Stabilizing a linear system with finite-state hybrid output feedback

Daniel Liberzon*
Dept. of Electrical Engineering
Yale University
New Haven, CT 06520-8267

Abstract

The purpose of this short note is to establish and explore a link between the problem of stabilizing a linear system using finite-state hybrid output feedback and the problem of finding a stabilizing switching sequence for a switched linear system with unstable individual matrices, each of which separately has recently received attention in the literature.

Keywords: Switched linear system; finite-state hybrid output feedback.

1 Introduction

Suppose that we are given a linear time-invariant control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and A, B and C are matrices of suitable dimensions. Suppose that the system (1) is *stabilizable* and *detectable*, i.e., there exist matrices F and K such that the eigenvalues of $A + BF$ and the eigenvalues of $A + KC$ have negative real parts. Then, as is well known, there exists a linear dynamic output feedback law that asymptotically stabilizes the system—see, e.g., Wolovich (1974, Section 6.4). In practice, however, such a continuous dynamic feedback might not be implementable, and a suitable discrete version of the controller is often desired. Recent references (Brockett and Liberzon, 1997; Hu and Michel, 1998; Litsyn *et al.*, 1998; McClamroch *et al.*, 1997; Sontag, 1999) discuss some issues related to control of continuous plants by various types of discontinuous feedback.

In particular, in (Litsyn *et al.*, 1998) it is shown that if the system (1) is controllable and observable, then it admits a stabilizing hybrid output feedback that uses a countable number of discrete states. A logical question to ask next is whether it is possible to stabilize (1) by using a hybrid output feedback with only a *finite* number of discrete states. Zvi Artstein explicitly raised this question in (Artstein, 1996) and discussed it in the context of a simple example (cf. Section 3 below). This problem seems to require a solution that is significantly different from the ones mentioned above because a finite-state stabilizing hybrid feedback is unlikely to be obtained from a continuous one by means of any discretization process.

In this note we propose an approach to the problem of stabilizing (1) via finite-state hybrid output feedback which is motivated by the following observation. Suppose that we are given a

*Email: liberzon@sysc.eng.yale.edu. This research was supported by ARO DAAH04-95-1-0114, NSF ECS 9634146, and AFOSR F49620-97-1-0108.

collection of $m \times p$ matrices $\{K_1, \dots, K_l\}$. Setting $u = K_i y$ for some $i \in \{1, \dots, l\}$, we obtain the system

$$\dot{x} = (A + BK_iC)x.$$

Thus the stabilization problem for the original system (1) will be solved if we can orchestrate the switching between the systems in the above form in such a way as to achieve asymptotic stability. Denoting $A + BK_iC$ by A_i for each $i \in \{1, \dots, l\}$, we are led to the following question: using the measurements of the output $y = Cx$, can we find a piecewise constant *switching signal* $\sigma : [0, +\infty) \to \{1, \dots, l\}$ such that the system

$$\dot{x} = A_\sigma x$$

is asymptotically stable? The value of σ at a given time t might just depend on t and/or $y(t)$, or a more general hybrid feedback with memory in the loop may be used. Below we discuss some paradigms, old and new, for constructing such a stabilizing switching signal. We are assuming, of course, that none of the matrices A_i are stable, as the existence of a matrix K such that the eigenvalues of $A + BKC$ have negative real parts would render the problem trivial.

2 Stabilizing switching signals

According to the main result of (Wicks *et al.*, 1994) and (Wicks *et al.*, 1998), if there exists a stable matrix of the form $A := \lambda A_i + \mu A_j$ for some $i, j \in \{1, \dots, l\}$ and $\lambda, \mu > 0$, then it is possible to construct a stabilizing switching signal (by using a single Lyapunov function that corresponds to A). In our case, the existence of such a stable linear combination would imply that the system (1) can be stabilized by the linear static output feedback $u = Ky$ with $K := \lambda K_i + \mu K_j$, contrary to the assumption made at the end of the previous section. Thus the techniques of (Wicks *et al.*, 1994; Wicks *et al.*, 1998) cannot be applied here. However, we will now demonstrate that linear combinations with positive coefficients are still useful in the present context. Assume for simplicity that $l = 2$, so that we are only given two matrices, $A_1 = A + BK_1C$ and $A_2 = A + BK_2C$. Let us further assume that the corresponding linear systems are second-order and have purely imaginary eigenvalues. In what follows, we show how to construct a switching signal such that the resulting switched system is asymptotically stable.

Since both individual linear systems are critically stable, there exist symmetric positive definite matrices P_1 and P_2 such that

$$A_i^T P_i + P_i A_i = 0, \qquad i = 1, 2.$$

Assume that we have $\langle P_1 x, P_2 x\rangle > 0$ and $\langle A_1 x, A_2 x\rangle > 0$ for all $x \neq 0$ (this is true, for example, if the matrices K_1 and K_2, and consequently the matrices A_1 and A_2, are "sufficiently close" to each other). Define $P := P_1 + P_2$. We have the following easy statement.

Lemma 1 *Under the above assumptions, for each $x \in \mathbb{R}^2 \setminus \{0\}$ one of the following is true:*

(i) *$A_1 x = \alpha A_2 x$ for some $\alpha > 0$*

(ii) *$x^T P A_1 x < 0$ and $x^T P A_2 x > 0$*

(iii) *$x^T P A_2 x < 0$ and $x^T P A_1 x > 0$*

Proof. Fix an arbitrary $x \neq 0$. Since $\langle P_1x, P_2x\rangle > 0$, $\langle A_1x, A_2x\rangle > 0$, and $\langle P_ix, A_ix\rangle = 0$ for $i = 1, 2$, there exists a 90° rotation matrix $\Theta \in SO(2)$ such that $\Theta P_1 x = A_1x\|P_1x\|/\|A_1x\|$ and $\Theta P_2x = A_2x\|P_2x\|/\|A_2x\|$. Let $\lambda := \|P_1x\|/\|A_1x\|$ and $\mu := \|P_2x\|/\|A_2x\|$. By linearity, $\Theta Px = \lambda A_1x + \mu A_2x$. Therefore, the matrix $A := \lambda A_1 + \mu A_2$ is such that $x^TPAx = 0$. It follows that either both quantities appearing in (ii) and (iii) are zero, or one of them is negative and the other one is positive. But the former is possible only when (i) holds. $\square$

By the above lemma, we can define the switching control policy as follows: at each time t, set $\sigma(t)$ equal to 1 if $x^TPA_2x \geq 0$, and set $\sigma(t)$ equal to 2 if $x^TPA_1x \geq 0$. Clearly, the switching will occur only when the state trajectory crosses the set of points on which (i) holds. This set is described by two lines intersecting at the origin (which are the coordinate axes corresponding to the basis in which P_1 and P_2 are simultaneously diagonalized), unless it is the entire $\mathbb{R}^2$ in which case the above switching strategy does not make sense. Observe that chattering cannot occur because the switching set is characterized by (i) with a positive α. To see that the resulting switched system is asymptotically stable, define the Lyapunov function $V(x) := x^TPx$. We have $\dot{V} \leq 0$, and LaSalle's principle easily leads to the desired conclusion.

It is important to notice that, since both systems being switched are linear time-invariant, the time between a crossing of the set $\{x : Cx = 0\}$ and the next crossing of the switching set can be explicitly calculated and is independent of the trajectory. In other words, the switching strategy can be implemented based just on the measurements of the output (and on the knowledge of the matrices A_1 and A_2).

Feron (1996) has shown that a switched linear system of the type studied here is quadratically stable only if there exists a stable linear combination of its matrices with nonnegative coefficients. This result implies that by using the above switching control policy we cannot achieve quadratic stability. Indeed, on the switching set we have $\dot{V} = 0$. Note also that the need to satisfy the assumptions of Lemma 1 poses a limitation on the speed of convergence of the switched system's trajectories to the origin: loosely speaking, as the matrices A_1 and A_2 come closer to one another, the convergence becomes slower.

The above method can be easily extended to the case of switching between *nonlinear* systems whose trajectories are closed orbits in $\mathbb{R}^2$. It is also applicable to certain special classes of higher-order systems.

3 Harmonic oscillator revisited

As an example that illustrates the above ideas, we present a modified version of the stabilizing switching strategy for the harmonic oscillator with position measurements described by Artstein (1996). Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u\,;$$

$$y = x_1$$

Although this system is both controllable and observable, it cannot be stabilized by (even discontinuous) static output feedback (Artstein, 1996). We will now apply to this system a switching control strategy along the lines described at the end of the previous section. Letting $u = -y$ we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2}$$

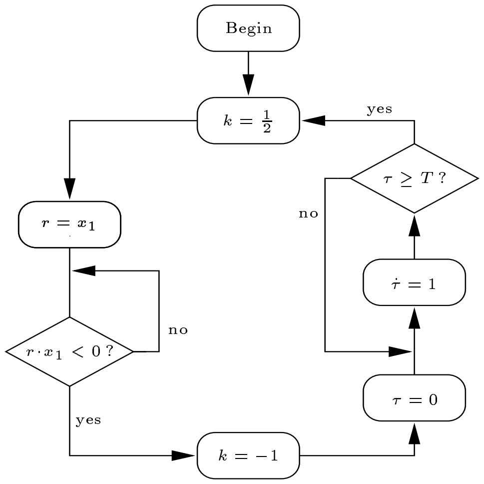
while letting $u = \frac{1}{2}y$ we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3}$$

The functions $V_1(x) := \frac{2}{3}x_1^2+\frac{1}{3}x_2^2$ and $V_2(x) := \frac{1}{3}x_1^2+\frac{2}{3}x_2^2$ are integrals of motion for (2) and (3), respectively. Define $V(x) := V_1(x)+V_2(x) = x_1^2+x_2^2$. This function decreases along the solutions of (2) when $x_1x_2 > 0$ and decreases along the solutions of (3) when $x_1x_2 < 0$. Therefore, if the system (2) is active in the 1st and 3rd quadrants while the system (3) is active in the 2nd and 4th quadrants, we will have $\dot{V} < 0$ whenever $x_1x_2 \neq 0$, hence the switched system is asymptotically stable by LaSalle's principle.

To define the hybrid output feedback strategy, let T be the time needed for a trajectory of the system (2) to pass through the 1st or the 3rd quadrant. The feedback control will be of the form $u = ky$, where the gain k will switch between the values -1 and $\frac{1}{2}$. When x_1 changes sign, we will let $k = -1$ for a period of time T, after which we let $u = \frac{1}{2}$, wait for the next change in sign of x_1, and so on. This strategy can be easily formalized according to the definition of hybrid feedback given in (Artstein, 1996). The hybrid automaton will have three discrete states (one responsible for control in the 1st and 3rd quadrants, one responsible for control in the 2nd quadrant and detecting the change of sign from positive to negative, and one responsible for control in the 4th quadrant and detecting the change of sign from negative to positive). It can be described by an appropriate state transition diagram—see (Artstein, 1996) for details.

The above hybrid control strategy can also be illustrated by the following computer-like diagram, similar to the ones used in (Morse, 1995). An auxiliary variable r is introduced to detect a change in sign of x_1 (the left branch), and a reset integrator is employed to determine the transition time (the right branch).



4 Multiple Lyapunov functions

In the previous sections the stability analysis of a switched system was based on a single Lyapunov function. In (Peleties and DeCarlo, 1991; Wicks and DeCarlo, 1997) the question of stabilizing switched systems has been addressed using *multiple* Lyapunov functions. In particular, (Peleties and DeCarlo, 1991) contains a worked example of stabilizing a system that switches between two unstable linear systems. The basic idea is to associate to each individual system a

Lyapunov-like function that decreases along the trajectories in a certain region. One then tries to orchestrate the switching in such a way that the value of each of these functions at the end of each interval on which the corresponding system is active exceeds the value at the end of the next such interval—see also (Branicky, 1997; Branicky, 1998; Hou *et al.*, 1996; Pettersson and Lennartson, 1996). In what follows, we will apply similar ideas to the problem at hand.

As before, let A_1 and A_2 be the two given matrices. Take a candidate Lyapunov function $V_1(x) = x^T P_1 x$ that decreases along solutions of the system $\dot{x} = A_1 x$ in some region (such a function always exists unless A_1 is a nonnegative multiple of the identity matrix). Similarly, take a candidate Lyapunov function $V_2(x) = x^T P_2 x$ that decreases along solutions of $\dot{x} = A_2 x$ in some region. Following (Wicks and DeCarlo, 1997), we consider the situation when the following condition holds:

CONDITION 1. $x^T(P_1A_1 + A_1^T P_1)x < 0$ whenever $x^T(P_1 - P_2)x \geq 0$ and $x \neq 0$, and $x^T(P_2A_2 + A_2^T P_2)x < 0$ whenever $x^T(P_2 - P_1)x \geq 0$ and $x \neq 0$.

If this condition is satisfied, then a stabilizing switching signal can be defined by $\sigma(t) := \arg\max\{V_i(x(t)) : i = 1, 2\}$. Indeed, the function V_σ will then be continuous and will decrease along solutions of the switched system, which guarantees asymptotic stability. A similar technique was used independently in (Malmborg *et al.*, 1996) in a more general, nonlinear context (that paper addresses an application to the interesting problem of stabilizing an inverted pendulum via a switching control strategy).

It is not hard to see that Condition 1 holds if the following condition is satisfied (by virtue of the S-procedure, the two conditions are actually equivalent provided that there exist $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T(P_1 - P_2)x_1 > 0$ and $x_2^T(P_2 - P_1)x_2 > 0$; see (Boyd *et al.*, 1994)).

CONDITION 2. There exist $\gamma_1, \gamma_2 \geq 0$ such that $-P_1A_1 - A_1^T P_1 + \gamma_1(P_2 - P_1) > 0$ and $-P_2A_2 - A_2^T P_2 + \gamma_2(P_1 - P_2) > 0$.

Alternatively, if the same condition is satisfied with $\gamma_1, \gamma_2 \leq 0$, then a stabilizing switching signal can be defined by $\sigma(t) := \arg\min\{V_i(x(t)) : i = 1, 2\}$. Now recall that in our present situation the given data is not the matrices A_1 and A_2, but rather the matrices A, B and C, and the problem is to find output feedback gains K_1 and K_2 such that the resulting matrices $A_1 = A + BK_1C$ and $A_2 = A + BK_2C$ satisfy the above requirements. This leads us to the following condition.

CONDITION 3. There exist two numbers γ_1 and γ_2, either both nonnegative or both nonpositive, such that

$$-P_1A - A^T P_1 + \gamma_1(P_2 - P_1) - P_1BK_1C - C^T K_1^T B^T P_1 > 0$$

and

$$-P_2A - A^T P_2 + \gamma_2(P_1 - P_2) - P_2BK_2C - C^T K_2^T B^T P_2 > 0.$$

Using the procedure for elimination of matrix variables described in Boyd *et al.* (1994, Section 2.6.2), one can show that the above inequalities are satisfied if and only if for some $\delta_1, \delta_2 \in \mathbb{R}$ we have

$$\begin{aligned} &- P_1A - A^T P_1 + \gamma_1(P_2 - P_1) - \delta_1 P_1 B B^T P_1^T > 0 \\ &- P_1A - A^T P_1 + \gamma_1(P_2 - P_1) - \delta_1 C^T C > 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} &- P_2A - A^T P_2 + \gamma_2(P_1 - P_2) - \delta_2 P_2 B B^T P_2^T > 0 \\ &- P_2A - A^T P_2 + \gamma_2(P_1 - P_2) - \delta_2 C^T C > 0 \end{aligned} \tag{5}$$

Suppose that our system is such that the above switching strategy can be implemented based just on the measurements of the output. As discussed in the previous sections, this is true, for example, when the individual systems are second-order critically stable. More generally, if $y \in \mathbb{R}^p$ with $p \geq n-1$ and if for some t we have $y(t) = 0$, then, since the systems being switched are linear time-invariant, the time T ($\leq +\infty$) until the next crossing of the switching set can be explicitly calculated and is independent of the trajectory. Also note that when (C, A) is an observable pair, we can reconstruct the entire state from n sampled measurements of the output, and determining the next switching time becomes an easy task. We arrive at the following statement.

Proposition 2 *If there exist two numbers γ_1 and γ_2, either both nonnegative or both nonpositive, and two numbers δ_1 and δ_2 such that the inequalities (4)–(5) are satisfied for some symmetric positive definite matrices P_1 and P_2, then the system (1) can be asymptotically stabilized by using hybrid output feedback with two discrete states.*

When $\gamma_1 = \gamma_2 = 0$, we recover LMIs that express conditions for stabilizability of (1) by static output feedback (and are equivalent to the ones given, e.g., in Syrmos *et al.* (1997, Theorem 3.8)). It would be interesting to compare the above bilinear matrix inequalities with the ones obtained in (Pogromsky *et al.*, 1998) as a characterization of stabilizability via switched *state* feedback, and also with the dynamic programming approach presented in (Savkin *et al.*, 1996; Savkin *et al.*, 1999).

5 Conclusions

We addressed the problem of stabilizing a linear system using finite-state hybrid output feedback. We proposed an approach to this problem that consists in switching between a finite number of constant linear gains, thereby reducing it to the problem of finding a stabilizing switching signal for a switched linear system with unstable individual matrices.

This note poses more questions than it provides answers. While available techniques that rely on the existence of stable linear combinations do not seem to be relevant in the present context, we have shown by way of examples how in some cases of interest a stabilizing switching signal can be constructed. The fact that the individual gains can be chosen as part of the design introduces considerable flexibility into the problem and is to be explored further. It remains to be seen whether the main results of (Wicks and DeCarlo, 1997), which relate the existence of a stabilizing switching signal to the eigenvalue locations of certain matrix operators, are useful in this regard. An interesting question left to consider is exactly what advantage is to be gained by switching between more than two linear systems. The developments of Section 4 make contact with the work reported in (Johansson and Rantzer, 1998) and (Pettersson and Lennartson, 1996) on LMI tests for piecewise quadratic Lyapunov functions for switched systems. Another possible source of interesting ideas, which suggests an altogether different approach to the problem, is the literature on periodic sampled-data output feedback control of linear systems—see, e.g., (Araki and Hagiwara, 1986; Aeyels and Willems, 1992; Francis and Georgiou, 1988).

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