \quad (10)$$

where $\boldsymbol{\lambda} : [0, t_f] \rightarrow \mathbb{R}^n$ is a costate satisfying the adjoint equation

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} - \sum_{k=1}^m \left(\frac{\partial \mathbf{g}_k}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} u_k. \quad (11)$$

More precisely, $\boldsymbol{\lambda}$ is identified with the linear functional on \mathbb{R}^n defined via the inner product $\langle \boldsymbol{\lambda}(t), \boldsymbol{\xi} \rangle = \boldsymbol{\lambda}^T(t)\boldsymbol{\xi}$.⁴

⁴In the more general formulation of the MP for manifolds, $\boldsymbol{\lambda}(t)$ is a linear functional on the tangent space at $\mathbf{x}(t)$; its evolution is described by the adjoint equation, which can be defined intrinsically and is given by (11) in local coordinates.

The necessary condition for optimality provided by the MP states that H must be pointwise maximized along the optimal trajectory. For $1 \leq i, j \leq m$, let us define the switching functions $\varphi_{ij} : [0, t_f] \rightarrow \mathbb{R}$ by

$$\varphi_{ij}(t) := \begin{cases} \boldsymbol{\lambda}^T(t) \mathbf{g}_i(\tilde{\mathbf{x}}(t)) & \text{if } i = j \\ \boldsymbol{\lambda}^T(t) (\mathbf{g}_i(\tilde{\mathbf{x}}(t)) - \mathbf{g}_j(\tilde{\mathbf{x}}(t))) & \text{if } i \neq j \end{cases} \quad (12)$$

Then the optimal control must satisfy the following condition almost everywhere: for each i ,

$$\begin{aligned} \tilde{u}_i(t) &= 0 & \text{if } \varphi_{ij}(t) < 0 & \text{ for some } j \\ \tilde{u}_i(t) &= 1 & \text{if } \varphi_{ij}(t) > 0 & \text{ for all } j \end{aligned} \quad (13)$$

If $\varphi_{ij}(t) = 0$ for some i, j , then the MP does not yield enough information to uniquely determine $\tilde{u}_i(t)$.

We say that a control \mathbf{u} is *bang-bang* if it takes values in the set of vertices of \mathcal{U} and has a finite number of switches on every bounded time interval. We also say that the control system (8) has the *bang-bang property* (with respect to Problem 1) if for every optimal solution, there exists a bang-bang control that transfers the same initial state to the same final state at time t_f (and thus achieves the same cost J). If, in addition, there is an upper bound N (independent of t_f) on the total number of switches of this bang-bang control, then we say that (8) has the *bang-bang property with N switches*.

It follows from the previous discussion that if each switching function φ_{ij} has isolated zeros, then we have the bang-bang property (because in this case every optimal control is bang-bang, modulo a modification on the set of measure 0 which does not change the trajectory). On the other hand, it may happen that some φ_{ij} identically vanishes on some time interval and gives rise to a *singular trajectory*, i.e., an optimal trajectory not obtained from a bang-bang control.

The connection between the bang-bang property and Lie brackets comes from the fact that the derivatives of the switching functions φ_{ij} , $1 \leq i, j \leq m$ are expressed in terms of iterated Lie brackets of $\mathbf{f}_0, \mathbf{g}_1, \dots, \mathbf{g}_m$. This is revealed in the proof of the next lemma, which will be crucial in establishing a bound on the total number of switches for bang-bang optimal controls.

Lemma 3 *Under the hypotheses of Theorem 1, all functions φ_{ij} , $1 \leq i, j \leq m$ are polynomial functions of time, with degrees not exceeding $r - 1$.*

Proof: Fix arbitrary indices i, j , and denote $\varphi := \varphi_{ij}$,

$$\mathbf{y}(\mathbf{x}) := \begin{cases} [\mathbf{f}_0, \mathbf{g}_i](\mathbf{x}) & \text{if } i = j \\ [\mathbf{f}_0, \mathbf{g}_i - \mathbf{g}_j](\mathbf{x}) & \text{if } i \neq j \end{cases}$$

and

$$\mathbf{z}_k(\mathbf{x}) := \begin{cases} [\mathbf{g}_k, \mathbf{g}_i](\mathbf{x}) & \text{if } i = j \\ [\mathbf{g}_k, \mathbf{g}_i - \mathbf{g}_j](\mathbf{x}) & \text{if } i \neq j \end{cases}$$

We know from (12) that φ is absolutely continuous. Differentiating it and using (8) and (11), we find that the derivative is given (almost everywhere) by

$$\dot{\varphi}(t) = \boldsymbol{\lambda}^T(t) \mathbf{y}(\tilde{\mathbf{x}}(t)) + \sum_{k=1}^m \boldsymbol{\lambda}^T(t) \mathbf{z}_k(\tilde{\mathbf{x}}(t)) u_k. \quad (14)$$

Recall the definition (9) and our convention that the conditions (5) and (6) hold with $l = 0$. Applying (6) with $s = 0$, we see that $[\mathbf{g}_k, \mathbf{g}_i](\mathbf{x}) \equiv \mathbf{0}$ for all i, k . Thus (14) simplifies to $\dot{\varphi}(t) = \boldsymbol{\lambda}^T(t) \mathbf{y}(\tilde{\mathbf{x}}(t))$. It follows from this expression that φ is absolutely continuous. Differentiating it, we obtain

$$\ddot{\varphi}(t) = \boldsymbol{\lambda}^T(t) [\mathbf{f}_0, \mathbf{y}](\tilde{\mathbf{x}}(t)) + \sum_{k=1}^m \boldsymbol{\lambda}^T(t) [\mathbf{g}_k, \mathbf{y}](\tilde{\mathbf{x}}(t)) u_k.$$

Applying (6) again, this time with $s = 1$, we see that the summation terms involving the control vanish as before. Proceeding in this fashion, we finally have

$$\begin{aligned} \varphi^{(r)}(t) &= \boldsymbol{\lambda}^T(t) (\text{ad } \mathbf{f}_0)^{r-1}(\mathbf{y})(\tilde{\mathbf{x}}(t)) \\ &\quad + \sum_{k=1}^m \boldsymbol{\lambda}^T(t) [\mathbf{g}_k, (\text{ad } \mathbf{f}_0)^{r-2}(\mathbf{y})](\tilde{\mathbf{x}}(t)) u_k. \end{aligned}$$

Now, the summation terms vanish by virtue of (6) applied with $s = r - 1$, and the remaining terms vanish because of (5). This means that $\varphi^{(r)}(t) \equiv 0$, and the claim follows (in view of absolute continuity of φ). ■

The main difficulty that remains is the possibility of singular optimal trajectories. As we show next, this problem can be overcome with the help of a construction due to Sussmann [23].

III. BANG-BANG PROPERTY

In this section we first summarize the basic steps of the construction given by Sussmann in [23], adopted to our context, and then show that the present assumptions lead to a more specific result which will play an important role in the proof of Theorem 1. We note that [23] is concerned with the time-optimal control problem. However, the relevant arguments from [23] carry over to the present setting with no significant changes. In fact, the Hamiltonians for Problem 1 and for the time-optimal problem are the same (modulo an additive constant), which makes their analysis quite similar. The reader is referred to [23] as well as [22] for more details on the terminology and concepts involved.

Let L denote the Lie algebra generated by the vector fields $\mathbf{f}_0, \mathbf{g}_1, \dots, \mathbf{g}_m$ (or, equivalently, the vector fields $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_m$). Under the assumption that these vector fields are analytic, \mathbb{R}^n is partitioned into *maximal integral manifolds* of L , whose tangent spaces are given by L . Let us again consider Problem 1 for some initial state $\mathbf{p} \in \mathbb{R}^n$ and final time $t_f < T_{\max}(|\mathbf{p}|)$. Let $\tilde{\mathbf{u}}$ be an optimal control and $\tilde{\mathbf{x}}$ be the corresponding state trajectory. Then this trajectory is entirely contained in the maximal integral manifold S of L passing through \mathbf{p} . For each $\mathbf{x} \in S$, the tangent space to S at \mathbf{x} is $L(\mathbf{x})$. The MP applied to

the restriction of the system (8) to S asserts the existence of a costate function $t \mapsto \lambda(t)$ which satisfies the adjoint equation and is nontrivial on $L(\tilde{x}(t))$ for all $t \in [0, t_f]$.

Next, let L_0 be the ideal in L generated by g_1, \dots, g_m and let S_0 be the maximal integral manifold of L_0 passing through p . According to [23], a trajectory x of (8) is called *strongly extremal* if it maximizes the Hamiltonian (10) for some costate function $t \mapsto \lambda(t)$ which satisfies the adjoint equation and has a nontrivial restriction to the subspace $L_0(x(t))$. The dimension of $L_0(x(t))$ is constant and equal to either $\dim L(x(t))$ or $\dim L(x(t)) - 1$. We see that unless $\dim L_0(\tilde{x}(t)) = \dim L(\tilde{x}(t))$, our optimal trajectory \tilde{x} is not necessarily strongly extremal. However, it turns out that \tilde{x} can be replaced by a concatenation of a strong extremal and a trajectory corresponding to an arbitrary constant control, such that the final state at $t = t_f$ remains the same. In fact, we have the following result, which is essentially a global version of [23, Lemma 2].

Lemma 4 *Assume that the hypotheses of Theorem 1 hold. Consider an optimal trajectory $\tilde{x}(t)$, $t \in [0, t_f]$ of the system (8) with $\tilde{x}(0) = p$, and take two points $a := \tilde{x}(t_1)$ and $b := \tilde{x}(t_2)$, $0 \leq t_1 < t_2 \leq t_f$ on it. Let $u^0 = (u_1^0, \dots, u_m^0)^T$ be an arbitrary vertex of \mathcal{U} . Then there exists a time $\tau \in [t_1, t_2]$ and an admissible control $\bar{v} : [t_1, \tau] \rightarrow \mathcal{U}$ with the following properties:*

1. *The trajectory $x(t; a, \bar{v})$, $t \in [t_1, \tau]$ of (8) is strongly extremal.*
2. *The solution of (8) corresponding to the control*

$$\tilde{v}(t) = \begin{cases} \bar{v}(t) & \text{if } t \in [t_1, \tau] \\ u^0 & \text{if } t \in (\tau, t_2] \end{cases} \quad (15)$$

satisfies $x(t_2; a, \tilde{v}) = b$.

Proof: Similar to the proof of [23, Lemma 2]. \blacksquare

In [23], Sussmann used the above result to derive a condition on the Lie brackets of f and g which guarantees the bang-bang property, with bounds on the number of switches on finite time intervals. Bang-bang theorems for multiple-input nonlinear control systems, along similar lines but with additional hypotheses, are also available; see [24], [25]. However, the bang-bang property of (8) is not enough for our purposes, because the number of switches may grow with the length of the time interval. To establish Theorem 1, we need to have a bound on the number of switches which is *independent of time*, i.e., we need to show that the control system (8) has the bang-bang property with N switches (as defined in Section II) for some N . The hypotheses of Theorem 1 are stronger than the conditions imposed in the above papers, and indeed allow us to reach such a conclusion.

Lemma 5 *Under the hypotheses of Theorem 1, the system (8) has the bang-bang property with $(r + 1)^m - 1$ switches.*

Proof: Consider, as in Lemma 4, two points a and b on an optimal trajectory \tilde{x} . We want to show that there is a control that steers the state of (8) from a to b , takes values in the vertices of \mathcal{U} , and has at most $(r + 1)^m - 1$ switches. This will imply the statement of the lemma. The number of vertices of \mathcal{U} is $m + 1$. Similarly to the proof of Theorem 8.1.2 in [18], we proceed by induction on m . If $m = 0$, then \mathcal{U} is a singleton and the statement is obvious. We now treat the case of an arbitrary $m > 0$, supposing that the above property with $m - 1$ instead of m would hold for two arbitrary points on an optimal trajectory if \mathcal{U} were a convex polyhedron with m vertices. Apply Lemma 4 to find another trajectory connecting a and b , which is a concatenation of a strongly extremal trajectory and a trajectory corresponding to a constant control u^0 , a vertex of \mathcal{U} . Let us denote the strongly extremal trajectory $x(t; a, \bar{v})$, $t \in [t_1, \tau]$ by \bar{x} . The corresponding costate function λ is nontrivial on $L_0(\bar{x})$. Using (5), (6), (9) and remembering that $l = 0$, we have

$$L_0 = \text{span} \{ (\text{ad } f_0)^s(g_k) : k = 1, \dots, m, s = 0, \dots, r-1 \}.$$

Considering the switching functions φ_{ij} , $1 \leq i, j \leq m$ defined by (12) and using the formulas for their derivatives obtained in the proof of Lemma 3, we see that these functions cannot all be identically zero along the strong extremal \bar{x} .

So, pick some i, j for which φ_{ij} is not identically zero along \bar{x} . We know from Lemma 3 that φ_{ij} is a polynomial function of degree at most $r - 1$. Hence, the time interval $[t_1, \tau]$ is divided into at most r subintervals, inside some of which we have $\varphi_{ij}(t) < 0$ and inside the others $\varphi_{ij}(t) > 0$. The formula (13) implies that the control \bar{v} satisfies $\bar{v}_i = 0$ on subintervals of the former type and $\bar{v}_j = 0$ on subintervals of the latter type (almost everywhere). Consider one such subinterval $[\tau_1, \tau_2] \subset [t_1, \tau]$. Let $\bar{a} := \bar{x}(\tau_1)$ and $\bar{b} := \bar{x}(\tau_2)$. Assume, for example, that φ_{ij} is negative on (τ_1, τ_2) . We have just shown that the restriction of \bar{x} to $[\tau_1, \tau_2]$ is a trajectory of (8) with controls taking values in the set $\mathcal{U}^i := \{u \in \mathcal{U} : u_i = 0\}$. This set is a simplex with m vertices (which are also vertices of \mathcal{U}). By the induction hypothesis, there is a control that steers the state of (8) from \bar{a} to \bar{b} , takes values in the set of vertices of \mathcal{U}^i , and is a concatenation of at most $(r + 1)^{m-1}$ constant controls. A similar conclusion holds for the restriction of \bar{x} to every other subinterval. This means that we can steer a to b by a bang-bang control which is a concatenation of at most $r(r + 1)^{m-1} + 1 \leq (r + 1)^m$ constant controls, and the induction step is complete. \blacksquare

IV. PROOF OF THEOREM 1

We are now ready to prove the main result. Consider the control system (8). Each vertex of \mathcal{U} corresponds to the closed-loop system $\dot{x} = f_i(x)$ for some index i between 0 and m . This system is globally asymptotically stable, hence its solutions satisfy $|x(t)| \leq \beta_i(|x(0)|, t)$ for some class \mathcal{KL}

function β_i . Let $\bar{\beta} := \max\{\beta_i : i = 0, \dots, m\} \in \mathcal{KL}$. Let α be the class \mathcal{K} function defined by $\alpha(r) := \bar{\beta}(r, 0)$.

Consider Problem 1 with arbitrary \mathbf{p} and $t_f < T_{\max}(|\mathbf{p}|)$. By Lemmas 2 and 5, there exists a piecewise constant optimal control $\tilde{\mathbf{u}} : [0, t_f] \rightarrow \mathcal{U}$ that takes values in the vertices of \mathcal{U} and has at most $(r + 1)^m - 1$ switches. The interval $[0, t_f]$ is thus divided into at most $(r + 1)^m$ subintervals, on each of which the optimal trajectory $\tilde{\mathbf{x}}$ satisfies one of the equations $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x})$, $i = 0, \dots, m$. The length of at least one of these subintervals is no smaller than $t_f/(r + 1)^m$. Considering all possible locations of this subinterval relative to the others, it is straightforward to check that $\tilde{\mathbf{x}}$ satisfies

$$|\tilde{\mathbf{x}}(t_f)| \leq \beta(|\mathbf{p}|, t_f) \quad (16)$$

where

$$\beta(r, t) := \max \left\{ \alpha^k(\bar{\beta}(\alpha^l(r), t/(r + 1)^m)) : k, l \geq 0, \right. \\ \left. k + l = (r + 1)^m - 1 \right\}$$

is a class \mathcal{KL} function. (Here α^k denotes the composition of α with itself k times.)

All the desired conclusions follow from the formula (16). Since $\tilde{\mathbf{x}}$ is an optimal trajectory for Problem 1 which consists in maximizing $|\mathbf{x}(t_f)|^2$, it is clear that every other solution of (8) with $\mathbf{x}(0) = \mathbf{p}$ also satisfies $|\mathbf{x}(t_f)| \leq \beta(|\mathbf{p}|, t_f)$. In view of the bound $\beta(|\mathbf{p}|, t_f) \leq \beta(|\mathbf{p}|, 0)$ and the fact that \mathbf{p} and $t_f < T_{\max}(|\mathbf{p}|)$ were arbitrary, we conclude that all solutions of (8) are bounded and so exist globally in time. In other words, t_f could be an arbitrary positive number, and all solutions of (8) satisfy the bound (4). We know from Lemma 1 that the same is automatically true for the solutions of the differential inclusion (3), which in turn include all solutions of the switched system (1). It remains to recall that (4) defines the stability properties that needed to be established. \square

V. CONCLUSIONS

We have formulated the first stability criterion for switched nonlinear systems (or, more generally, differential inclusions) that involves Lie brackets of the individual vector fields but does not require that these vector fields commute (Theorem 1). As a special case, we have shown that a switched system generated by two globally asymptotically stable nonlinear vector fields whose third-order Lie brackets vanish is globally uniformly asymptotically stable (Corollary 1). This represents a nonlinear extension of an earlier result from [12] and a promising step toward a solution of the open problem described in [1]. In contrast with the methods previously used to obtain results of this kind, our approach relied on reducing the stability analysis problem to the “worst-case” optimal control problem and then proving, via the MP, the existence of an optimal control which is bang-bang with a bound on the total number of switches.

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