15th Triennial World Congress, Barcelona, Spain

STABILIZATION BY QUANTIZED STATE OR OUTPUT FEEDBACK: A HYBRID CONTROL APPROACH

Daniel Liberzon [*]

[*] *Coordinated Science Laboratory*
University of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A.
liberzon@uiuc.edu

Abstract:
This paper deals with global asymptotic stabilization of continuous-time systems with quantized signals. A hybrid control strategy originating in earlier work relies on the possibility of making discrete on-line adjustments of quantizer parameters. We explore this method here for general nonlinear systems with general types of quantizers affecting the state of the system or the measured output.

Keywords: Quantization, hybrid control, Lyapunov function, input-to-state stability.

1. INTRODUCTION

In the classical feedback control setting, the output of the process is assumed to be passed directly to the controller, which generates the control input and in turn passes it directly back to the process. In practice, however, this paradigm often needs to be reexamined because the interface between the controller and the process features some additional information-processing devices. These considerations arise, for example, in networked control systems; see the articles in (Bushnell, 2001) and the references therein.

One important aspect to take into account in such situations is signal quantization. We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values. Quantization may affect the process output (this happens, for example, when the output measurements to be used for feedback are obtained by using a digital camera, stored in the memory of a digital computer, or transmitted over a digital communication channel). It may also affect the control input (examples include the standard PWM amplifier and the manual transmission on a car).

We assume that the given system evolves in continuous time. In the presence of quantization, the state space (or the input space) of the system is divided into a finite number of *quantization regions*, each corresponding to a fixed value of the quantizer. At the time of passage from one quantization region to another, the dynamics of the system change abruptly. Therefore, systems with quantization can be naturally viewed as *hybrid* systems, i.e., systems described by a coupling between continuous and discrete dynamics.

There are two well-studied phenomena which account for changes in the system's behavior caused by quantization. The first one is saturation: if the signal is outside the range of the quantizer, then the quantization error is large, and the control law designed for the ideal case of no quantization leads to instability. The second one is deterioration of performance near the equilibrium: as the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is impossible. These phenomena manifest themselves in the existence of two nested invariant regions such that all

[1] Supported by NSF grant ECS-0114725.

trajectories of the quantized system starting in the bigger region approach the smaller one, while no further convergence guarantees can be given.

A standard assumption made in the literature is that parameters of the quantizer are fixed in advance and cannot be changed by the control designer; see, among many sources, (Delchamps, 1990; Feng and Loparo, 1997; Lunze *et al.*, 1999; Raisch, 1995; Sur and Paden, 1998; Wong and Brockett, 1999). There has been some research concerned with the question of how the choice of quantization parameters affects the behavior of the system (Åström and Bernhardsson, 1999; Elia and Mitter, 2001; Ishii and Francis, 2002; Liberzon and Brockett, 2000). In this paper, building on the earlier work reported in (Brockett and Liberzon, 2000; Liberzon, 2000), we adopt the approach that it is possible to vary some parameters of the quantizer *on line*, on the basis of collected data. (In the example where a quantizer is used to represent a camera, this corresponds to zooming in or out, i.e., varying the focal length, while the number of pixels of course remains fixed.) When such manipulations are feasible, they allow one to change the range of the quantizer and the quantization error as the system evolves, thereby helping to overcome the two difficulties described above.

The quantization parameters will be updated at discrete instants of time (these *switching events* will be triggered by the values of a suitable Lyapunov function). This results in a *hybrid quantized feedback control policy*. There are several reasons for adopting a hybrid control approach rather than varying the quantization parameters continuously. First, in specific situations there may be some constraints on how many values these parameters are allowed to take and how frequently they can be adjusted. Thus a discrete adjustment policy is more natural and easier to implement than a continuous one. Secondly, the analysis of hybrid systems obtained in this way appears to be more tractable than that of systems resulting from continuous parameter tuning. In fact, we will see that invariant regions defined by level sets of a Lyapunov function provide a simple and effective tool for studying the behavior of the closed-loop system. This also implies that precise computation of the switching times is not essential, which makes our hybrid control policies robust with respect to time delays.

The recent paper (Brockett and Liberzon, 2000) thoroughly investigates the hybrid control methodology outlined above in the context of the feedback stabilization problem for linear control systems with output (or state) quantization. It is shown there that if a linear system can be stabilized by a linear feedback law, then it can also be *globally asymptotically stabilized* by a hybrid quantized feedback control policy. The control strategy is usually composed of two stages. The first, “zooming-out” stage consists in increasing the range of the quantizer until the state of the system can be adequately measured. The second, “zooming-in” stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. The developments of (Brockett and Liberzon, 2000) were restricted to quantizers that give rise to rectilinear quantization regions.

The present work generalizes the contributions of the paper (Brockett and Liberzon, 2000) in two directions. First, we consider more general types of quantizers, with quantization regions having *arbitrary shapes* as in (Lunze *et al.*, 1999). This extension is important for applications. For example, in the context of vision-based feedback control mentioned earlier, the image plane of the camera is divided into rectilinear regions, but the shapes of the quantization regions in the state space which result from computing inverse images of these rectangles can be rather complicated. We will demonstrate that the principal findings of (Brockett and Liberzon, 2000) are still valid in this more general setting.

The second goal of this paper is to address the quantized feedback stabilization problem for *nonlinear systems*. It can be shown via a linearization argument that by using the approach of (Brockett and Liberzon, 2000) one can obtain local asymptotic stability for a nonlinear system, provided that the corresponding linearized system is stabilizable; see (Hu *et al.*, 1999). Here we are concerned with achieving global stability [2] results. We will show that the techniques developed in (Brockett and Liberzon, 2000) can be extended in a natural way to those nonlinear systems that are *input-to-state stabilizable with respect to measurement disturbances*. We thus reveal an interesting interplay between the problem of quantized feedback stabilization, the theory of hybrid systems, and topics of current interest in nonlinear control design. A preliminary investigation of these questions has been reported in (Liberzon, 2000), but only for state quantizers with rectilinear quantization regions.

2. QUANTIZER

By a *quantizer* we mean a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. This leads to a partition of $\mathbb{R}^n$ into a finite number of *quantization regions* of the form $\{z \in \mathbb{R}^n : q(z) = l\}$, $l \in \mathcal{Q}$. The shapes of these quantization regions are arbitrary.

When z does not belong to the union of quantization regions of finite size, the quantizer *saturates*. More precisely, we assume that there exist positive real num-

[2] Working with a given nonlinear system directly, one gains an advantage even if only local asymptotic stability is sought, because the linearization of a stabilizable nonlinear system may fail to be stabilizable.

bers M and Δ such that the following two conditions hold:

(1) If
$$|z| \le M \tag{1}$$
then
$$|z - q(z)| \le \Delta. \tag{2}$$
(2) If
$$|z| > M$$
then
$$|q(z)| > M - \Delta.$$

Condition 1 gives a bound on the quantization error when the quantizer does not saturate. Condition 2 provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* of q and the *quantization error*, respectively. To preserve the equilibrium at the origin, we also assume that $q(0) = 0$. An example of a quantizer satisfying the above requirements is provided by the quantizer with rectangular quantization regions considered in earlier work (Brockett and Liberzon, 2000; Liberzon, 2000).

In the control strategies to be developed below, we will use quantized measurements of the form
$$\mu q\left(\frac{z}{\mu}\right)$$
where $\mu > 0$. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can think of μ as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error. We will update μ at discrete instants of time, so it will be the discrete state of the resulting hybrid closed-loop system. In the camera model mentioned in the Introduction, μ corresponds to the inverse of the focal length[3] f. It is possible to introduce more general, nonlinear scaling of the quantized variable, as in $\nu \circ q \circ \nu^{-1}(z)$ where ν is some invertible function from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\circ$ denotes composition; however, this does not seem to introduce any significant advantages in the context of the problems studied here.

3. STATE QUANTIZATION

To fix ideas, we treat linear systems first.

3.1 *Linear systems*

Consider the linear system
$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{3}$$

[3] We prefer to work with $\mu = 1/f$ rather than with f to avoid system signals that grow unbounded, although this is merely a formal distinction.

Suppose that (3) is *stabilizable*, so that for some matrix K the eigenvalues of $A + BK$ have negative real parts. By the standard Lyapunov stability theory, there exist positive definite symmetric matrices P and Q such that
$$(A + BK)^T P + P(A + BK) = -Q. \tag{4}$$
We will let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. The inequality
$$\lambda_{\min}(P)|x|^2 \le x^T Px \le \lambda_{\max}(P)|x|^2$$
will be used repeatedly below. We will assume that $B \ne 0$ and $K \ne 0$; this is no loss of generality because the case of interest is when A is not a stability matrix.

In this section we are interested in the situation where only quantized measurements of the state are available. Since the state feedback law $u = Kx$ is not implementable, we apply the "certainty equivalence" quantized feedback control law
$$u = K\mu q\left(\frac{x}{\mu}\right). \tag{5}$$
The closed-loop system can be written as
$$\dot{x} = (A + BK)x - BK\mu\left(\frac{x}{\mu} - q\left(\frac{x}{\mu}\right)\right). \tag{6}$$
The behavior of trajectories of the system (6) for a fixed μ is characterized by the following result.

Lemma 1. Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have
$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon) \tag{7}$$
where
$$\Theta_x := \frac{2\|PBK\|}{\lambda_{\min}(Q)} > 0.$$
Then the ellipsoids
$$\mathscr{R}_1 := \{x : x^T Px \le \lambda_{\min}(P)M^2\mu^2\} \tag{8}$$
and
$$\mathscr{R}_2 := \{x : x^T Px \le \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{9}$$
are invariant regions for the system (6). Moreover, all solutions of (6) that start in the ellipsoid $\mathscr{R}_1$ enter the smaller ellipsoid $\mathscr{R}_2$ in finite time.

PROOF. Whenever the inequality (1), and consequently (2), hold with $z = x/\mu$, the derivative of $x^T Px$ along solutions of (6) satisfies
$$\begin{aligned}\frac{d}{dt}x^T Px &= -x^T Qx - 2x^T PBK\mu\left(\frac{x}{\mu} - q\left(\frac{x}{\mu}\right)\right)\\ &\le -\lambda_{\min}(Q)|x|^2 + 2|x|\|PBK\|\Delta\mu\\ &= -|x|\lambda_{\min}(Q)(|x| - \Theta_x\Delta\mu)\end{aligned}$$
This implies the following formula:
$$\begin{gathered}\Theta_x\Delta(1+\varepsilon)\mu \le |x| \le M\mu\\ \Downarrow\\ \frac{d}{dt}x^T Px \le -|x|\lambda_{\min}(Q)\Theta_x\Delta\varepsilon\mu.\end{gathered} \tag{10}$$

Define the balls

$$\mathscr{B}_1 := \{x : |x| \leq M\mu\}$$

and

$$\mathscr{B}_2 := \{x : |x| \leq \Theta_x\Delta(1+\varepsilon)\mu\}.$$

In view of the inequality (7), we have

$$\mathscr{B}_2 \subset \mathscr{R}_2 \subset \mathscr{R}_1 \subset \mathscr{B}_1.$$

Combined with (10), this immediately implies that the ellipsoids $\mathscr{R}_1$ and $\mathscr{R}_2$ are both invariant. The fact that the trajectories starting in $\mathscr{R}_1$ approach $\mathscr{R}_2$ in finite time follows from the bound on the derivative of $x^T Px$ given by (10). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathscr{R}_1$ and if we let

$$T := \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2}{\Theta_x^2\Delta^2(1+\varepsilon)\lambda_{\min}(Q)\varepsilon} \tag{11}$$

then $x(t_0+T)$ is guaranteed to belong to $\mathscr{R}_2$. □

As we explained before, a hybrid quantized feedback control policy involves updating the value of μ at discrete instants of time. Using this idea and Lemma 1, it is possible to achieve global asymptotic stability, as we now show.

Theorem 1. Assume that M is large enough compared to Δ so that we have

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta\max\left\{1, \frac{\|PBK\|}{\lambda_{\min}(Q)}\right\}. \tag{12}$$

Then there exists a hybrid quantized feedback control policy that makes the system (6) globally asymptotically stable.

PROOF. The control strategy is divided into two stages.

The "zooming-out" stage. Set u equal to 0. Let $\mu(0) = 1$. Then increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $\|e^{At}\|$. For example, one can fix a positive number τ and let $\mu(t) = 1$ for $t \in [0,\tau)$, $\mu(t) = \tau e^{2\|A\|\tau}$ for $t \in [\tau, 2\tau)$, $\mu(t) = 2\tau e^{2\|A\|2\tau}$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \geq 0$ such that

$$\left|\frac{x(t)}{\mu(t)}\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - 2\Delta$$

(by (12), the right-hand side of this inequality is positive). In view of condition 1 imposed in Section 2, this implies

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - \Delta. \tag{13}$$

We can thus pick a time t_0 such that (13) holds with $t = t_0$. Therefore, in view of conditions 1 and 2 of Section 2, we have

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M$$

hence $x(t_0)$ belongs to the ellipsoid $\mathscr{R}_1$ given by (8) with $\mu = \mu(t_0)$. Note that this event can be detected using only the available quantized measurements.

The "zooming-in" stage. Choose an $\varepsilon > 0$ such that the inequality (7) is satisfied; this is possible because of (12). We know that $x(t_0)$ belongs to $\mathscr{R}_1$ with $\mu = \mu(t_0)$. We now apply the control law (5). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0+T)$, where T is given by the formula (11). Then $x(t_0+T)$ belongs to the ellipsoid $\mathscr{R}_2$ given by (9) with $\mu = \mu(t_0)$. For $t \in [t_0+T, t_0+2T)$, let

$$\mu(t) = \Omega\mu(t_0)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(P)}M}.$$

We have $\Omega < 1$ by (7), hence $\mu(t_0+T) < \mu(t_0)$. The ellipsoid $\mathscr{R}_2$ with the old value $\mu = \mu(t_0)$ is the same as the ellipsoid $\mathscr{R}_1$ with the new value $\mu = \mu(t_0+T)$. This means that we can continue the analysis for $t \geq t_0 + T$ as before. Namely, $x(t_0+2T)$ belongs to the ellipsoid $\mathscr{R}_2$ defined by (9) with $\mu = \mu(t_0+T)$. For $t \in [t_0+2T, t_0+3T)$, let $\mu(t) = \Omega\mu(t_0+T)$. Repeating this procedure, we obtain the desired control policy. Indeed, it is not hard to show that the equilibrium $x = 0$ of the continuous dynamics is stable in the sense of Lyapunov. Moreover, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. □

3.2 *Nonlinear systems*

Consider the system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{14}$$

It is natural to assume that there exists a state feedback law $u = k(x)$ that makes the closed-loop system globally asymptotically stable. Actually, we need to assume that k satisfies the following stronger condition: there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathscr{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x, e \in \mathbb{R}^n$ we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{15}$$

and

$$|x| \geq \rho(|e|) \Rightarrow \nabla V(x) f(x, k(x+e)) \leq -\alpha_3(|x|). \tag{16}$$

According to the results of (Sontag, 1989; Sontag and Wang, 1995), this is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x+e)) \tag{17}$$

is *input-to-state stable* (ISS) with respect to the measurement disturbance input e.

Since only quantized measurements of the state are available, we again consider the "certainty equivalence" quantized feedback control law, which in this case is given by

$$u = k\left(\mu q\left(\frac{x}{\mu}\right)\right). \tag{18}$$

The closed-loop system is

$$\dot{x} = f\left(x, k\left(\mu q\left(\frac{x}{\mu}\right)\right)\right) \tag{19}$$

and this takes the form (17) with

$$e = \mu q\left(\frac{x}{\mu}\right) - x. \tag{20}$$

The behavior of trajectories of (19) for a fixed value of μ is characterized by the following lemma.

Lemma 2. Assume that we have

$$\alpha_1(M\mu) > \alpha_2 \circ \rho(\Delta\mu). \tag{21}$$

Then the sets

$$\mathscr{R}_1 := \{x : V(x) \le \alpha_1(M\mu)\} \tag{22}$$

and

$$\mathscr{R}_2 := \{x : V(x) \le \alpha_2 \circ \rho(\Delta\mu)\} \tag{23}$$

are invariant regions for the system (19). Moreover, all solutions of (19) that start in the set $\mathscr{R}_1$ enter the smaller set $\mathscr{R}_2$ in finite time.

The next result is a nonlinear counterpart of Theorem 1.

Theorem 2. Assume that the system $\dot{x} = f(x,0)$ is forward complete and that for all $\mu > 0$ we have

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \max\{\rho(\Delta\mu), \chi(\mu) + 2\Delta\mu\} \tag{24}$$

where χ is some class $\mathscr{K}_\infty$ function. Then there exists a hybrid quantized feedback control policy that makes the system (19) globally asymptotically stable.

EXAMPLE. Consider the following system, which is a simplified version of the system treated in the example on page 811 in (Jiang *et al.*, 1999):

$$\dot{x} = x^3 + xu, \qquad x, u \in \mathbb{R}.$$

In (Jiang *et al.*, 1999) it is shown how to construct a feedback law k such that the closed-loop system

$$\dot{x} = x^3 + xk(x+e)$$

is ISS with respect to e. It follows from the analysis of (Jiang *et al.*, 1999) that the inequalities (15) and (16) hold with $V(x) = x^2/2$, $\alpha_1(r) = \alpha_2(r) = r^2/2$, $\alpha_3(r) = r^2$, and $\rho(r) = cr$ for an arbitrary $c > 1$. We have $(\alpha_2^{-1} \circ \alpha_1)(r) = r$, so (24) is valid for every $M > \Delta \max\{c, 2\}$. □

4. OUTPUT QUANTIZATION

We now extend some of the above results to linear systems with output feedback. The developments that follow are essentially based on the ideas from (Brockett and Liberzon, 2000, Section 5). Other approaches are also possible; see (Delchamps, 1989; Sur and Paden, 1998).

Consider the linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. Suppose that (A,B) is a stabilizable pair and (C,A) is an observable pair. This implies that there exist a feedback matrix K and an output injection matrix L such that the eigenvalues of $A+BK$ and $A+LC$ have negative real parts. The eigenvalues of the matrix

$$\bar{A} := \begin{pmatrix} A+BK & -BK \\ 0 & A+LC \end{pmatrix}$$

then also have negative real parts, and so there exist positive definite symmetric $2n \times 2n$ matrices $\bar{P}$ and $\bar{Q}$ such that

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} = -\bar{Q}.$$

In this section we are interested in the situation where only quantized measurements of the output y are available. We therefore consider the following dynamic output feedback law, which is based on the standard Luenberger observer but uses $\mu q\left(\frac{y}{\mu}\right)$ instead of y:

$$\begin{aligned} \dot{\hat{x}} &= (A+LC)\hat{x} + Bu - L\mu q\left(\frac{y}{\mu}\right) \\ u &= K\hat{x} \end{aligned} \tag{25}$$

where $\hat{x} \in \mathbb{R}^n$. The closed-loop system takes the form

$$\dot{x} = Ax + BK\hat{x}$$
$$\dot{\hat{x}} = (A+LC)\hat{x} + BK\hat{x} - L\mu q\left(\frac{y}{\mu}\right)$$

In the coordinates given by

$$\bar{x} := \begin{pmatrix} x \\ x - \hat{x} \end{pmatrix} \in \mathbb{R}^{2n}$$

we can rewrite this system more compactly as

$$\dot{\bar{x}} = \bar{A}\bar{x} - L\mu \begin{pmatrix} 0 \\ \frac{y}{\mu} - q\left(\frac{y}{\mu}\right) \end{pmatrix}. \tag{26}$$

For a fixed value of μ, the behavior of this system is characterized by the following result.

Lemma 3. Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have

$$\sqrt{\lambda_{\min}(\bar{P})}M > \sqrt{\lambda_{\max}(\bar{P})}\Theta_y\|C\|\Delta(1+\varepsilon) \tag{27}$$

where

$$\Theta_y := \frac{2\|\bar{P}L\|}{\lambda_{\min}(\bar{Q})}.$$

Then the ellipsoids

$$\mathscr{R}_1 := \{\bar{x} : \bar{x}^T\bar{P}\bar{x} \le \lambda_{\min}(\bar{P})M^2\mu^2/\|C\|^2\} \tag{28}$$

and

$$\mathscr{R}_2 := \{\bar{x} : \bar{x}^T\bar{P}\bar{x} \le \lambda_{\max}(\bar{P})\Theta_y^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{29}$$

are invariant regions for the system (26). Moreover, all solutions of (26) that start in the ellipsoid $\mathscr{R}_1$ enter the smaller ellipsoid $\mathscr{R}_2$ in finite time.

Combining the dynamic output feedback law (25) with the idea of updating the value of μ at discrete instants of time as before, we arrive at the following result.

Theorem 3. Assume that M is large enough compared to Δ so that we have

$$\sqrt{\frac{\lambda_{\min}(\bar{P})}{\lambda_{\max}(P)}}M > \max\left\{3\Delta, 2\Delta\frac{\|\bar{P}L\|\|C\|}{\lambda_{\min}(Q)}\right\}. \qquad (30)$$

Then there exists a hybrid quantized feedback control policy that makes the system (26) globally asymptotically stable.

5. REFERENCES

Åström, K. J. and B. Bernhardsson (1999). Comparison of periodic and event based sampling for first-order stochastic systems. In: *Proc. 14th IFAC World Congress*. pp. 301–306.

Brockett, R. W. and D. Liberzon (2000). Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control* **45**, 1279–1289.

Bushnell, L., Ed.) (2001). *Special section on networks and control*. pp. 22–99. Vol. 21 of *IEEE Control Systems Magazine*.

Delchamps, D. F. (1989). Extracting state information from a quantized output record. *Systems Control Lett.* **13**, 365–372.

Delchamps, D. F. (1990). Stabilizing a linear system with quantized state feedback. *IEEE Trans. Automat. Control* **35**, 916–924.

Elia, N. and S. K. Mitter (2001). Stabilization of linear systems with limited information. *IEEE Trans. Automat. Control* **46**, 1384–1400.

Feng, X. and K. A. Loparo (1997). Active probing for information in control systems with quantized state measurements: a minimum entropy approach. *IEEE Trans. Automat. Control* **42**, 216–238.

Hu, B., Z. Feng and A. N. Michel (1999). Quantized sampled-data feedback stabilization for linear and nonlinear control systems. In: *Proc. 38th IEEE Conf. on Decision and Control*. pp. 4392–4397.

Ishii, H. and B. A. Francis (2002). Stabilizing a linear system by switching control with dwell time. *IEEE Trans. Automat. Control*. To appear.

Jiang, Z.-P., I. Mareels and D. Hill (1999). Robust control of uncertain nonlinear systems via measurement feedback. *IEEE Trans. Automat. Control* **44**, 807–812.

Liberzon, D. (2000). Nonlinear stabilization by hybrid quantized feedback. In: *Proc. Third International Workshop on Hybrid Systems: Computation and Control* (N. Lynch and B. H. Krogh, Eds.). Vol. 1790 of *Lecture Notes in Computer Science*. pp. 243–257. Springer.

Liberzon, D. and R. W. Brockett (2000). Nonlinear feedback systems perturbed by noise: steady-state probability distributions and optimal control. *IEEE Trans. Automat. Control* **45**, 1116–1130.

Lunze, J., B. Nixdorf and J. Schröder (1999). Deterministic discrete-event representations of linear continuous-variable systems. *Automatica* **35**, 395–406.

Raisch, J. (1995). Control of continuous plants by symbolic output feedback. In: *Hybrid Systems II* (P. Antsaklis et al., Eds.). pp. 370–390. Springer. Berlin.

Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control* **34**, 435–443.

Sontag, E. D. and Y. Wang (1995). On characterizations of the input-to-state stability property. *Systems Control Lett.* **24**, 351–359.

Sur, J. and B. E. Paden (1998). State observer for linear time-invariant systems with quantized output. *ASME J. Dynamic Systems, Measurement, and Control* **120**, 423–426.

Wong, W. S. and R. W. Brockett (1999). Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback. *IEEE Trans. Automat. Control* **44**, 1049–1053.