INTEGRAL-INPUT-TO-STATE STABILIZATION

Joint work with
Eduardo D. Sontag (Rutgers Univ.) and
Yuan Wang (Florida Atlantic Univ.)

Asymptotic Stability

$$\boxed{\dot{x} = f(x)}$$

GAS $\Leftrightarrow$ $\exists$ Lyap. fcn: $\dot{V} < 0$ for all $x \neq 0$

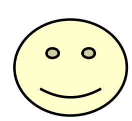
$$\boxed{\dot{x} = f(x) + g(x)u}$$

CLF: $\inf\limits_{u} \{L_f V + L_g V u\} < 0$ for all $x \neq 0$

Sontag's formula:

$$u = \begin{cases} -\dfrac{a + \sqrt{a^2 + b^4}}{b^2} b & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

where $a := L_f V$ and $b := L_g V$



Input-to-State Stability

$$\boxed{\dot{x} = f(x,d)}$$

ISS: for some $\beta \in KL$ and $\gamma \in K_\infty$

$$x(t) \leq \beta(|x(0)|, t) + \gamma\Big(\sup_{0\leq s\leq t} |d(s)|\Big)$$

(Sontag '89)

K_∞ function

Characterization: ISS-Lyap. fcn

$$\dot{V} \leq -\alpha(|x|) + \sigma(|d|)$$

where $\alpha, \sigma \in K_\infty$

Input-to-State Stabilization

$$\boxed{\dot{x} = f(x) + g_1(x)d + g_2(x)u}$$

ISS-CLF: for some $\alpha, \sigma \in K_\infty$

$$\inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} \leq -\alpha(|x|) + \sigma(|d|)$$

V is an ISS-CLF iff $\exists\, \rho \in K_\infty$ s. t.

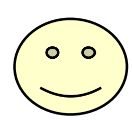
$$|x| \geq \rho(|d|)$$
$$\Downarrow$$
$$\inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} < 0 \quad \forall x \neq 0$$

Equivalently

$$\inf_u \{L_f V + |L_{g_1} V| \rho^{-1}(|x|) + L_{g_2} V u\} < 0 \quad \forall x \neq 0$$

Now can apply Sontag’s formula

(Krstiȁ et al.; Wang)



Integral-Input-to-State Stability

$$\boxed{\dot{x} = f(x,d)}$$

iISS: for some $\beta \in KL$ and $\gamma \in K_\infty$

$$x(t) \le \beta(|x(0)|,t) + \int_0^t \gamma(|d(s)|)ds$$

(Sontag '98)

Characterization: iISS-Lyap. fcn

$$\dot{V} \le -\alpha(|x|) + \sigma(|d|)$$

where $\sigma \in K_\infty$ and α is pos def (not K_∞!)

(Angeli-Sontag-Wang '98)

Example of an iISS but not ISS system:

$$\dot{x} = -x + xd = (-1+d)x$$

Integral-Input-to-State Stabilization

$$\boxed{\dot{x} = f(x) + g_1(x)d + g_2(x)u}$$

iISS-CLF: for some $\sigma \in K_\infty$ and α pos def

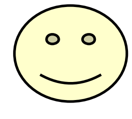
$$\inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} \leq -\alpha(|x|) + \sigma(|d|)$$

Assume $\dfrac{\sigma(r)}{r} \to \infty$ as $r \to \infty$

Then V is an iISS-CLF iff

$$\inf_u \{L_f V + y(x) + L_{g_2} V u\} < 0 \quad \forall x \neq 0$$

where $y(x) := \max_r \{|L_{g_1} V| r - \sigma(r)\}$

Now can apply Sontag’s formula 

Example

$$\dot{x} = -x + xd + u - x^2 d$$

$$\dot{y} = -y + yd - u + x^2 d$$

If $d = 2$ then $\frac{d}{dt}(x + y) = x + y$ for any u

so we cannot have ISS

But: setting $u = x^2 d$ we get

$$\dot{x} = -x + xd$$
$$\dot{y} = -y + yd$$

This system is iISS

State feedback $u = k(x)$ also exists

Backstepping

If the system

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u$$

is integral-input-to-state stabilizable with a smooth control law $u = k(x)$ satisfying $k(0) = 0$

then so is the augmented system

$$\dot{x} = f(x) + g_1(x)d + g_2(x)\xi$$

$$\dot{\xi} = u + h(x,\xi)d$$

Concluding Remarks

- Alternative for ISS
- $\dot{x} = f(x, d) + g(x)u$
- Restricted control spaces

Application: supervisory control of uncertain nonlinear systems (Hespanha-Morse '99)