Lie-algebraic conditions for stability of switched systems

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THE PROBLEM

A family of as. stable systems

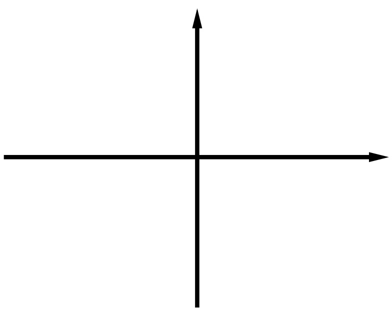
$$\{\dot{x} = f_p(x),\ \ p \in \mathcal{P}\}$$

Switched system:

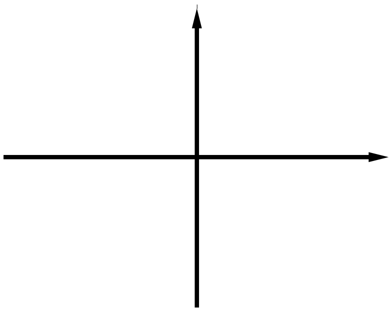
$$\boxed{\dot{x} = f_\sigma(x)}$$

$\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant *switching signal*

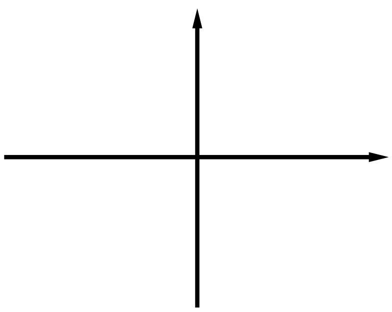
Conditions for stability under arbitrary switching – ?



stable



stable



unstable

$$g := \{f_p\,,\ \ p \in \mathcal{P}\}_{LA}$$

SWITCHED LINEAR SYSTEM

$$\dot{x} = A_\sigma x$$

where $\{A_p\,,\ p \in \mathcal{P}\}$ is a compact set of stable matrices in $\mathbb{R}^{n\times n}$

$$g := \{A_p\,,\ p \in \mathcal{P}\}_{LA}$$

Case 1: all matrices A_p commute pairwise

$[A_p, A_q] = A_pA_q - A_qA_p = 0 \Rightarrow e^{A_p}e^{A_q} = e^{A_q}e^{A_p}$

$$\mathcal{P} = \{1, \ldots, m\}$$

$$\Downarrow$$

$x(t) = e^{A_1t_1}e^{A_2t_2}\ldots e^{A_mt_m}x(0)$ where $t_1 + \cdots + t_m = t$

$$\Downarrow$$

$$x(t) \to 0 \text{ as } t \to \infty$$

QUADRATIC COMMON LYAPUNOV FUNCTION

Narendra-Balakrishnan'94:

$$A_1^T P_1 + P_1 A_1 = -I$$

$$A_2^T P_2 + P_2 A_2 = -P_1$$

$$\dots$$

$$A_m^T P_m + P_m A_m = -P_{m-1}$$

Then $V(x) := x^T P_m x$ is a common Lyap. fcn:

$$P_m = \int_0^\infty e^{A_m^T t_m} \dots \left(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1 \right) \dots e^{A_m t_m} dt_m$$

For each $i \in \{1, \dots, m\}$ we have

$$P_m = \int_0^\infty e^{A_i^T t_i} Q_i \, e^{A_i t_i} \, dt_i \quad \text{with } Q_i > 0$$

NILPOTENT LIE ALGEBRA

Case 2: g is nilpotent

($g^1 := g$, $g^{k+1} := [g, g^k]$. *Nilpotent*: $g^k = 0$ for some k
$g^{(1)} := g$, $g^{(k+1)} := [g^{(k)}, g^{(k)}]$. *Solvable*: $g^{(k)} = 0$ for some k)

Conjecture (Gurvits'95):

g is nilpotent $\Rightarrow$ the switched linear system

$$x(k+1) = A_{\sigma(k)} x(k)$$

is as. stable for any σ

A_1, A_2 nonsingular, $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$

Baker-Campbell-Hausdorff formula:

$$x(k) = A_1^{k_1} A_2^{k_1} A_1^{k_3} A_2^{k_4} A_1^{k_5} x(0) \text{ where } k_1 + \cdots + k_5 = k$$

$$\Downarrow$$

$$x(k) \to 0 \text{ as } k \to \infty$$

SOLVABLE LIE ALGEBRA

Case 3: g is solvable

Lie's Theorem:

$\exists$ a basis in which all matrices are upper-triangular

Example:

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$$

$$\dot{x} = A_\sigma x$$

$$\begin{aligned} \dot{x}_1 &= a_\sigma x_1 + b_\sigma x_2 \\ \dot{x}_2 &= c_\sigma x_2 \end{aligned}$$

MAIN RESULT

Quad. common Lyap. fcn: $V(x) = x^T D\, x$

Theorem. *If* $\{A_p,\ p \in \mathcal{P}\}$ *is a compact set of stable matrices in* $\mathbb{R}^{n\times n}$ *and the Lie algebra* $\{A_p\,,\ p \in \mathcal{P}\}_{LA}$ *is solvable, then the switched linear system* $\dot{x} = A_\sigma x$ *is globally uniformly exponentially stable:*

$$|x(t)| \le ce^{-\lambda t}|x(0)| \qquad \forall t \ge 0,\ \forall x(0),\ \forall \sigma(\cdot)$$

ROBUSTNESS

Exp. stability is a robust property

A_p upper-triangular

$$A_p^T P + P A_p = -Q_p < 0 \qquad \forall p$$

E_p lower-triangular :

$$\sigma_{max}(E_p) < \frac{\sigma_{min}(Q_p)}{2\sigma_{max}(P)}$$

Then

$$(A_p + E_p)^T P + P(A_p + E_p) < 0 \qquad \forall p$$

Lie-algebraic condition is not robust

NECESSITY

Molchanov-Pyatnitskiy '89:

as. stability for any σ $\Rightarrow$ common Lyap. fcn

It may not be quadratic (Martin-Dayawansa'96)

Conditions that lead to quadratic common Lyap. fcn cannot be necessary

NONLINEAR CASE

$$\dot{x} = f_\sigma(x)$$

Commuting vector fields: $[f_p, f_q](x) = 0 \quad \forall p, q \in \mathcal{P}$

Local common Lyap. fcn (Shim-Noh-Seo'98):

$\mathcal{P} = \{1, \dots, m\}$

$\varphi_p(t, x)$ is solution of $\dot{x} = f_p(x)$ with i.c. x

$$V_1(x) := \int_0^T |\varphi_1(s, x)|^2 ds$$
$$V_2(x) := \int_0^T V_1(\varphi_2(s, x)) ds$$
$$\dots$$
$$V_m(x) := \int_0^T V_{m-1}(\varphi_m(s, x)) ds$$

So we have local as. stability for any σ

LINEARIZATION

Consider linearizations:

$$F_p := \left.\frac{\partial f_p}{\partial x}\right|_{x=0}$$

If $\{F_p,\, p \in \mathcal{P}\}_{LA}$ is solvable, then the switched system $\dot{x} = f_\sigma(x)$ is locally as. stable for any σ

Original Lie algebra – ?

CONCLUSIONS

$$\dot{x} = f_\sigma(x)$$

Summary:

- Stability $\longleftrightarrow$ structure of $g := \{f_p,\, p \in \mathcal{P}\}_{LA}$
- g is solvable $\Rightarrow$ as. stability for any σ

Open questions:

- Robustness
- Nonlinear case: global results
- Wei-Norman equations