Invertibility of Nonlinear Switched Systems

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***Abstract*— This article addresses the invertibility problem for switched nonlinear systems affine in controls. The problem is concerned with finding the input and switching signal uniquely from given output and initial state. We extend the concept of switch-singular pairs, introduced recently, to nonlinear systems and develop a formula for checking if given state and output form a switch-singular pair. We give a necessary and sufficient condition for a switched system to be invertible, which says that the subsystems should be invertible and there should be no switch-singular pairs. When all the subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs. Detailed examples are included to illustrate these newly developed concepts.**

I. INTRODUCTION

Switched systems consist of a family of dynamical subsystems together with a switching signal that determines the active subsystem at each time instant. Switching behaviors can come from controller design, such as in switching supervisory control [1] or gain scheduling control. Switching can also be inherent by nature, such as when a physical plant has the capability of undergoing several operational modes (*e.g.*, an aircraft during different thrust modes, a walking robot during leg impact and leg swing modes, different formations of a group of vehicles). Also, switched systems may be viewed as higher-level abstractions of hybrid systems obtained by neglecting the details of the discrete behavior and instead considering switching signals from a suitable class. As a result, switched systems have been a focus of ongoing research and several results related to stability, controllability, observability, and input-to-state stability of such systems have been published; see [1] for references. More recently, Vu and Liberzon introduced the problem of invertibility of switched linear systems in [2]. In this paper, we extend their methodology to study the problem of invertibility of continuous-time switched nonlinear systems, which concerns with the following question: *What is the condition on the subsystems of a switched system so that, given an initial state x_0 and the corresponding output y generated with some switching signal σ and input u, we can recover the switching signal σ and the input u uniquely?* The problem statement is analogous to the classical invertibility problem for non-switched systems. In fact, for every control system with an output, we have an input-output map and the question of left (resp. right) invertibility is, roughly speaking, that of the injectivity (surjectivity) of this map.

System invertibility problems are of great importance from theoretical and practical viewpoint and have been studied extensively for fifty years, after being pioneered by Brockett-Mesarovic [3]. The systematic study of invertibility for non-switched nonlinear systems began with Hirschorn, who first studied the single-input single-output (SISO) case [4], and then generalized Silverman's structure algorithm to multiple-input multiple-output (MIMO) nonlinear systems [5]. Singh [6] then modified the algorithm to cover a larger class of systems. Isidori and Moog [7] used this algorithm to calculate zero-output constrained dynamics and reduced inverse system dynamics. The algorithm is also closely related to the dynamic extension algorithm used to solve the dynamic state feedback input-output decoupling problem [8, Sections 8.2 and 11.3]. The geometric methods have also been used by Nijmeijer [9]. A higher-level interpretation given by a linear-algebraic framework, which also establishes links between these algorithms and geometric approach, is presented by Di Benedetto et al. in [10]. We also recommend [11, Chapter 5] for a useful survey on various invertibility techniques.

The problem of invertibility for switched linear systems was introduced very recently in [2] where the authors used Silverman's structure algorithm to formulate conditions for the invertibility of switched systems with continuous dynamics. The problem of invertibility for discrete time switched linear systems has been discussed in [12], [13] but here, the authors assume that the switching sequence is known and find the corrsponding input. In this paper, we make no such assumption and adopt an approach similar to [2] to study the invertibility problem for continuous-time switched nonlinear systems, affine in controls, using Singh's nonlinear structure algorithm.[1] Although the form of the main condition (invertibility of subsystems plus no switch-singular pairs) is essentially similar to [2], the technical details of checking the conditions are different because we work with the nonlinear structure algorithm.

The basic idea is to do mode identification by utilizing relationship among the outputs and the states of the subsystems and then use the nonlinear structure algorithm for corresponding subsystem to recover the input. We can think of non-switched systems as switched systems with constant switching signals. In this regard, the invertibility problem for switched systems is an extension of the non-switched counterpart in the sense that we have to recover the switching

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[1] A related problem is discussed in [14] but it doesn't follow the same theoretical approach we do, and instead uses a heuristic approach with the purpose of studying a specific application.

signal in addition to the input, based on the output and the initial state.

The paper is organized as follows. Section II contains the definitions of invertibility and the formal problem statement. The main result on left invertibility is presented in Section III. We then give a characterization of switch-singular pairs and the construction of inverse systems in Section IV. An algorithm for output generation is given in Section V along with an example. We conclude the article with some remarks on further research directions.

II. PRELIMINARIES

A. Nonlinear Non-switched Systems

The dynamics of a square nonlinear system affine in controls are given by:

$$\Gamma := \begin{cases} \dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathbb{M}$, an n-dimensional real connected smooth manifold, for example $\mathbb{R}^n$; and f, g_i are smooth vector fields on $\mathbb{M}$, $h : \mathbb{M} \to \mathbb{R}^m$ is a smooth function.

We start off by reviewing classical definitions of invertibility for such systems. For that, consider the input-output map $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ for some input function space $\mathcal{U}$ and the corresponding output function space $\mathcal{Y}$. Since nonlinear systems exhibit finite blow-up times, some input signals may not have a well defined image in the output space, over the same length of interval, under this map. We don't give a rigorous definition of H_{x_0} but use it, nevertheless, for better illustration. It is assumed that the outputs exist on the intervals considered. We do not specify the input space $\mathcal{U}$ at this moment as it depends upon the system under consideration (see Section IV). Denote by $\Gamma_{x_0}(u)$ the trajectory of the corresponding system with the initial state x_0 and the input u, and the corresponding output by $\Gamma^O_{x_0}(u)$. In case of non-switched systems, the following notion of invertibility[2] was introduced in [5].

Definition 1: Consider the interval $[t_0, T]$ and the inputs u_1 and u_2 that are well-defined over this interval. The system (1) is *invertible at a point* $x_0 := x(t_0) \in \mathbb{M}$ if $\Gamma^O_{x_0}(u_{1[t_0,T]}) = \Gamma^O_{x_0}(u_{2[t_0,T]})$ implies that $\exists\, \varepsilon > 0$ such that $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$. The system is *strongly invertible at a point* x_0 if it is invertible for each $x \in N(x_0)$, where N is some open neighborhood of x_0. The system is *strongly invertible* if there exists an open and dense submanifold $\mathbb{M}^\alpha$ (called inverse submanifold) such that $\forall x_0 \in \mathbb{M}^\alpha$, the system is strongly invertible at x_0. ◁

In general, the inverses of nonlinear dynamical systems are not defined globally. An open and dense subset of $\mathbb{M}$ on which the dynamics of a nonlinear system are invertible is called the *inverse submanifold* and is denoted by $\mathbb{M}^\alpha$. If $x \in \mathbb{M} \backslash \mathbb{M}^\alpha$, we call it a *singular point* as it is not possible to invert the system starting from such an initial condition. In the most general construction of inverse systems as the one given by Singh [6], there exists a set of *singular outputs* $\mathcal{Y}^s$ such that the system is not invertible for $y \in \mathcal{Y}^s$; and its complement $\mathcal{Y}^\alpha := \mathcal{Y} \backslash \mathcal{Y}^s$ is the set of all outputs on which the system is strongly invertible. All these notions will be developed formally in Section IV but here we give an example to illustrate their usage.

Example 1: Consider a non-switched nonlinear system with two inputs and outputs,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 u_1 \\ x_3 u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbb{M} = \mathbb{R}^3$$

We then have

$$\dot{y}_1 = x_1 u_1 \tag{2a}$$

$$\ddot{y}_2 = \frac{x_3 \ddot{y}_1 - \dot{y}_1 \dot{y}_2 + \dot{y}_1 u_2}{x_1} \tag{2b}$$

So, y_1 is in one-to-one correspondence with u_1 if $x_1 \neq 0$, and y_2 is in one-to-one relation with u_2 if $\dot{y}_1 \neq 0$. Consequently, the set $\mathbb{M}^\alpha = \{x \in \mathbb{R}^3 \,|\, x_1 \neq 0\}$ is the inverse submanifold and the set of singular outputs is $\mathcal{Y}^s = \{y(t) \in \mathbb{R}^2 \,|\, \dot{y}_1(t) = 0\}$. Note that $\dot{y}_1(t) = 0$ implies that $x_1(t) = 0$ or $u_1(t) = 0$; so, the definition of $\mathcal{Y}^s$ essentially depends upon the value of state trajectories and input signals. If $x(t) \in \mathbb{M}^\alpha$ and $y(t) \notin \mathcal{Y}^s$ for all $t \in [t_0, t_0 + \varepsilon)$, then there is a one-to-one relation between the output and input signals provided their domains are restricted to $[t_0, t_0 + \varepsilon)$. By using Definition 1, we deduce that the system is strongly invertible. ◁

From Example 1, it is clear that the system (1) is invertible at every $x \in \mathbb{M}^\alpha$ for the class of inputs u such that along the trajectory of the system (1), the resulting motion $(x(t), y(t)) \in \mathbb{M}^\alpha \times \mathcal{Y}^\alpha$. Hence, invertibility is achieved when the domain of signals is restricted to $[t_0, T)$ with $T \in [t_0, \bar{t})$ and $\bar{t} := \min\{t > t_0 : (x(t), y(t)) \notin \mathbb{M}^\alpha \times \mathcal{Y}^\alpha\}$. In other words, a nonlinear system is invertible at x_0 if for a given output y over the time interval $[t_0, T']$, one can find $T \in (t_0, T']$ and a unique input u over $[t_0, T)$ such that $\Gamma^O_{x_0}(u_{[t_0,T)}) = y_{[t_0,T)}$. Since T can be arbitrarily small, this explains why we require arbitrarily small time domains in Definition 1. We will generalize this notion of local invertibility to the switched systems.

B. Switched Systems

A finite family of systems defined by (1) generates a switched system and in this paper we will consider such switched nonlinear systems, affine in controls, that have the following structure:

$$\Gamma_\sigma : \begin{cases} \dot{x} = f_\sigma(x) + G_\sigma(x)u = f_\sigma(x) + \sum_{i=1}^{m} (g_i)_\sigma(x)u_i, \\ y = h_\sigma(x) \end{cases} \tag{3}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is the switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some finite index set, and f_p, G_p, h_p, where $p \in \mathcal{P}$, define the dynamics of individual subsystems. The state space $\mathbb{M}$ is a connected real smooth manifold of dimension n, for example $\mathbb{R}^n$; f_p, $(g_i)_p$ are real smooth vector fields on $\mathbb{M}$, and $h_p : \mathbb{M} \to \mathbb{R}^m$ is a smooth function. A switching signal, as defined in [1], is a piecewise constant and everywhere right-continuous function

[2]Throughout the paper invertibility refers to the left invertibility.

that has a finite number of discontinuities τ_i, which we call *switching times*, on every bounded time interval and thus $\sigma(t) = p \in \mathcal{P}$, $\forall\, t \in [\tau_i, \tau_{i+1})$. We assume that all the subsystems are equi-dimensional, they live in the same state space $\mathbb{M}$, and that there is no state jump at switching times. For any initial state x_0, a switching signal σ, and a piecewise continuous input u on any time domain, a solution of (3) over the same domain always exists (in Carathéodory sense) and is unique, provided the flow of the active subsystem is defined $\forall\, t \in [\tau_i, \tau_{i+1})$. In case of no switching this condition is equivalent to forward completeness of the flow and we assume that each subsystem satisfies this condition. For $p \in \mathcal{P}$, denote by $\Gamma_{p,x_0}(u)$ the trajectory of the corresponding subsystem with the initial state x_0 and the input u, and the corresponding output by $\Gamma^O_{p,x_0}(u)$. Since switching signals are right-continuous, the outputs are also right-continuous and whenever we take derivative of the output, we assume it is the right derivative. We will use $\mathcal{F}^{pc}$ to denote the space of piecewise continuous functions, and $\oplus$ for concatenation of signals.

In case of switched systems (3), the "map" H_{x_0} has an augmented domain, that is, now we have a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$, where $\mathcal{S}$ is a switching signal set. Let us first extend the definition of invertibility of non-switched systems to define the invertibility of the map H_{x_0} for switched systems.

Definition 2: Consider the interval $[t_0, T]$ and the inputs u_1 and u_2 that are well-defined over this interval. A *switched system is invertible at a point* $x_0 := x(t_0)$ if $\mathrm{H}_{x_0}(\sigma_{1[t_0,T]}, u_{1[t_0,T]}) = \mathrm{H}_{x_0}(\sigma_{2[t_0,T]}, u_{2[t_0,T]}) = y_{[t_0,T]}$, implies that $\exists\ \varepsilon > 0$ such that $\sigma_{1[t_0,t_0+\varepsilon)} = \sigma_{2[t_0,t_0+\varepsilon)}$ and $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$; that is, the pre-image of H_{x_0} is unique on some interval for given x_0 and y. A *switched system is strongly invertible at a point* x_0 if it is invertible at each $x \in N(x_0)$, where N is some open neighborhood of x_0. A *switched system is strongly invertible* if there exists an open and dense submanifold $\mathbb{M}^\alpha$ of $\mathbb{M}$ such that $\forall x_0 \in \mathbb{M}^\alpha$, the system is strongly invertible at x_0 for given $y \in \mathcal{Y}$. $\triangleleft$

The reason we have a different notion of invertibility is because in switched systems, if a subsystem is invertible at x_0 for a given non-singular output y, then it is possible that another subsystem might produce the same output starting from the same initial condition. This means that the pre-image of H_{x_0} at such $(x_0,\, y)$ is not unique and hence the switched system is not invertible at x_0 if such pairs $(x_0,\, y)$ exist. We call all such pairs *switch-singular pairs*[3]. The concept of switch-singular pairs for switched systems basically refers to the ability of more than one subsystem to produce a segment of the desired output starting from the same initial condition. The formal definition is given below:

Definition 3: Let $x_0 \in \mathbb{M}$ and $y \in \mathcal{Y}$ on some time interval. The pair $(x_0,\, y)$ is a *switch-singular pair* of the two subsystems $\Gamma_p,\ \Gamma_q$ if there exist $u_1,\ u_2$ such that $\Gamma^O_{p,x_0}(u_1) = \Gamma^O_{q,x_0}(u_2) = y$. $\triangleleft$

[3]This is similar to the concept of singular pairs conceived in [2]. We use the term "switch-singular pair" to avoid conflict with the singularities of individual nonlinear subsystems.

The invertibility problem for switched nonlinear systems is now formally defined as:

The invertibility problem: Consider a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ for the switched system (3). Find the largest possible set $\mathcal{Y}$, an open dense set in $\mathbb{M}$ and a condition on the subsystems such that for a given output $y \in \mathcal{Y}$ over a finite time interval $[t_0,\, T']$, there exist $T \in (t_0, T']$ and a unique $(\sigma,\, u)$ over $[t_0,\, T)$ having the property that $\mathrm{H}_{x_0}(\sigma_{[t_0,T)}, u_{[t_0,T)}) = y_{[t_0,T)}$.

III. CHARACTERIZATION OF INVERTIBILITY

We now give conditions on the subsystem dynamics so that the map H_{x_0} is injective for some sets $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{Y}$. We do not explicitly specify what the input sets $\mathcal{U}$ and $\mathcal{S}$ are but instead we specify the set $\mathcal{Y}$ and then $\mathcal{U}$ will be the corresponding set which, together with $\mathcal{S}$, generates $\mathcal{Y}$.

For all $p \in \mathcal{P}$, let $\mathbb{M}^\alpha_p$ be the inverse submanifold of Γ_p, $\mathcal{Y}_p$ be the set of sufficiently smooth[4] outputs that can be generated by Γ_p, $\mathcal{Y}^s_p$ be the set of singular outputs of Γ_p, and $\mathcal{Y}^\alpha_p$ be the set of outputs on which Γ_p is strongly invertible. Define $\mathcal{Y}^s := \cup_{p\in\mathcal{P}} \mathcal{Y}^s_p$ as the collection of all singular outputs and let $\mathcal{Y}^{all}$ be the set of outputs generated by all the possible concatenations of all elements of $\mathcal{Y}_p,\ \forall p \in \mathcal{P}$. Then $\mathcal{Y}^\alpha := \mathcal{Y}^{all} \backslash \mathcal{Y}^s$ is a set of outputs on which every subsystem is strongly invertible. We consider outputs $y \in \mathcal{Y}^\alpha$ over a finite interval $[t_0,\, T']$ and seek invertibility on a subinterval $[t_0,\, T) \subset [t_0,\, T']$ such that $(\sigma_{[t_0,\,T)}, u_{[t_0,\,T)})$ is a unique preimage of $y_{[t_0,\,T)}$. The first main result is about strong invertibility at some $x_0 \in \mathbb{M}$.

Theorem 1: *Consider the switched system (3) and the output set $\mathcal{Y}^\alpha$. The switched system is strongly invertible at $x_0 \in \mathbb{M}$ for given $y \in \mathcal{Y}^\alpha$ if and only if there exists a neighborhood $N(x_0)$ such that each subsystem is invertible at every $x \in N(x_0)$, and for all $x \in N(x_0)$, $y \in \mathcal{Y}^\alpha$, the pairs (x, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q$, and $p, q \in \mathcal{P}$.*

To get some intuition behind the result, note that for any output y that is generated by the switched system, at any time instant τ, there exists $p \in \mathcal{P}$ such that $y(\tau) \in \overline{\mathcal{Y}} \cap \mathcal{Y}^\alpha_p$. Since each subsystem Γ_p is invertible, there exists a unique input which produces that output. Non existence of switch-singular pairs implies that no other system can produce the same output even with different input. Hence, $\mathrm{H}^{-1}_{x_0}(y) = (\sigma,\, u)$ is unique.

Proof. *Necessity*: We show that if any of the subsystems is not strongly invertible at x_0 or if there exist switch-singular pairs, then the switched system is not invertible.

Suppose that a subsystem Γ_p, $p \in \mathcal{P}$ is not invertible at some x in arbitrary $N(x_0)$, then there exist $y \in \mathcal{Y}^\alpha \cap \mathcal{Y}^\alpha_p$, and inputs $u_1 \neq u_2$ over time interval $[t_0, t_0 + \varepsilon) \subset [t_0, T']$, for some $\varepsilon > 0$ such that $\Gamma^O_{p,x}(u_1) = \Gamma^O_{p,x}(u_2) = y_{[t_0,t_0+\varepsilon)}$. This implies that $\mathrm{H}_x(\sigma^p, u_1) = \mathrm{H}_x(\sigma^p, u_2) = y$, and the map H_x is not injective for given y. Hence, the switched system is not invertible at x. Since there exists such x in every

[4]This assumption can be relaxed depending upon the system under consideration as discussed in Section IV.

neighborhood of x_0, it follows that the switched system is not strongly invertible at x_0.

For necessity of the second condition, suppose that $\exists\, x \in N(x_0)$, $y \in \mathcal{Y}^\alpha \cap \mathcal{C}^\infty$ such that $(x,\, y)$ is a switch-singular pair of Γ_p, Γ_q, $p \neq q$. This means that both subsystems, even though invertible at x, can produce this output over the interval $[t_0, t_0+\varepsilon) \subset [t_0, T']$, $\forall \varepsilon > 0$. Consequently, $\exists\ u_1$, u_2 (possibly same) on the corresponding interval such that $\Gamma^O_{p,x}(u_1) = \Gamma^O_{q,x}(u_2) = y$. Hence, we have $\mathrm{H}_x(\sigma^p, u_1) = \mathrm{H}_x(\sigma^q, u_2) = y$, that is the preimage of y is not unique as $\sigma^p \neq \sigma^q$. This implies that the switched system is not invertible at x for given $y \in \mathcal{Y}^\alpha$. Since there exists such x in every neighborhood of x_0, it follows that the switched system is not strongly invertible at x_0.

Sufficiency: Suppose that for given $x_0 \in \mathbb{M}$, there exist some inputs $u_1,\, u_2$ and switching signals $\sigma_1,\, \sigma_2$ such that $\mathrm{H}_{x_0}(\sigma_1,\, u_1) = \mathrm{H}_{x_0}(\sigma_2,\, u_2) = y \in \mathcal{Y}^\alpha$ over $[t_0, T']$. Initially, we have $\sigma_1(t_0) = \sigma_2(t_0) = p$ because $(x_0,\, y)$ is not a switch-singular pair. Since $y \in \mathcal{Y}^\alpha_p$, and Γ_p is invertible at every $x \in N(x_0)$, $\exists\ \varepsilon_1 > 0$ such that $u_{1[t_0,t_0+\varepsilon_1)} = u_{2[t_0,t_0+\varepsilon_1)} = \Gamma^{-1,O}_{p,x}(y_{[t_0,t_0+\varepsilon_1)})$, the output of the inverse subsystem. As there are no switch-singular pairs in $N(x_0)$, $\exists\ \varepsilon_2 > 0$ such that $\sigma_{1[t_0,t_0+\varepsilon_2)} = \sigma_{2[t_0,t_0+\varepsilon_2)}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then it follows from Definition 2 that the switched system is invertible at every $x \in N(x_0)$ and hence is strongly invertible at x_0. □

Based on the result of Theorem 1, the conditions for strong invertibility of switched systems can be developed. Let $\mathbb{M}^\alpha := \bigcap_{p\in\mathcal{P}} \mathbb{M}^\alpha_p$, then $\mathbb{M}^\alpha$ is an open and dense subset of $\mathbb{M}$ because it is a finite intersection of open and dense subsets. Since, every subsystem is strongly invertible on $\mathbb{M}^\alpha$, we have the following result.

Corollary 1: *The switched system (3) is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$ and for $y \in \mathcal{Y}^\alpha$ if and only if Γ_p, $\forall\, p \in \mathcal{P}$, is strongly invertible at every $x_0 \in \mathbb{M}^\alpha_p$ and the subsystem dynamics are such that the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q$, $p, q \in \mathcal{P}$, $\forall x_0 \in \mathbb{M}^\alpha$, $y \in \mathcal{Y}^\alpha$.* ◁

It follows from the proof of the sufficiency part in Theorem 1 that the switched system is strongly invertible over the interval $[t_0, T)$, where $T \in [t_0, \bar{t})$ and $\bar{t} := \min\{t > t_0 : (x(t), y(t)) \notin \mathbb{M}^\alpha \times \mathcal{Y}^\alpha\}$. If the output y loses continuity over the interval $[t_0,\, T)$ because of switching, then $(\sigma_{[t_0,T)}, u_{[t_0,T)}) = (\sigma_{[t_0,\tau_1)}, u_{[t_0,\tau_1)}) \oplus \cdots \oplus (\sigma_{[\tau_k,T)}, u_{[\tau_k,T)})$, where k is the total number of switches in the interval $[t_0,\, T)$ and τ_i, $i = 1, \cdots, k$, are the switching instants.

Remark 1: For the switched system (3), if all subsystems are globally invertible in addition to the hypothesis of Corollary 1, that is, $\mathbb{M}^\alpha = \mathbb{M}$ and $\mathcal{Y}^s = \emptyset$, then the domain of signals can be arbitrary such as $[t_0, \infty)$ and the switched system is strongly invertible on $[t_0,\, T)$, $\forall\, T > t_0$. ◁

Remark 2: If a subsystem has more inputs than outputs, then it cannot be (left) invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case. ◁

IV. CHECKING INVERTIBILITY

In this section, we address the computational aspect of the concepts introduced in previous sections and develop algebraic criteria for checking the invertibility of switched systems. The first condition in Theorem 1 asks for invertibility of subsystems and is verified by the structure algorithm. To put everything into perspective, we provide appropriate background related to the invertibility of non-switched systems and use it to develop the concept of functional reproducibility. To check if $(x_0,\, y)$ is a switch-singular pair, we develop a formula using the functional reproducibility criteria of non-switched systems. Based on these two mathematical characterizations and the result in Theorem 1, we will be able to construct a switched inverse system for recovering the original input and switching signal uniquely.

A. Single-Input Single-Output (SISO) Systems

We start off with the case when all the subsystems are SISO because it gives more insight into computations and helps understand the concepts which we will later generalize to multivariable systems. To this end, consider a SISO nonlinear system affine in controls (1) with $m = 1$ and assume it has a relative degree r at x_0 [15], *i.e*, $\exists$ a neighborhood $N(x_0)$ such that $L_g L_f^{r-1} h(x) \neq 0\ \forall\, x \in N(x_0)$, where $L_f^k h(x) = \frac{\partial (L_f^{k-1}h(x))}{\partial x} f(x)$ and $L_f^0 h(x) = h(x)$. To check if the subsystem is invertible or not, we first derive an explicit expression for the input u in terms of the output y by computing the derivatives of y as follows:

$$
\begin{aligned}
y(t) &= h(x(t)) && \text{(4a)}\\
\dot{y}(t) &= L_f h(x(t)) && \text{(4b)}\\
&\ \ \vdots \\
y^{(r)}(t) &= L_f^r h(x) + L_g L_f^{r-1} h(x) u(t) && \text{(4c)}
\end{aligned}
$$

From the last equation, we can derive an expression for $u(t)$:

$$
u(t) = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)} + \frac{1}{L_g L_f^{r-1} h(x)} y^{(r)}(t) \tag{5}
$$

Hence, u can be determined explicitly in terms of measured output y. On substituting the expression for u from (5) in equation (1), one gets the dynamics for the inverse system:

$$
\begin{aligned}
\dot{z} &= f(z) + g(z)\Big(-\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)}(t) \Big),\\
u(t) &= -\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)}(t)
\end{aligned} \tag{6}
$$

The dynamics of this inverse subsystem evolve on the set $\mathbb{M}^\alpha := \{z \in \mathbb{M} \mid L_g L_f^{r-1} h(z) \neq 0\}$. Since the inverse system dynamics are driven by $y^{(r)}(t)$ which satisfies equation (4c), it is not hard to see that the state trajectories of the inverse system satisfy the differential equation of the original system (1). So if the inverse system is initialized with the same initial condition as that of the plant, then both of the systems follow exactly the same trajectory. The discussion motivates the following result, given in [4]:

Lemma 1: *A SISO system is strongly invertible at x_0 if and only if the system has a finite relative degree r at x_0.*

We developed the proof of the sufficiency part. The necessity part, although intuitively clear, is proved rigorously in [4].

For SISO systems, the input u appears in the r-th derivative of the output (4). Thus the differentiability/ smoothness of u will not affect the existence of the first $r-1$ derivatives of y. If $u : [0,t) \to \mathbb{R}$ is a locally essentially bounded, Lebesgue measurable function, then $y^{(r)}(t)$ exists almost everywhere and $y^{(r-1)}(t)$ is absolutely continuous [16]. So for SISO nonlinear non-switched systems, $\mathcal{U}$ can be defined as the space of functions which are locally essentially bounded and Lebesgue measurable; and $\mathcal{Y}$ can be the set of corresponding outputs.

We now turn to the concept of functional reproducibility, which in broad terms means the ability to follow a given reference signal. This concept will help us study the existence of switch-singular pairs. We look at the conditions under which a system can produce the desired output y_d over some interval $[t_0, T)$ starting from a particular initial state x_0. To be precise, given the system (1) with $m = 1$ and initial state x_0, we want to find out if there exists a control u such that $\Gamma^O_{x_0}(u) = y_d(\cdot)$. The following result was given in [4]:

Lemma 2: *If the system (1), with $m = 1$ and $x(t_0) = x_0$, has a relative degree $r < \infty$ at x_0, then there exists a control input u such that $\Gamma^O_{x_0}(u) = y_d(\cdot)$ if and only if*

$$y_d^{(k)}(t_0) = L_f^k h(x_0) \qquad \forall\, k = 0, 1, \cdots, r-1 \tag{7}$$

This result is easy to comprehend by looking at the expressions for the output derivatives (4). As control $u(t)$ does not directly affect $y^{(k)}(t),\ \forall k = 1, \cdots, r-1$, their values at t_0 are determined by the initial state. The control u, for which $\Gamma^O_{x_0}(u) = y_d(\cdot)$, is given by (5) with y replaced by y_d in that formula. We can now easily check for the switch-singular pairs among Γ_p, Γ_q with relative degrees r_p, r_q respectively, where $p,\, q \in \mathcal{P}$.

Lemma 3: *For SISO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p and Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa-1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(x_0) \\ \vdots \\ L_{f_\kappa}^{r_\kappa-1} h_\kappa(x_0) \end{pmatrix}, \quad \kappa = p, q \in \mathcal{P} \tag{8}$$

The example below illustrates the use of these concepts.

Example 2: Consider a SISO switched system with two modes

$$\Gamma_p := \begin{cases} \dot{x} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^3 \\ y = x_1 \end{cases}$$

$$\Gamma_q := \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ x_2 x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^3 \\ y = 2x_1 \end{cases}$$

If Γ_p is active, then $\dot{y} = x_1 + x_2$; if Γ_q is active, then $\dot{y} = 2x_2$. Both subsystems have relative degree 2 on $\mathbb{R}^3$ which can be verified by taking second derivative of the output. If there exists $x \in \mathbb{R}^3$ which forms a switch-singular pair with $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$, then the following equality must be satisfied

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

which gives $x_1 = x_2 = 0$. This state constraint yields $y = \dot{y} = 0$. If we let $\overline{\mathcal{Y}}^\alpha := \Big\{ y \in \mathcal{F}^{pc} \ : \ \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \neq 0 \ \forall t \Big\}$, then there exists no switch-singular pair between $x \in \mathbb{R}^3$ and $y \in \overline{\mathcal{Y}}^\alpha$. Theorem 1 and Lemma 1 infer that the switched system generated by $\{\Gamma_p, \Gamma_q\}$ is strongly invertible on $\overline{\mathcal{Y}}^\alpha$, $\forall x_0 \in \mathbb{R}^3$. $\triangleleft$

We now have the tool set to check for the invertibility conditions given in Theorem 1. If these conditions are satisfied and the switched system is strongly invertible, a switched inverse system can be constructed to recover the input and switching signal σ from given output and initial state. For the switched inverse system, define the *index inversion function* $\overline{\Sigma}^{-1} : \mathbb{M}^\alpha \times \mathcal{Y}^\alpha \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = p \ : \ y \in \mathcal{Y}_p \text{ and } y^{(k)}(t_0) = L_{f_p}^k h_p(x_0) \tag{9}$$

where $k = 0, 1, \cdots, r_p - 1$, t_0 is the initial time of y, and $x_0 = x(t_0)$. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there are no switch-singular pairs. The existence of p is guaranteed because it is assumed that $y \in \mathcal{Y}^\alpha$ is an output. Thus, an inverse switched system Γ_σ^{-1} is:

$$\begin{aligned} &\sigma(t) = \overline{\Sigma}^{-1}(z(t), y(t)), \\ &\dot{z} = f_\sigma(z) + g_\sigma(z)\Big(- \frac{L_{f_\sigma}^{r_\sigma} h_\sigma(z)}{L_{g_\sigma} L_{f_\sigma}^{r_\sigma-1} h_\sigma(z)} + \frac{y^{(r_\sigma)}(t)}{L_{g_\sigma} L_{f_\sigma}^{r_\sigma-1} h_\sigma(z)} \Big), \\ &u(t) = - \frac{L_{f_\sigma}^{r_\sigma} h_\sigma(z)}{L_{g_\sigma} L_{f_\sigma}^{r_\sigma-1} h_\sigma(z)} + \frac{y^{(r_\sigma)}(t)}{L_{g_\sigma} L_{f_\sigma}^{r_\sigma-1} h_\sigma(z)} \end{aligned}$$

with the initial condition $z(t_0) = x_0$. The notation $(\cdot)_\sigma$ denotes the object calculated for the subsystem with index $\sigma(t)$. The initial condition $\sigma(t_0)$ determines the initial active subsystem at the initial time t_0, from which time onwards, the active subsystem indexes and the input as well as the state are determined uniquely and simultaneously.

B. Multiple-Input Multiple-Output (MIMO) Systems

For multiple-input multiple-output (MIMO) nonlinear systems affine in controls (1), one uses the *structure algorithm* to compute the inverse. When a system is invertible, the structure algorithm, or Singh's inversion algorithm, allows us to express the input as a function of the output, its derivatives and possibly some states.

The Structure Algorithm: This version of the algorithm closely follows the construction given in [10], which is a slightly modified version of the algorithm in [6].

Step 1: Calculate

$$\dot{y} = L_f h(x) + L_G h(x) u = \frac{\partial h}{\partial x}[f(x) + G(x)u]$$

and write it as $\dot{y} =: a_1(x) + b_1(x)u$. Define $s_1 := \operatorname{rank} b_1(x)$, which is the maximal rank of $b_1(x)$ in some neighborhood of x_0, denoted as $N_1(x_0)$. Permute, if necessary, the components of the output so that the first s_1 rows of $b_1(x)$ are linearly dependent. Decompose y as

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix}$$

where $\dot{\tilde{y}}_1$ consists of the first s_1 rows of $\dot{y}$. Since the last $m - s_1$ rows of $b_1(x)$ are linearly dependent upon the first s_1 rows, there exists a matrix $F_1(x)$ such that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ \dot{\hat{y}}_1 &= \hat{h}^1(x, \dot{\tilde{y}}_1) = \hat{a}_1(x) + F_1(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x)) \end{aligned} \tag{10}$$

where the last equation is affine in $\dot{\tilde{y}}_1$. Finally, set $\tilde{B}_1(x) := \tilde{b}_1(x)$.

Step k+1: Suppose that in steps 1 through k, $\dot{\tilde{y}}_1, \ldots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}) \\ &\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\})u, \\ \hat{y}_k^{(k)} &= \hat{h}^k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\}) \end{aligned}$$

where all the expressions on the right-hand side are rational functions of $\tilde{y}_i^{(j)}$. Suppose also that the matrix $\tilde{B}_k := [\tilde{b}_1^T, \ldots, \tilde{b}_k^T]^T$ (vertical stacking of the linearly independent rows obtained at each step) has full rank equal to s_k in $N_k(x_0)$. Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial}{\partial x}\hat{h}^k[f(x) + G(x)u] + \sum_{i=1}^{k}\sum_{j=i}^{k} \frac{\partial \hat{h}^k}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)}$$

and write it as

$$\begin{aligned} \hat{y}_k^{(k+1)} &= a_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + b_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u \end{aligned} \tag{11}$$

Define $B_{k+1} := [\tilde{B}_k^T, b_{k+1}^T]^T$, and $s_{k+1} := \operatorname{rank} B_{k+1}$. Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that the first s_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix}$$

where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $(s_{k+1} - s_k)$ rows. Since the last rows of $B_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})$ are linearly dependent on the first s_{k+1} rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u, \\ \hat{y}_{k+1}^{(k+1)} &= \hat{h}^{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k+1, i \le j \le k+1\}) \end{aligned}$$

where once again everything is rational in $\tilde{y}_i^{(j)}$. Finally, set $\tilde{B}_{k+1} := [\tilde{B}_k^T, \tilde{b}_{k+1}^T]^T$, which has full rank equal to s_{k+1} locally.

End of Step $k + 1$.

By construction, $s_1 \le s_2 \le \cdots \le m$. If for some integer α we have $s_\alpha = m$, then the algorithm terminates. We call α the *relative order*[5] of the system. If $s_n < m$, then such an α does not exist. The closed form expression for u is derived from the α-th step of the structure algorithm which gives an invertible matrix $\tilde{B}_\alpha := [\tilde{b}_1^T, \ldots, \tilde{b}_\alpha^T]^T$ having full rank equal to m in a neighborhood $N_\alpha(x_0) =: N(x_0)$.

$$u(t) = \tilde{B}_\alpha^{-1}\left[\begin{pmatrix} \dot{\tilde{y}}_1 \\ \vdots \\ \dot{\tilde{y}}_\alpha \end{pmatrix} - \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_\alpha \end{pmatrix}\right] =: \tilde{B}_\alpha^{-1}[\tilde{Y}_\alpha - \tilde{A}_\alpha] \tag{12}$$

Note that the entries of the matrix $\tilde{B}_\alpha$ are rational functions of the derivatives of the output and there may exist an output for which the rank of $\tilde{B}_\alpha$ drops. All such outputs are called singular outputs and we define $\mathcal{Y}_p^s := \{y \in \mathcal{Y}_p \mid \operatorname{rank} \tilde{B}_\alpha(x, y) < m, x \in N(x_0)\}$. Hence, we work with u such that $\Gamma_{x_0}^O(u) \notin \mathcal{Y}_p^s$ for any time instant. Comparing to the SISO case, we had $\tilde{B}_\alpha = L_g L_f^{r-1} h(x)$ which is a function of the state only and thus there exists no singular output for SISO systems. Another useful class of systems for which $\mathcal{Y}_p^s = \emptyset$ was discussed in [5] by Hirschorn. As was the case in SISO systems, substitution of the expression for u from (12) in (1) gives the dynamics of the inverse system. These dynamics are defined on an open and dense set $\mathbb{M}^\alpha := \{x \in \mathbb{M} \mid \operatorname{rank} \tilde{B}_\alpha(x, y) = m, y \notin \mathcal{Y}_p^s\}$.

However, unlike in the SISO case, we need some differentiability assumptions on the input signals to characterize the input space for MIMO systems. In the structure algorithm, Step 1 gives $\dot{\tilde{y}}_1$ that has already u on the right-hand side and the α-th step of the algorithm involves $\{\tilde{y}_i^{(j)} \mid 1 \le i \le \alpha - 1, i \le j \le \alpha\}$. Thus $\tilde{y}_i^{(\alpha-1)}$ must be absolutely continuous so that $\tilde{y}_i^{(\alpha)}$ exists almost everywhere. For the input space, it means that $u^{(\alpha-1)}$ must be Lebesgue measurable and locally essentially bounded. These constraints characterize the input space $\mathcal{U}$ for MIMO case.

Based on the structure algorithm, we now study the conditions for functional reproducibility of multivariable

[5]The term was coined in [5] and is weaker than the notion of vector relative degree.

nonlinear systems. Using the notation derived in the structure algorithm, denote by Z the vector

$$Z\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) := \begin{pmatrix} h(x) \\ \hat{h}^1(x, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}^{\alpha-1}\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) \end{pmatrix} \tag{13}$$

and let

$$\hat{y} := \begin{pmatrix} y \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{\alpha-1}^{(\alpha-1)} \end{pmatrix}, \qquad \hat{y}_d := \begin{pmatrix} y_d \\ \hat{y}_{d_1} \\ \vdots \\ \hat{y}_{d_{\alpha-1}}^{(\alpha-1)} \end{pmatrix} \tag{14}$$

So Z is basically a concatenation of the expressions appearing at each step of Singh's structure algorithm which get differentiated and $\hat{y}$ is the concatenation of the corresponding expressions on the left-hand side so that

$$Z\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) - \hat{y} = 0$$

The following result is along the same line as Lemma 2.

Lemma 4: *If the system given by equation (1) with $x(t_0) = x_0$ has a relative order $\alpha < \infty$, then there exists a control input u such that $\Gamma_{x_0}^O(u) = y_d(\cdot)$ if and only if*

$$\hat{y}_d(t_0) = Z\big(x_0, \dot{\tilde{y}}_{d_1}(t_0), \cdots, \tilde{y}_{d_k}^{(k)}(t_0)\big) \;\; \forall k = 0, 1, \cdots, \alpha-1 \tag{15}$$

where $\hat{y}_d$ is defined as in (14).

An insightful geometric version of this result in terms of jet spaces is given in [11]. Similarly to the SISO case, the idea is that the portion of output which is not directly affected by u is determined initially by the value of state variables; and the input u, for which $\Gamma_{x_0}^O(u) = y_d(\cdot)$, is given by (12) with y replaced by y_d in that formula.

Example 3: Consider the system given in Example 1. The vector $\hat{y}$ is the portion of the output that gets differentiated and therefore,

$$\hat{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dot{y}_2 \end{pmatrix} \quad \Rightarrow \quad \hat{y}_d = \begin{pmatrix} y_{d_1} \\ y_{d_2} \\ \dot{y}_{d_2} \end{pmatrix}$$

and the vector $Z(x, y_1, y_2, \dot{y}_{d_1})$ is given by

$$Z(x, y_1, y_2, \dot{y}_{d_1}) = \begin{pmatrix} x_1 \\ x_2 \\ \dot{y}_{d_1}(x_3/x_1) \end{pmatrix}$$

By Lemma 4 and equation (2), if we have $\hat{y}_d(t_0) = Z\big((x_0, y_1(t_0), y_2(t_0), \dot{y}_{d_1}(t_0)\big)$ then the control which produces y_d as an output, on a small interval, is given by

$$u_1 = \frac{\dot{y}_{d_1}}{x_1}$$
$$u_2 = \frac{x_1 \ddot{y}_{d_2} - x_3 \ddot{y}_{d_1} + \dot{y}_{d_1}\dot{y}_{d_2}}{\dot{y}_{d_1}}$$

If $y_d(t) \notin \mathcal{Y}^s$ for all time instants and the corresponding state trajectory $x(t) \in \mathbb{M}^\alpha$, then the system can produce y_d as an output over arbitrary time interval. ◁

This result gives the following condition for the verification of switch-singular pairs.

Lemma 5: *For MIMO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p, Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}_{(\alpha_\kappa-1)}^{\alpha_\kappa-1} \end{pmatrix} = \begin{pmatrix} h_\kappa(x_0) \\ \hat{h}_\kappa^1(x_0, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}_\kappa^{\alpha_\kappa-1}(x_0, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha_\kappa-1}^{(\alpha_\kappa-1)}) \end{pmatrix} \tag{16}$$

where α_κ denotes the relative order of subsystems Γ_κ and $\kappa = p, q$.

The procedure for constructing the inverse from this point onwards is exactly the same as discussed earlier for the SISO case.

V. OUTPUT TRACKING

In the previous section, we considered the question of left invertibility where the objective was to recover $(\sigma,\ u)$ uniquely for all y in some output set $\mathcal{Y}^\alpha$. In this section, we address a different problem which concerns with finding $(\sigma,\ u)$ (that may not be unique) such that $\mathrm{H}_{x_0}(\sigma,\ u) = y_d$ for a given function y_d and a state x_0. For the invertibility problem, we found conditions on the subsystems and the output set $\mathcal{Y}$ so that the map H_{x_0} is injective for all x_0 in some subset. Here, we are given one particular $(x_0,\ y_d)$ and wish to find its preimage under the map H_{x_0}. For the switched system (3), denote by $\mathrm{H}_{x_0}^{-1}$ the preimage of a function y_d,

$$\mathrm{H}_{x_0}^{-1} := \{(\sigma,\ u) : \mathrm{H}_{x_0}(\sigma,\ u) = y_d\} \tag{17}$$

If y_d is not in the image set of H_{x_0} then by convention $\mathrm{H}_{x_0}^{-1} = \emptyset$. When $\mathrm{H}_{x_0}^{-1}(y_d)$ is a singleton, the map H_{x_0} is invertible at y. We want to find conditions and an algorithm to generate $\mathrm{H}_{x_0}^{-1}(y_d)$ when $\mathrm{H}_{x_0}^{-1}(y_d)$ is a finite set.

We require the individual subsystems to be strongly invertible because if this is not the case, then the set $\mathrm{H}_{x_0}^{-1}(y_d)$ may be infinite. For a non-invertible non-switched nonlinear system[6], the matrix $\tilde{B}_\alpha^{-1}$ in (12) is not defined and the expression for u is modified to:

$$u(t) = \tilde{B}_\alpha^\dagger[\tilde{Y}_\alpha - \tilde{A}_\alpha] + K(x, \tilde{Y}_{\alpha-1})v \tag{18}$$

where K is a matrix whose columns form a basis for the null space of $\tilde{B}_\alpha$ and $\tilde{B}_\alpha^\dagger := \tilde{B}_\alpha^T(\tilde{B}_\alpha\tilde{B}_\alpha^T)^{-1}$ is a right pseudo-inverse of $\tilde{B}_\alpha$. If an output is generated by some input u obtained from (18) with some initial state, then due to arbitrary choice of v, there always exist infinitely many different inputs that generate the same output with the same initial state. Hence to avoid infinite loop reasoning, we will assume that the individual subsystems Γ_p are strongly invertible for all $p \in \mathcal{P}$. However, we do not assume that the switched system is invertible as the subsystems may have switch-singular pairs. We will only consider the functions

[6] A non-switched system is not invertible if it has more inputs than outputs or it doesn't satisfy the structure algorithm criteria.

$y_d(t)$ over finite time intervals so that there is only a finite number of switches under consideration.

We now present a switching inversion algorithm for switched systems similar to the one given in [2]. The algorithm takes the parameters $x_0 \in \mathbb{M}$, $y_d \in \mathcal{F}^{pc}$ (defined over a finite interval) and returns the set $\mathrm{H}_{x_0}^{-1}(y_d)$. It uses the *index-matching map*[7] $\Sigma^{-1} : \mathbb{M} \times \mathcal{F}^{pc} \to 2^{\mathcal{P}}$ defined as $\Sigma^{-1}(x_0, y_d) := p$ such that $y_d \in \mathcal{Y}_p^\alpha$ and y_d satisfies (15), obtained via the structure algorithm of Γ_p. The index-matching map returns the indexes of the subsystems that are capable of generating y_d starting from x_0. If the returned set is empty, no subsystem is able to generate that y_d starting from x_0. In the algorithm, $\Gamma_{p,x_0}^{-1,O}(y_d)$ denotes the output of the inverse subsystem Γ_p^{-1}. The concatenation of an element η and a set S is $\eta \oplus S := \{\eta \oplus \zeta, \zeta \in S\}$. By convention, $\eta \oplus \emptyset = \emptyset, \forall \eta$. Finally, the concatenation of two sets S and T is $S \oplus T := \{\eta \oplus \zeta, \eta \in S, \zeta \in T\}$.

begin $\mathrm{H}_{x_0}^{-1}(y_d)$
  Let the domain of y_d be $[t_0, T)$.
  Let $\overline{\mathcal{P}} := \{p \in \mathcal{P} : y_{d[t_0, t_0+\varepsilon)} \in \mathcal{Y}_p^\alpha$ and $x_0 \in \mathbb{M}_p^\alpha, \varepsilon > 0\}$
  Let $t^* := \min\{t \in [t_0, T) : y_{d[t, t+\varepsilon)} \notin \mathcal{Y}_p^\alpha$ for some $p \in \overline{\mathcal{P}},\ \varepsilon > 0\}$ otherwise $t^* = T$.
  Let $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{d[t_0, t_0+\varepsilon)})$.
  if $\mathcal{P}^* \neq \emptyset$ **then**
    Let $\mathcal{A} := \emptyset$
    foreach $p \in \mathcal{P}^*$ **do**
      Let $x := \Gamma_{p,x_0}^{-1}(y_{d[t_0, t^*)})$
      if $x \in \mathbb{M}_p^\alpha$ *and* $y_{d[t_0,t^*)} \in \mathcal{Y}_p^\alpha$ **then**
        Let $u := \Gamma_{p,x_0}^{-1,O}(y_{d[t_0,t^*)})$
        $\mathcal{T} := \{t \in (t_0, t^*) : (x(t), y_d(t))$ is a switch-singular pair of Γ_p, Γ_q for some $q \neq p\}$.
        if $\mathcal{T}$ *is a finite set* **then**
          foreach $\tau \in \mathcal{T}$ **do**
            let $\xi := \Gamma_p(u)(\tau)$
            $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,\tau)}, u_{[t_0,\tau)}) \oplus \mathrm{H}_\xi^{-1}(y_{d[\tau,T)})\}$
        else if $\mathcal{T} = \emptyset$ *and* $t^* < T$ **then**
          let $\xi := \Gamma_p(u)(t^*)$
          $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,t^*)}, u) \oplus \mathrm{H}_\xi^{-1}(y_{d[t^*,T)})\}$
        else if $\mathcal{T} = \emptyset$ *and* $t^* = T$ **then**
          $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,T)}, u)\}$
        else
          $\mathcal{A} := \emptyset$
      else
        $\mathcal{A} := \emptyset$
  else
    $\mathcal{A} := \emptyset$
  return $\mathrm{H}_{x_0}^{-1}(y_d) := \mathcal{A}$
end

If the return is a non-empty set, the set must be finite and contains pairs of switching signals and inputs that generate the given y_d starting from x_0. If the return is an empty set, it means that there is no switching signal and input that generate y_d, or there is an infinite number of possible switching times. Also by our concatenation notation: if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $\eta \oplus \emptyset = \emptyset$.

Based on the semigroup property for the trajectories of dynamical systems, the algorithm determines the preimage on a subinterval $[t_0, t)$ of $[t_0, T)$ and then concatenates these with the corresponding preimage on the rest of the interval $[t, T)$. If t is the first switching time after t_0, then we can find $H_{x_0}^{-1}(y_{d[t_0,t)})$ by singling out which subsystems are capable of generating $y_{d[t_0,t)}$ using the index-matching map. The obvious candidate for first switching time, denoted by t^* in the algorithm, is the time at which the output loses smoothness. Note that in the SISO case, t^* is the time at which one of the first $r-1$ derivatives of the output lose continuity (see Section IV-A). But, it is entirely possible that we have a switching and the output is still smooth at that switching time because of a switch-singular pair (see Example 4). The algorithm takes that into account and uses a switch at a later time to recover a "hidden switch" earlier (e.g. a switch at which the output is smooth). This makes the switching inversion algorithm a recursive procedure calling itself with different parameters within the main algorithm (*e.g.* the function $\mathrm{H}_{x_0}^{-1}(y_d)$ uses the returns of $\mathrm{H}_\xi^{-1}(y_{d[t^*,T)})$).

The following example will help understand this algorithm.

Example 4: Consider a switched system with two modes

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^2 \\ y = x_2 \end{cases}$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^2 \\ y = x_1 \end{cases}$$

We wish to reconstruct the switching signal $\sigma(t)$ and the input $u(t)$ which will generate the following output:

$$y_d(t) = \begin{cases} \cos t & \text{if } t \in [0, t^*) \\ 2\cos t & \text{if } t \in [t^*, T) \end{cases}$$

where $t^* = \pi$ and $T = 4.5$, with the given initial state $x_0 = (0, 1)^T$.

In this example, any state x lying on the diagonal, $\Delta := \{(x_1, x_2)^T : x_1 = x_2\}$ forms a switch-singular pair with the output whose corresponding state trajectory hits the same state x at any time.

We now use the above switching inversion algorithm to find (σ, u) such that $\Gamma_{x_0,\sigma}^O(u) = y_d$. We have $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{d[0,t^*)}) = \{1\}$ by using the index-matching map with given x_0 and $y_d(0) = 1$. The inverse of Γ_1 on $[0, t^*)$ is

$$\Gamma_1^{-1} : \begin{cases} \dot{z} = \begin{pmatrix} z_1 z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{y}_d, \quad \mathbb{M}_1^\alpha = \mathbb{R}^2 \\ u(t) = -z_2 + \dot{y}_d \end{cases}$$

[7]The set $2^{\mathcal{P}}$ denotes the set of all subsets of the set $\mathcal{P}$

with $z(0) = x_0$, which then gives

$$\begin{aligned} z(t) &= \begin{pmatrix} 0 \\ \cos t \end{pmatrix} =: x(t) \\ u(t) &= -\cos t - \sin t \end{aligned} \qquad t \in [0, t^*). \tag{19}$$

We want to find $\mathcal{T} := \{t \le t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of Γ_1, Γ_2 $\}$, which is equivalent to solving

$$\cos t = x_1(t) = 0, \quad t \in (0,\, t^*).$$

This equation has a solution $t = \pi/2 =: t_1 < t^*$, and hence $\mathcal{T} = \{t_1\}$, a finite set. We have $\xi = x(t_1) = (0, 0)^T$ and we repeat the procedure for the initial state ξ and the output $y_{d[t_1,T)}$ with $\mathcal{P}^* := \Sigma^{-1}(\xi,\, y_{d[t_1,t^*)}) = \{1, 2\}$. We analyze these two cases:

Case 1: $p = 1$. This implies t_1 is not a switching time and $u(t)$, $x(t)$ are still given by (19) for $t_1 \le t < t^*$. Repeating the procedure with $\xi = x(t^*) = (0,\, 0)^T$ and $y_{d[t^*,T)}$ and $y_d(t^*) = -2$, we observe that $y_d(t^*) \neq x_1(t^*)$ and also $y_d(t^*) \neq x_2(t^*)$, thus the index-matching map returns an empty set, $\Sigma^{-1}(\xi, y_{d[t^*,T)}) = \emptyset$.

Case 2: $p = 2$, which means that t_1 is a switching instant. So we work with the inverse system of Γ_2,

$$\Gamma_2^{-1} : \begin{cases} \dot{z} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \dot{y}_d, \quad \mathbb{M}_1^\alpha = \mathbb{R}^2 \\ u(t) = e^{-z_2}\, \dot{y}_d \end{cases}$$

with initial state $z(t_1) = \xi$, which gives

$$\begin{aligned} z(t) &= \begin{pmatrix} \cos t \\ \cos t + \sin t - 1 \end{pmatrix} =: x(t) \\ u(t) &= -e^{\cos t + \sin t}\ \sin t \end{aligned} \qquad t \ge t_1.$$

We find $\mathcal{T} = \{t_1 < t \le t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving for

$$\cos t = x_2(t) = \cos t + \sin t - 1, \qquad \frac{\pi}{2} = t_1 < t \le t^* = \pi$$

It is easy to see that this equation has no solution and thus there exist no switch-singular pairs in interval $(t_1,\, t^*)$. So, we let $\xi = x(t^*) = (-1,\, -2)^T$ and repeat the procedure with ξ and $y_{d[t^*,T)}$, which gives the unique solution $\sigma_{[t^*,T)} = 1$ and $u_{[t^*,T)} = -2(\cos t + \sin t)$.

Thus, the switching inversion algorithm returns $(\sigma,\, u)$, where

$$(\sigma,\, u) = \begin{cases} (1,\ -\cos t - \sin t), & \text{if } 0 \le t < t_1 \\ (2,\ -e^{\cos t + \sin t}\ \sin t), & \text{if } t_1 \le t < t^* \\ (1,\ -2(\cos t + \sin t)), & \text{if } t^* \le t \le T \end{cases}$$

In this example, the output only loses smoothness at t^* and t^* is a switching instant. However, there is another switching at t_1 where the output doesn't lose smoothness. Without the concept of switch-singular pairs, one might falsely conclude that there is no switching signal and input that generates $y_d(t)$ but instead the use of the switching inversion algorithm allows us to recover the input and switching signal. $\lhd$

VI. CONCLUSIONS

In this paper, we addressed the invertibility problem of switched nonlinear systems. The concepts introduced in [2] for the linear systems were extended to the nonlinear systems. A necessary and sufficient condition for the invertibility of switched systems was given which required the invertibility of subsystems and the non-existence of switch-singular pairs. We developed formulae for checking if $(x_0,\, y)$ is a switch-singular pair of two subsystems and then gave an algorithm to recover the input and switching signal from the given output and initial state.

For future work, one interesting problem is to develop conditions for checking the existence of switch-singular pairs which are more constructive as it is in general not feasible to verify (16) for every output and state. Another research direction is to approach the problem geometrically and investigate characterizations equivalent to non-existence of switch-singular pairs to obtain geometric criteria for left invertibility of switched systems.

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