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Brief Paper

# Hysteresis-based switching algorithms for supervisory control of uncertain systems<sup>☆</sup>

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## Abstract

We address the problem of controlling a linear system with unknown parameters ranging over a continuum by means of switching among a finite family of candidate controllers. We present a new hysteresis-based switching logic, designed specifically for this purpose, and derive a bound on the number of switches produced by this logic on an arbitrary time interval. The resulting switching control algorithm is shown to provide stability and robustness to arbitrary bounded noise and disturbances and sufficiently small unmodeled dynamics.

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## 1. Introduction

Suppose that a given process admits a model containing unknown parameters, and the goal is to design a feedback controller that achieves some desired behavior in the face of noise, disturbances, and unmodeled dynamics. The kind of control algorithm that we have in mind is the one that relies on switching among a family of candidate controllers, and bases controller selection on certainty equivalence. In this framework, one associates to each possible value of the unknown parameters a *monitoring signal*, designed in such a way that a small value of this signal indicates a high likelihood that the corresponding parameters are close to the actual unknown values. The switching algorithm then selects, from time to time, a controller that has been designed for the parameter values associated with the smallest monitoring signal.

Relevant questions in this context are: How to pick the individual controllers? How to design the monitoring

signals? Which controller to switch to? When to switch? In addressing the first question, the methods considered here allow one to rely on conventional techniques from linear robust control theory. An answer to the second question emerges from the *supervisory control* architecture (Morse, 1996; Hocherman-Frommer, Kulkarni, & Ramadge, 1998; Mosca, Capecchi, & Casavola, 2001; Pait & Kassab, 1997) reviewed later in this paper. The certainty equivalence principle provides one way to settle the third question. To deal with the last question, one needs to specify a *switching logic*. It is this last question that will be of primary concern in what follows.

One way to orchestrate the switching is provided by the *dwell-time switching logic*. In this logic, consecutive switching instants are separated by (at least) a prespecified time interval, called the *dwell time*, which is sufficiently large so that the switching does not destabilize the system. Dwell-time switching was used in Morse (1996, 1997) to design set-point supervisory control algorithms for linear systems with a continuum of parametric uncertainty, noise, disturbances, and unmodeled dynamics.

However, dwell-time switching has some significant disadvantages. With a prespecified dwell time, the performance of the currently active controller might deteriorate to an unacceptable level before the next switch is permitted. If the uncertain process is nonlinear, the trajectories may even

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escape to infinity in finite time. These considerations motivate the study of switching algorithms which do not rely on a fixed dwell time.

An example of such a switching logic is the *hysteresis switching logic* described in Middleton, Goodwin, Hill, and Mayne (1988), Morse, Mayne, and Goodwin (1992). According to this logic, a switch occurs when the monitoring signal that corresponds to the controller currently in the feedback loop exceeds the smallest monitoring signal by a prespecified positive number, called the *hysteresis constant*. Using a multiplicative hysteresis constant instead of an additive one, one arrives at the *scale-independent hysteresis switching logic*, which was studied and applied to control of uncertain nonlinear systems in Hespanha (1998) and elsewhere. For a finite family of monitoring signals satisfying suitable assumptions, these logics guarantee that switching stops in finite time. This allows one to design supervisory control algorithms which are effective when the unknown parameters take values in a finite set and there are no noise, disturbances, or unmodeled dynamics.

We continue to study the scale-independent hysteresis switching logic in this paper by establishing a bound on the number of switches on an arbitrary finite interval. Combining this with the results of Hespanha and Morse (1999) on stability of switched systems with *average dwell time*, it is possible to analyze the correctness of the supervisory control algorithm for the case when the unknown parameters belong to a finite set, without relying on the termination of switching. Details of this analysis are provided in Hespanha et al. (2001).

The main contribution of the present work is to deal with the case when the unknown parameters belong to a continuum. To do this, we need a new switching logic, which we develop here and call *hierarchical hysteresis switching*. It relies on a partition of the parametric uncertainty set into a finite number of subsets. The name of the logic comes from the fact that the minimization of the monitoring signals is carried out on two levels: first, the smallest one is taken for each of the subsets that form the partition, and then the smallest signal among these is chosen. We derive a bound on the number of switches produced by the hierarchical hysteresis switching logic, using the corresponding result for scale-independent hysteresis switching.

In the context of supervisory control with hierarchical hysteresis switching, the subsets in the partition used by the switching logic are chosen to be sufficiently small so that there exists a robustly stabilizing controller for each subset. We show that this leads to a supervisory control algorithm whose stability can be analyzed in the presence of noise, disturbances, and unmodeled dynamics. Thus we are able to handle the same class of systems as that treated in Morse (1996, 1997), without requiring a fixed dwell time.

The scale-independent hysteresis switching logic is defined and studied in the next section. The hierarchical

hysteresis switching logic and its properties are presented in Section 3. The supervisory control system is described in Section 4. Its analysis is given in Section 5.

## 2. Scale-independent hysteresis switching logic

The functioning of the *scale-independent hysteresis switching logic*, introduced in Hespanha (1998), is as follows (see Figure 1). Inputs to the logic are continuous signals  $\mu_p : [0, \infty) \rightarrow \mathbb{R}$ ,  $p \in \mathcal{P}$ , where the index set  $\mathcal{P}$  is taken throughout the paper to be a compact subset of a finite-dimensional normed linear vector space. The output of the logic will be a *switching signal*  $\sigma$  taking values in  $\mathcal{P}$ . Let us pick a number  $h > 0$  called the *hysteresis constant*. First, we set  $\sigma(0) = \arg \min_{p \in \mathcal{P}} \{\mu_p(0)\}$ . Suppose that at a certain time  $t_i$  the value of  $\sigma$  has just switched to some  $q \in \mathcal{P}$ . We then keep  $\sigma$  fixed until a time  $t_{i+1} > t_i$  such that  $(1 + h) \min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\} \leq \mu_q(t_{i+1})$ , at which point we set  $\sigma(t_{i+1}) = \arg \min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\}$ .

Repeating the above steps, we generate a piecewise constant signal  $\sigma$  which is continuous from the right everywhere. If  $\mu_p$ ,  $p \in \mathcal{P}$  are uniformly bounded away from zero, i.e., if for some  $\varepsilon > 0$  we have  $\mu_p(t) \geq \varepsilon$  for all  $p \in \mathcal{P}$  and all  $t \geq 0$ , then infinitely fast switching (chattering) is avoided. In fact, there must be an interval  $[0, T_{\max})$  of maximal length on which  $\sigma$  exists, and there can only be a finite number of switches on each proper subinterval of  $[0, T_{\max})$ ; see Hespanha (1998) for details.

When the indicated argmin is not unique, a particular value for  $\sigma$  among those that achieve the minimum can be chosen arbitrarily. Carrying out the minimization over  $\mathcal{P}$  is a trivial task if  $\mathcal{P}$  is a finite set. If  $\mathcal{P}$  is a continuum, in many cases of interest (including those when the unknown parameters enter the model of the process linearly) the problem reduces to solving a polynomial equation in  $p$  on-line. Tractability issues regarding the above minimization procedure in the context of supervisory control are further discussed in Morse (1996).

It is not hard to see from the definition of the above switching logic that its output  $\sigma$  would not be affected if we replaced the signals  $\mu_p$ ,  $p \in \mathcal{P}$  by their scaled versions

$$\tilde{\mu}_p(t) := \Theta(t)\mu_p(t), \quad p \in \mathcal{P}, \quad (1)$$

where  $\Theta$  is some positive function of time. This is where the term “scale-independent” comes from. One can choose  $\Theta$  so that the scaled signals  $\tilde{\mu}_p$ ,  $p \in \mathcal{P}$  have some additional properties (such as monotonicity) and then use these signals for analysis purposes (cf. below).

For  $0 \leq t_0 < t < T_{\max}$ , we will denote by  $N_\sigma(t, t_0)$  the number of discontinuities of  $\sigma$  on the interval  $(t_0, t)$ . For the case of a finite  $\mathcal{P}$ , we have the following result.

**Lemma 1** (Scale-independent hysteresis switching lemma). *Let  $\mathcal{P} = \{1, 2, \dots, m\}$  with  $m$  a positive integer. Suppose that the signals  $\tilde{\mu}_p$ ,  $p \in \mathcal{P}$  are continuous and monotone*

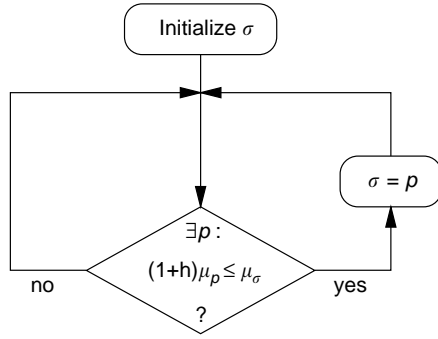


Fig. 1. Scale-independent hysteresis switching logic.

increasing, and that there exists a number  $\varepsilon > 0$  such that  $\bar{\mu}_p(0) \geq \varepsilon \forall p \in \mathcal{P}$ . Then, for an arbitrary index  $l \in \mathcal{P}$  and arbitrary numbers  $t_0$  and  $t$  satisfying  $0 \leq t_0 < t < T_{\max}$ , we have

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\mu}_l(t)}{\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)}\right) \quad (2)$$

and

$$\sum_{k=0}^{N_\sigma(t, t_0)} (\bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k)) \leq m \left( (1+h)\bar{\mu}_l(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0) \right) \quad (3)$$

where  $t_1 < t_2 < \dots < t_{N_\sigma(t, t_0)}$  are the discontinuities of  $\sigma$  on  $(t_0, t)$  and  $t_{N_\sigma(t, t_0)+1} := t$ .

**Remark 1.** The left-hand side of inequality (3) can be thought of as the variation of  $\bar{\mu}_\sigma$  over the interval  $[t_0, t]$ . If the signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  are (piecewise) differentiable, then the left-hand side of (3) equals the integral  $\int_{t_0}^t \dot{\bar{\mu}}_{\sigma(\tau)}(\tau) d\tau$ , which is to be interpreted as the sum of integrals over intervals on which  $\sigma$  is constant.

**Proof.** For each  $k \in \{0, 1, \dots, N_\sigma(t, t_0)\}$ , the value of  $\sigma$  remains constant and equal to  $\sigma(t_k)$  on  $[t_k, t_{k+1})$ . Therefore,

$$\bar{\mu}_{\sigma(t_k)}(t) \leq (1+h)\bar{\mu}_p(t) \quad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \{0, 1, \dots, N_\sigma(t, t_0)\}, \quad \forall p \in \mathcal{P}. \quad (4)$$

Since  $\sigma$  switched to  $\sigma(t_k)$  at time  $t_k$ ,  $k \in \{1, 2, \dots, N_\sigma(t, t_0)\}$ , we also have

$$\bar{\mu}_{\sigma(t_k)}(t_k) \leq \bar{\mu}_p(t_k) \quad \forall k \in \{1, 2, \dots, N_\sigma(t, t_0)\}, \quad \forall p \in \mathcal{P}. \quad (5)$$

Moreover,  $\sigma$  switched from  $\sigma(t_k)$  to  $\sigma(t_{k+1})$  at time  $t_{k+1}$ ,  $k < N_\sigma(t, t_0) - 1$ , thus, because of the continuity of  $\bar{\mu}_{\sigma(t_k)}$ ,

$$\bar{\mu}_{\sigma(t_k)}(t_{k+1}) = (1+h)\bar{\mu}_{\sigma(t_{k+1})}(t_{k+1}) \quad \forall k \in \{0, 1, \dots, N_\sigma(t, t_0) - 1\}. \quad (6)$$

Inequalities (4)–(6) summarize all the properties of the switching logic that are relevant for this proof.

Since  $\mathcal{P}$  has  $m$  elements, there must be a  $q \in \mathcal{P}$  such that  $\sigma = q$  on at least  $\bar{N} := \lceil (N_\sigma(t, t_0) - 1)/m \rceil$  of the intervals  $[t_1, t_2), [t_2, t_3), \dots, [t_{N_\sigma(t, t_0)-1}, t_{N_\sigma(t, t_0)})$ . (7)

If  $\bar{N} \leq 1$ , then we must have  $N_\sigma(t, t_0) \leq 1 + m$  and therefore (2) automatically holds. Suppose now that  $\bar{N} \geq 2$  and let  $[t_{k_1}, t_{k_1+1}), [t_{k_2}, t_{k_2+1}), \dots, [t_{k_{\bar{N}}}, t_{k_{\bar{N}}+1})$  be the  $\bar{N}$  intervals on which  $\sigma = q$ . Picking any  $i \in \{1, 2, \dots, \bar{N} - 1\}$ , in view of (6), the assumed monotonicity of  $\bar{\mu}_{\sigma(t_{k_i+1})}$ , and (5), we conclude that

$$\begin{aligned} \bar{\mu}_q(t_{k_i+1}) &= (1+h)\bar{\mu}_{\sigma(t_{k_i+1})}(t_{k_i+1}) \\ &\geq (1+h)\bar{\mu}_{\sigma(t_{k_i+1})}(t_{k_i}) \\ &\geq (1+h)\bar{\mu}_q(t_{k_i}). \end{aligned} \quad (8)$$

Moreover, because the intervals are nonoverlapping,  $t_{k_{i+1}} \geq t_{k_i+1}$ , hence  $\bar{\mu}_q(t_{k_{i+1}}) \geq \bar{\mu}_q(t_{k_i+1})$ . From this and (8) we obtain  $(1+h)\bar{\mu}_q(t_{k_i}) \leq \bar{\mu}_q(t_{k_{i+1}})$  for  $i \in \{1, 2, \dots, \bar{N} - 1\}$ . Iterating the last inequality from  $i = 1$  to  $\bar{N} - 1$  yields  $(1+h)^{\bar{N}-1}\bar{\mu}_q(t_{k_1}) \leq \bar{\mu}_q(t_{k_{\bar{N}}})$  and therefore  $(1+h)^{\bar{N}-1}\bar{\mu}_q(t_{k_1}) \leq \bar{\mu}_l(t_{k_{\bar{N}}})$  for all  $l \in \mathcal{P}$  because of (5). Using the monotonicity of the  $\bar{\mu}_p$ , we conclude that  $(1+h)^{\bar{N}-1}\bar{\mu}_q(t_0) \leq \bar{\mu}_l(t)$  for all  $l \in \mathcal{P}$ . In light of this, we must have  $(1+h)^{\bar{N}-1}\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0) \leq \bar{\mu}_l(t)$  for all  $l \in \mathcal{P}$ , from which the inequality (2) follows by virtue of the fact that  $\bar{N} \geq (N_\sigma(t, t_0) - 1)/m$ .

It remains to show that inequality (3) holds. Grouping all the terms in the summation on the left-hand side of (3) for which  $\sigma(t_k)$  is the same, we obtain

$$\begin{aligned} &\sum_{k=0}^{N_\sigma(t, t_0)} (\bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k)) \\ &= \sum_{q \in \mathcal{P}} \sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t, t_0)} (\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)). \end{aligned} \quad (9)$$

Take some value  $q \in \mathcal{P}$  that  $\sigma$  takes on  $(t_0, t)$ . Since intervals (7) are nonoverlapping, it follows from the monotonicity of  $\bar{\mu}_q$  that

$$\sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t, t_0)} (\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)) \leq \bar{\mu}_q(t_{k_q+1}) - \bar{\mu}_q(t_0), \quad (10)$$

where  $k_q$  denotes the largest index  $k \in \{0, 1, \dots, N_\sigma(t, t_0)\}$  for which  $\sigma(t_k) = q$ . Now, because of (4),  $\bar{\mu}_q(t_{k_q+1}) \leq (1+h)\bar{\mu}_l(t_{k_q+1})$  for all  $l \in \mathcal{P}$ . From this, (10), and the

<sup>1</sup> Given a scalar  $a$ , we denote by  $\lceil a \rceil$  the smallest integer larger than or equal to  $a$ .

monotonicity of the  $\bar{\mu}_p$ , we conclude that

$$\sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t,t_0)} (\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)) \\ \leq (1+h)\bar{\mu}_l(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0) \quad \forall l \in \mathcal{P}.$$

Inequality (3) follows from this, (9), and the fact that  $\mathcal{P}$  has  $m$  elements.  $\square$

### 3. Hierarchical hysteresis switching

We now describe the hierarchical hysteresis switching logic. Its inputs are continuous signals  $\mu_p$ ,  $p \in \mathcal{P}$ , where  $\mathcal{P}$  is a compact index set as before. Fix a positive integer  $m$  and denote by  $\mathcal{M}$  the set  $\{1, 2, \dots, m\}$ . Assumed given is a family of closed subsets  $\mathcal{D}_i$ ,  $i \in \mathcal{M}$  of  $\mathcal{P}$ , whose union is the entire  $\mathcal{P}$ . The output of the switching logic will be a *switching signal*  $\sigma$  taking values in  $\mathcal{M}$ . Pick a number  $h > 0$ , called the *hysteresis constant*. First, we select some  $j_0 \in \mathcal{M}$  such that  $\mathcal{D}_{j_0}$  contains  $\arg \min_{p \in \mathcal{P}} \{\mu_p(0)\}$ , and set  $\sigma(0) = j_0$ . Suppose that at a certain time  $t_i$  the value of  $\sigma$  has just switched to some  $j_i \in \mathcal{M}$ . We then keep  $\sigma$  fixed until a time  $t_{i+1} > t_i$  such that the following inequality is satisfied:

$$(1+h) \min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\} \leq \min_{p \in \mathcal{D}_{j_i}} \{\mu_p(t_{i+1})\}.$$

At this point, we select some  $j_{i+1} \in \mathcal{M}$  such that  $\mathcal{D}_{j_{i+1}}$  contains  $\arg \min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\}$ , and set  $\sigma(t_{i+1}) = j_{i+1}$ .

The above procedure yields a piecewise constant signal  $\sigma$  which is continuous from the right everywhere. By the same argument as in Hespanha (1998), one can show that chattering is avoided if  $\mu_p$ ,  $p \in \mathcal{P}$  are uniformly bounded away from zero. In fact, there exists a maximal interval  $[0, T_{\max})$  on which  $\sigma$  is defined, and there can only be a finite number of switches on each proper subinterval of  $[0, T_{\max})$ . In the supervisory control application treated below, we will always have  $T_{\max} = \infty$ . When the indicated  $\arg \min$  is not unique, a particular index among those that achieve the minimum can be chosen arbitrarily. The understanding here is that minimization over  $\mathcal{D}_i$ 's is computationally tractable if these sets are sufficiently small.

The above switching logic is also scale independent, i.e., its output would not be affected if we replaced the signals  $\mu_p$ ,  $p \in \mathcal{P}$  by their scaled versions (1), where  $\Theta$  is some positive function of time. In the supervisory control context to be discussed below, we will arrange matters in such a way that it is possible to find a function  $\Theta$  which makes the scaled signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  strictly positive and monotone increasing. For analysis purposes, we will always use the scaled signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  that have this property, while the actual inputs to the switching logic are the original signals  $\mu_p$ ,  $p \in \mathcal{P}$ .

**Definition.** We say that a piecewise constant signal  $\zeta$  taking values in  $\mathcal{P}$  is  $\{\mathcal{D}_i\}$ -consistent with a given switching signal  $\sigma$  on an interval  $[t_0, t]$  if:

1. For all  $s \in [t_0, t]$  we have  $\zeta(s) \in \mathcal{D}_{\sigma(s)}$ .
2. The set of discontinuities of  $\zeta$  on  $[t_0, t]$  is a subset of the set of discontinuities of  $\sigma$ .

We now note the following fact, which follows immediately from the definitions of the switching logics considered in this section and the previous one.

**Remark 2.** The signal  $\sigma$  produced by the hierarchical hysteresis switching logic coincides with the signal that would be produced by the scale-independent hysteresis switching logic with inputs  $\min_{p \in \mathcal{D}_i} \{\mu_p(t)\}$ ,  $i \in \mathcal{M}$ .

The result presented below is a consequence of this observation. As before, for  $0 \leq t_0 < t < T_{\max}$  we denote by  $N_\sigma(t, t_0)$  the number of discontinuities of  $\sigma$  on the interval  $(t_0, t)$ .

**Lemma 2** (Hierarchical hysteresis switching lemma).

Suppose that the signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  are continuous and monotone increasing, and that there exists a number  $\varepsilon > 0$  such that  $\bar{\mu}_p(0) \geq \varepsilon \forall p \in \mathcal{P}$ . Then, for an arbitrary index  $l \in \mathcal{P}$  and arbitrary numbers  $t_0$  and  $t$  satisfying  $0 \leq t_0 < t < T_{\max}$ , we have

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log \left( \frac{\bar{\mu}_l(t)}{\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)} \right). \quad (11)$$

In addition, there exists a signal  $\zeta$ , which is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$  on  $[t_0, t]$ , such that

$$\sum_{k=0}^{N_\sigma(t,t_0)} (\bar{\mu}_{\zeta(t_k)}(t_{k+1}) - \bar{\mu}_{\zeta(t_k)}(t_k)) \\ \leq m \left( (1+h)\bar{\mu}_l(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0) \right), \quad (12)$$

where  $t_1 < t_2 < \dots < t_{N_\sigma(t,t_0)}$  are the discontinuities of  $\sigma$  on  $(t_0, t)$  and  $t_{N_\sigma(t,t_0)+1} := t$ .

**Proof.** Inequality (11) follows at once from the scale-independent hysteresis switching lemma (Lemma 1) and Remark 2. A signal  $\zeta$  that satisfies the second statement of the lemma can be defined as follows: for each  $s \in [t_0, t]$ , let  $\zeta(s) := \arg \min_{p \in \mathcal{D}_{\sigma(s)}} \{\bar{\mu}_p(t_{k+1})\}$ , where  $k$  is the largest index in the set  $\{0, 1, \dots, N_\sigma(t, t_0)\}$  for which  $\sigma(t_k) = \sigma(s)$ . In other words,  $\zeta(s) = \arg \min_{p \in \mathcal{D}_{\sigma(s)}} \{\bar{\mu}_p(\tau)\}$ , where  $\tau$  is the right endpoint of the last subinterval of  $[t_0, t]$  on which  $\sigma$  equals  $\sigma(s)$ . Then  $\zeta$  is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$  on  $[t_0, t]$  by construction. Grouping all the terms in the summation on the left-hand side of (12) for which  $\sigma$  is the same, and reasoning exactly as in the proof of Lemma 1, we arrive at (12).  $\square$



**Remark 3.** The signal  $\zeta$  depends on the choice of the time  $t$ . As before, if the signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  are (piecewise) differentiable, then the left-hand side of inequality (12) equals the integral  $\int_{t_0}^t \bar{\mu}_{\zeta(\tau)}(\tau) d\tau$ , which is to be interpreted as the sum of integrals over intervals on which  $\zeta$  is constant.

#### 4. Supervisory control system

Suppose that the uncertain process  $\mathbb{P}$  to be controlled admits the model of a SISO finite-dimensional stabilizable and detectable linear system with control input  $u$  and measured output  $y$ , perturbed by a bounded disturbance input  $d$  and a bounded output noise  $n$ . It is assumed known that the transfer function of  $\mathbb{P}$  from  $u$  to  $y$  belongs to a family of admissible process model transfer functions  $\bigcup_{p \in \mathcal{P}} \mathcal{F}(p)$ , where  $p$  is a parameter taking values in a compact index set  $\mathcal{P}$ . Here each  $\mathcal{F}(p)$  denotes a family of transfer functions “centered” around some known *nominal* process model transfer function  $v_p$ .

The problem of interest is to design a feedback controller that achieves output regulation, i.e., drives the output  $y$  of  $\mathbb{P}$  to zero, whenever the noise and disturbance signals are zero. Moreover, all system signals must remain bounded in response to arbitrary bounded noise and disturbance inputs. Everything that follows can be readily extended to the more general problem of set-point control with the help of adding an integrator in the feedback loop, as in Morse (1996, 1997). Extensions to MIMO systems are also possible.

Uncertainty of the kind considered here may be associated with unpredictable changes in operating environment, component failures, or various external influences. The set  $\mathcal{P}$  represents the range of parametric uncertainty, while for each fixed  $p \in \mathcal{P}$  the subfamily  $\mathcal{F}(p)$  accounts for unmodeled dynamics. There are several ways to specify allowable unmodeled dynamics around the nominal process model transfer functions  $v_p$  (see Hespanha et al. (2001)). For example, take two arbitrary numbers  $\delta \geq 0$  and  $\lambda_u > 0$ . Then for each  $p \in \mathcal{P}$  we can define

$$\mathcal{F}(p) := \{v_p(1 + \delta_p^m) + \delta_p^a : \|\delta_p^m\|_{\infty, \lambda_u} \leq \delta, \|\delta_p^a\|_{\infty, \lambda_u} \leq \delta\},$$

where  $\|\cdot\|_{\infty, \lambda_u}$  denotes the  $e^{\lambda_u t}$ -weighted  $\mathcal{H}_\infty$  norm of a transfer function, i.e.,  $\|v\|_{\infty, \lambda_u} = \sup_{\omega \in \mathbb{R}} |v(j\omega - \lambda_u)|$ . This yields the class of admissible process models treated in Morse (1996, 1997). In the sequel, we assume for concreteness that unmodeled dynamics are specified in this way; we will refer to the parameter  $\delta$  as the *unmodeled dynamics bound*. The results presented below can be extended to other classes of admissible process models, such as those studied in Hespanha et al. (2001).

Typically, no single controller is capable of solving the regulation problem for the entire family of admissible process models. Therefore, one needs to develop a controller whose dynamics can change on the basis of available real-time data. Within the framework of supervisory control discussed here, this task is carried out by a “high-level”

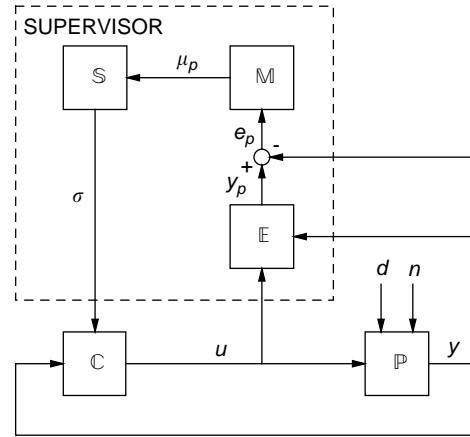


Fig. 2. Supervisory control architecture.

controller, called a *supervisor*, whose purpose is to orchestrate the switching among a parameterized family of *candidate controllers*  $\{C_q : q \in \mathcal{Q}\}$ , where  $\mathcal{Q}$  is an index set. This leads to a switched controller which we call the *multi-controller* and denote by  $\mathbb{C}$ . We require the above controller family to be sufficiently rich so that every admissible process model can be stabilized by placing in the feedback loop the controller  $C_q$  for some index  $q \in \mathcal{Q}$ . In this paper, we focus on the case when  $\mathcal{Q}$  is a finite set. Without loss of generality, we take  $\mathcal{Q} = \mathcal{M} = \{1, 2, \dots, m\}$  where  $m$  is a positive integer.

The supervisor consists of three subsystems (Fig. 2):

**Multi-estimator  $\mathbb{E}$ :** A dynamical system whose inputs are the output  $y$  and the input  $u$  of the process  $\mathbb{P}$  and whose outputs are the signals  $y_p$ ,  $p \in \mathcal{P}$ . Each  $y_p$  would converge to  $y$  asymptotically if the transfer function of  $\mathbb{P}$  were equal to the nominal process model transfer function  $v_p$  and there were no noise or disturbances.

**Monitoring signal generator  $\mathbb{M}$ :** A dynamical system whose inputs are the *estimation errors*

$$e_p := y_p - y, \quad p \in \mathcal{P}$$

and whose outputs  $\mu_p$ ,  $p \in \mathcal{P}$  are suitably defined integral norms of the estimation errors, called *monitoring signals*.

**Switching logic  $\mathbb{S}$ :** A switched system whose inputs are the monitoring signals  $\mu_p$ ,  $p \in \mathcal{P}$  and whose output is a *switching signal*  $\sigma$ , taking values in  $\mathcal{M}$ , which is used to define the control law  $u$ .

We now briefly recall from Morse (1996) the key state-space equations for the different subsystems appearing in Fig. 1. As  $i$  ranges over  $\mathcal{M}$ , let

$$\dot{x}_{\mathbb{C}} = A_i x_{\mathbb{C}} + b_i y,$$

$$u = k_i x_{\mathbb{C}} + r_i y$$

be realizations of the transfer functions of the candidate controllers, all sharing the same state  $x_{\mathbb{C}}$ . See Morse (1996) for more details on constructing such realizations. We then

describe the multi-controller  $\mathbb{C}$  by the equations

$$\begin{aligned}\dot{x}_{\mathbb{C}} &= A_{\sigma}x_{\mathbb{C}} + b_{\sigma}y, \\ u &= k_{\sigma}x_{\mathbb{C}} + r_{\sigma}y.\end{aligned}$$

We assume that the multi-estimator is also realized in a state-shared fashion, as given by

$$\begin{aligned}\dot{x}_{\mathbb{E}} &= A_{\mathbb{E}}x_{\mathbb{E}} + b_{\mathbb{E}}y + d_{\mathbb{E}}u, \\ y_p &= c_p x_{\mathbb{E}}, \quad p \in \mathcal{P}\end{aligned}$$

with  $A_{\mathbb{E}}$  a stable matrix. This type of structure is quite common in adaptive control. We denote by  $x$  the composite state  $(x'_{\mathbb{E}}, x'_{\mathbb{C}})'$  of the multi-estimator and the multi-controller, and by  $p^*$  an (unknown) element of  $\mathcal{P}$  such that the transfer function of  $\mathbb{P}$  belongs to  $\mathcal{F}(p^*)$ , i.e., the “true” parameter value (due to possibly overlapping sets of unmodeled dynamics,  $p^*$  may not be unique). For every  $l \in \mathcal{P}$ , the evolution of  $x$  can then be described by a system of the form

$$\dot{x} = A_{\sigma l}x + d_{\sigma}e_l, \quad (13)$$

$$y = (c_{p^*}0)x - e_{p^*}, \quad (14)$$

$$u = f_{\sigma}x + g_{\sigma}e_{p^*}. \quad (15)$$

We assume that a partition  $\mathcal{P} = \bigcup_{i \in \mathcal{M}} \mathcal{D}_i$  is given, such that the matrices  $A_{ip}$ ,  $i \in \mathcal{M}$ ,  $p \in \mathcal{P}$  have the following *stability margin property*: for every  $i \in \mathcal{M}$  and every  $p \in \mathcal{D}_i$  the matrix  $A_{ip} + \lambda_0 I$  is stable, where  $\lambda_0$  is a fixed positive number. It follows from the developments of Morse (1996) that such a partition exists, provided that the sets  $\mathcal{D}_i$ ,  $i \in \mathcal{M}$  are sufficiently small and each  $\mathbb{C}_i$  stabilizes the  $p$ th nominal process model whenever  $p \in \mathcal{D}_i$ . We take the sets  $\mathcal{D}_i$  to be closed.

Fix a number  $\lambda \in (0, \min\{\lambda_u, \lambda_0\})$ . As shown in Morse (1996, 1997), there exist positive constants  $\delta_1, \delta_2$  that only depend on the unmodeled dynamics bound  $\delta$  and go to zero as  $\delta$  goes to zero; positive constants  $B_1, B_2$  that only depend on the noise and disturbance bounds and go to zero as these bounds go to zero; and positive constants  $C_1, C_2$  that only depend on the system's parameters and on initial conditions, such that along all solutions of the closed-loop system we have

$$\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau) d\tau \leq B_1 e^{2\lambda t} + C_1 + \delta_1 \int_0^t e^{2\lambda\tau} u^2(\tau) d\tau \quad (16)$$

and

$$\begin{aligned}|e_{p^*}(t)| &\leq B_2 + C_2 e^{-\lambda t} \\ &\quad + \delta_2 e^{-\lambda t} \sqrt{\int_0^t e^{2\lambda\tau} u^2(\tau) d\tau}.\end{aligned} \quad (17)$$

The above inequalities represent the basic requirements being placed on the multi-controller and the multi-estimator, upon which the subsequent analysis depends.

The constant  $\lambda$  will play the role of a “weighting” design parameter in the definition of the monitoring signals. Fix an arbitrary constant  $\varepsilon_{\mu} \geq 0$  (its role will become clear later). We generate the monitoring signals  $\mu_p$ ,  $p \in \mathcal{P}$  by the equations

$$\begin{aligned}\dot{W} &= -2\lambda W + \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix} \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix}', \quad W(0) \geq 0, \\ \mu_p &:= (c_p - 1)W(c_p - 1)' + \varepsilon_{\mu}, \quad p \in \mathcal{P},\end{aligned} \quad (18)$$

where  $W(t)$  is a symmetric nonnegative-definite  $k \times k$  matrix,  $k := \dim(x_{\mathbb{E}}) + 1$ . Since  $c_p x_{\mathbb{E}} - y = e_p \forall p \in \mathcal{P}$ , this yields

$$\begin{aligned}\mu_p(t) &= e^{-2\lambda t} \tilde{\mu}_p(0) \\ &\quad + \int_0^t e^{-2\lambda(t-\tau)} e_p^2(\tau) d\tau + \varepsilon_{\mu}, \quad p \in \mathcal{P},\end{aligned}$$

where  $\tilde{\mu}_p(0) := (c_p - 1)W(0)(c_p - 1)'$ . Note that the dynamics of the monitoring signal generator, as well as those of the multi-estimator, are finite-dimensional even if  $\mathcal{P}$  is an infinite set.

Finally, we define the switching signal using the hierarchical hysteresis switching logic described in Section 3, where the sets  $\mathcal{D}_i$ ,  $i \in \mathcal{M}$  are chosen as explained earlier. Setting  $\Theta(t) := e^{2\lambda t}$  in (1), we see that the signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  are monotone increasing, because they satisfy

$$\dot{\bar{\mu}}_p(t) = \tilde{\mu}_p(0) + \int_0^t e^{2\lambda\tau} e_p^2(\tau) d\tau + \varepsilon_{\mu} e^{2\lambda t}, \quad p \in \mathcal{P}. \quad (19)$$

Moreover, it is easy to ensure that  $\mu_p$ ,  $p \in \mathcal{P}$  are uniformly bounded away from zero, either by setting  $\varepsilon_{\mu} > 0$  or by requiring  $W(0)$  to be positive definite. Therefore, we can apply Lemma 2 and conclude that inequalities (11) and (12) are valid. Since in this case the signals  $\bar{\mu}_p$ ,  $p \in \mathcal{P}$  are differentiable, the left-hand side of inequality (12) equals  $\int_{t_0}^t \dot{\bar{\mu}}_{\zeta(\tau)}(\tau) d\tau$  (see Remark 3). From (19) we have the following formula:

$$\dot{\bar{\mu}}_p(t) = e^{2\lambda t} e_p^2(t) + 2\lambda \varepsilon_{\mu} e^{2\lambda t}, \quad p \in \mathcal{P}. \quad (20)$$

## 5. Analysis

We now proceed to the analysis of the supervisory control system defined by (13)–(15), (18), and the hierarchical hysteresis switching logic. We will sometimes appeal to the state of the uncertain process  $\mathbb{P}$ , which we denote by  $x_{\mathbb{P}}$ . However, the analysis centers around system (13) and its stability margin property described earlier. Consider the system obtained from (13) by

substituting  $\zeta$  for  $l$ , where  $\zeta$  is some signal that is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$ :

$$\dot{x} = A_{\sigma\zeta}x + d_{\sigma}e_{\zeta}. \quad (21)$$

The stability margin property implies that for every fixed time  $s \geq 0$ , the matrix  $A_{\sigma(s)\zeta(s)}$  is stable, with stability margin  $\lambda_0$ . Note that this is true even if the controller  $C_{\sigma(s)}$  does not stabilize the process. (In this case, the estimation error  $e_{\zeta(s)}$  may be large. The underlying property of the closed-loop system is detectability with respect to the estimation error; cf. Morse (1996).)

Following Hespanha and Morse (1999b), we say that the switching signal  $\sigma$  has an average dwell time  $\tau_{AD} > 0$  if there exists a nonnegative number  $N_0$  such that the number of discontinuities of  $\sigma$  on an arbitrary interval  $(t_0, t)$ , denoted by  $N_{\sigma}(t, t_0)$ , satisfies

$$N_{\sigma}(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_{AD}}. \quad (22)$$

We need the following result, which in view of the present assumptions is a straightforward corollary of Theorem 1 of Hespanha and Morse (1999b). It states that the switched system

$$\dot{x} = A_{\sigma\zeta}x \quad (23)$$

is exponentially stable with stability margin  $\lambda$ , uniformly over all  $\sigma$  with sufficiently large average dwell time and all  $\zeta$  that are  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$ .

**Lemma 3.** *There exist positive constants  $\tau^*$  and  $c$  such that for every switching signal  $\sigma$  with an average dwell time  $\tau_{AD} \geq \tau^*$  and every signal  $\zeta$  that is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$  on a given interval  $[t_0, t]$ , all solutions of (23) satisfy  $|x(t)| \leq ce^{-\lambda(t-t_0)}|x(t_0)|$ .*

Let  $\tau^*$  be the number specified by this lemma; it can be calculated explicitly from the proof of (Hespanha and Morse, 1999b, Theorem 1). An immediate corollary of Lemma 3 is that system (21) has a finite  $e^{\lambda t}$ -weighted  $\mathcal{L}_2$ -to- $\mathcal{L}_{\infty}$  induced norm, uniform over all  $\sigma$  with average dwell time no smaller than  $\tau^*$  and all  $\zeta$  that are  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$ .

**Corollary 4.** *There exist positive constants  $g$  and  $g_0$  such that for every  $t > 0$ , every switching signal  $\sigma$  with an average dwell time  $\tau_{AD} \geq \tau^*$ , and every signal  $\zeta$  that is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$  on  $[0, t]$ , all solutions of (21) satisfy*

$$e^{2\lambda t}|x(t)|^2 \leq g \int_0^t e^{2\lambda\tau} e_{\zeta(\tau)}^2(\tau) d\tau + g_0|x(0)|^2. \quad (24)$$

With these results in place, the analysis is similar to that given in Hespanha et al. (2001), where a supervisory control system employing the scale-independent hysteresis

switching logic was studied with the help of Lemma 1; some details will be omitted.

### 5.1. No noise, disturbances, or unmodeled dynamics

We first consider the simple situation where the noise and disturbance signals are zero ( $n = d \equiv 0$ ) and there are no unmodeled dynamics ( $\delta = 0$ ), i.e., the process  $\mathbb{P}$  exactly matches one of the nominal process models. In this case, the constants  $B_1, B_2, \delta_1, \delta_2$  in (16) and (17) are all zero. Let us take  $\varepsilon_{\mu}$  in the definition of the monitoring signals to be zero as well ( $W(0)$  must then be taken positive definite). Inequality (16) gives  $\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau) d\tau \leq C_1$ , which together with (19) implies  $\bar{\mu}_{p^*} \leq \tilde{\mu}_{p^*}(0) + C_1$ . It follows from (11), applied with  $l = p^*$ , that  $N_{\sigma}(t, t_0)$  is bounded by a fixed constant for arbitrary  $t > t_0 \geq 0$ . This means that switching stops in finite time, i.e., there exist a time  $T^*$  and an index  $i^* \in \mathcal{M}$  such that  $\sigma(t) = i^*$  for  $t \geq T^*$ . Thus, (22) holds for every  $\tau_{AD}$  if  $N_0$  is large enough. Fix an arbitrary  $t > 0$ . In view of Lemma 2, Remark 3, and formula (20), there exists a signal  $\zeta$  which is  $\{\mathcal{D}_i\}$ -consistent with  $\sigma$  on  $[0, t]$  and satisfies  $\int_0^t e^{2\lambda\tau} e_{\zeta(\tau)}^2(\tau) d\tau \leq m(1+h)(\tilde{\mu}_{p^*}(0) + C_1)$ . Using (24), we have  $e^{2\lambda t}|x(t)|^2 \leq gm(1+h)(\tilde{\mu}_{p^*}(0) + C_1) + g_0|x(0)|^2$ , thus  $x \rightarrow 0$ . Since  $e_{p^*} \rightarrow 0$  by virtue of (17), we conclude from (14) that  $y \rightarrow 0$ . Therefore, the output regulation problem is solved. Also note that since the evolution of  $x_{\mathbb{P}}$  and  $x$  for  $t \geq T^*$  is described by a linear time-invariant system, the rate of convergence is actually exponential. In light of (15), (18), and detectability of  $\mathbb{P}$ , all the other signals remain bounded for all  $t \geq 0$  (in fact, here the state converges to 0, but this would no longer be true in the case of set-point control). We summarize this as follows.

**Proposition 5.** *Suppose that the noise and disturbance signals are zero and there are no unmodeled dynamics, and set  $\varepsilon_{\mu} = 0$ . Then all the signals in the supervisory control system remain bounded for every set of initial conditions such that  $W(0) > 0$ . Moreover, switching stops in finite time, after which  $y(t)$  converges to 0 exponentially fast.*

### 5.2. Noise and disturbances, no unmodeled dynamics

We now assume that bounded noise  $n$  and disturbance  $d$  are present, but there are no unmodeled dynamics. In this case the switching typically will not stop in finite time. Inequalities (16) and (17) hold with some unknown but finite constants  $B_1$  and  $B_2$ . The parameters  $\delta_1$  and  $\delta_2$  are still zero, and  $C_1$  and  $C_2$  are positive constants as before. We take  $\varepsilon_{\mu}$  to be a positive number. From (16) and (19) we have

$$\bar{\mu}_{p^*}(t) \leq \tilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \varepsilon_{\mu} e^{2\lambda t}. \quad (25)$$

Applying (11) with  $l = p^*$  and using the bound  $\tilde{\mu}_{p^*}(t_0) \geq \varepsilon_\mu e^{2\lambda t_0}$  provided by (19) yields

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_{AD}},$$

where

$$\tau_{AD} = \frac{\log(1+h)}{2\lambda m}$$

and

$$N_0 = 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\tilde{\mu}_{p^*}(0) + B_1 + C_1 + \varepsilon_\mu}{\varepsilon_\mu}\right).$$

We can guarantee that  $\tau_{AD} \geq \tau^*$  by increasing the hysteresis constant  $h$  and/or decreasing the weighting constant  $\lambda$  if necessary. In the sequel, we assume that  $h$  and  $\lambda$  have been chosen in this way.

Using (12) with  $l = p^*$  and  $t_0 = 0$ , (20), and (25), we obtain

$$\int_0^t e^{2\lambda\tau} e_{\zeta(\tau)}^2(\tau) d\tau \leq m((1+h)(\tilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \varepsilon_\mu e^{2\lambda t}) - \varepsilon_\mu e^{2\lambda t_0}),$$

where  $t > 0$  is arbitrary and  $\zeta$  is the signal provided by Lemma 2. Together with (24) this implies that

$$|x(t)|^2 \leq (gm(1+h)(\tilde{\mu}_{p^*}(0) + C_1) + g_0|x(0)|^2)e^{-2\lambda t} + gm(1+h)(B_1 + \varepsilon_\mu).$$

Two conclusions can be drawn from the last formula. First,  $x$  is bounded and, as in the previous subsection we can easily deduce from (15), (17), (18), and detectability of  $\mathbb{P}$  that all system signals remain bounded. Note that the choice of the design parameters  $\lambda$ ,  $h$  and  $\varepsilon_\mu$  did not depend on the noise or disturbance bounds, in other words, explicit knowledge of these bounds is not necessary (we are merely requiring that such bounds exist). Secondly, if  $n$  and  $d$  equal or converge to zero, then  $x$  will approach a neighborhood of the origin whose size is proportional to  $g\varepsilon_\mu$ . A close examination of the last quantity reveals that it decreases to 0 as  $\varepsilon_\mu$  goes to 0, which means that we can make this neighborhood as small as desired by choosing  $\varepsilon_\mu$  sufficiently small. Moreover,  $e_{p^*}$  will converge to zero because of (17), hence  $y$  will also become arbitrarily small in view of (14). We arrive at the following result.

**Proposition 6.** *Suppose that the noise and disturbance signals are bounded and there are no unmodeled dynamics. Then for an arbitrary  $\varepsilon_\mu > 0$  all the signals in the supervisory control system remain bounded for every set of initial conditions. Moreover, for every number  $\varepsilon_y > 0$  there is a value of  $\varepsilon_\mu$  leading to the property that if the noise and disturbance signals converge to zero, then for each solution there is a time  $\bar{T}$  such that  $|y(t)| \leq \varepsilon_y$  for all  $t \geq \bar{T}$ .*

Note that we cannot simply let  $\varepsilon_\mu = 0$ , as this would invalidate the above analysis even if  $W(0) > 0$ . However, by

decreasing  $\varepsilon_\mu$  on-line (e.g., in a piecewise constant fashion), it is possible to recover asymptotic convergence of  $y$  to zero when the noise and disturbance signals converge to zero.

**Remark 4.** The above analysis relies on the property that the average dwell time is sufficiently large. However, using techniques described in (Morse (1996), Section VIII), it is possible to establish the same result for an arbitrary average dwell time, thus removing the restrictions on the design parameters  $h$  and  $\lambda$ . We do not pursue this issue here due to lack of space.

### 5.3. Noise, disturbances, and unmodeled dynamics

If unmodeled dynamics are present, i.e., if the parameter  $\delta$  is positive, then  $\delta_1$  and  $\delta_2$  in (16) and (17) are also positive. In this case, the analysis becomes more complicated, because we can no longer deduce from (16) that the switched system must possess an average dwell time. However, it is possible to prove that the above control algorithm, without any modification, is robust with respect to unmodeled dynamics in the following, “semi-global”, sense. The proof uses a small-gain argument, and is almost identical to the proof of Theorem 4 in Hespanha et al. (2001).

**Theorem 7.** *For arbitrary bounds on the noise and disturbance signals, the supervisory control system has the following properties:*

1. *For every positive value of  $\varepsilon_\mu$  and every number  $E > 0$  there exists a number  $\bar{\delta} > 0$  such that if the unmodeled dynamics bound  $\delta$  is smaller than  $\bar{\delta}$ , then all signals remain bounded for every set of initial conditions such that  $|x_{\mathbb{P}}(0)|, |x(0)| \leq E$ .*
2. *For arbitrary positive numbers  $E$  and  $\varepsilon_y$  there exist a value of  $\varepsilon_\mu$  and a number  $\bar{\delta} > 0$  such that if the noise and disturbance signals converge to zero and the unmodeled dynamics bound  $\delta$  is smaller than  $\bar{\delta}$ , then for each solution with  $|x_{\mathbb{P}}(0)|, |x(0)| \leq E$  there is a time  $\bar{T}$  such that  $|y(t)| \leq \varepsilon_y$  for all  $t \geq \bar{T}$ .*

## 6. Concluding remarks

The main contribution of the paper was to develop a new hysteresis-based switching logic, together with appropriate analysis tools, which can be used in the case of infinite parametric uncertainty (ranging over a continuum) to design provably correct supervisory control algorithms. The results obtained here for linear systems with noise, disturbances, and unmodeled dynamics extend those available previously for systems with unknown parameters taking values in finite sets.

For linear systems, other types of switching logics—particularly those based on dwell time—can be applied in similar situations. However, as explained in the



Introduction, dwell-time switching logics have inherent limitations when one tries to use them in the nonlinear context. Hysteresis-based switching logics, on the other hand, have proved to be successful in dealing with nonlinear systems whose parametric uncertainty is described by a finite set (Hespanha & Morse, 1999a; Hespanha, Liberzon, & Morse, 2002). The work reported here is motivated by these results and serves as an important step towards extending the supervisory control techniques to nonlinear systems with parameters taking values in a continuum. In fact, the hierarchical hysteresis switching logic presented above can be applied to nonlinear systems without any changes; difficulties lie elsewhere, particularly in designing candidate controllers which achieve a suitable nonlinear counterpart of the stability margin property defined in Section 4 and exploited in Section 5.

Another direction for future research is to understand relative advantages of the algorithm presented here and existing schemes for linear systems, from the implementation and performance points of view. This issue fell outside the scope of the present paper. For some examples of how supervisory control methods of the kind considered here can be applied to specific problems, as well as comparison with more traditional adaptive control techniques, we refer the reader to Hespanha, Liberzon, and Morse, (to appear).

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