LIMITED-INFORMATION CONTROL of SWITCHED and HYBRID SYSTEMS via PROPAGATION of REACHABLE SETS

Daniel Liberzon

Coordinated Science Laboratory
Electrical & Computer Engineering
Univ. of Illinois, Urbana-Champaign
PROBLEM FORMULATION

Switched system: \[
\dot{x} = A_\sigma x + B_\sigma u
\]

\{(A_p, B_p) : p \in \mathcal{P}\} are (stabilizable) modes,
\mathcal{P} is a (finite) index set, \(\sigma : [0, \infty) \to \mathcal{P}\) is a switching signal
(can be state-dependent, realizing discrete state in hybrid system)

Information structure:

Sampling: state \(x\) is measured at times \(t_k = k\tau_s, k = 0, 1, \ldots\)

Quantization: each \(x(t_k)\) is encoded by an integer from 0 to \(N^n\)
and sent to the controller, along with \(\sigma(t_k) \in \mathcal{P}\) \((n = \dim x)\)

Data rate: \[
\frac{\log_2(N^n + 1) + \log_2|\mathcal{P}|}{\tau_s}
\]

Objective: design an encoding & control strategy s.t. \(x(t) \to 0\)
based on this limited information about \(x(\cdot)\) and \(\sigma(\cdot)\)
MOTIVATION

Switching:
• ubiquitous in realistic system models
• lots of research on stability & stabilization under switching
• tools used: common & multiple Lyapunov functions, slow switching assumptions

Quantization:
• coarse sensing (low cost, limited power, hard-to-reach areas)
• limited communication (shared network resources, security)
• theoretical interest (how much info is needed for a control task)
• tools used: Lyapunov analysis, data-rate / MATI bounds

Commonality of tools is encouraging
Almost no prior work on quantized control of switched systems (except quantized MJLS [Nair et. al. 2003, Dullerud et. al. 2009])
NON-SWITCHED CASE

Quantized control of a single LTI system: [Baillieul, Brockett-L, Hespanha, Nair-Evans, Petersen-Savkin, Tatikonda]

Crucial step: obtaining a reachable set over-approximation at next sampling instant

System is stabilized if

\[
\frac{\|e^{A\tau_s}\|_\infty}{\|N\|} > \text{growth factor on } [t_k, t_{k+1}]
\]

Crucial step: obtaining a reachable set over-approximation at next sampling instant

How to do this for switched systems?
REACHABLE SET ALGORITHMS

Many computational (on-line) methods for hybrid systems

• Puri–Varaiya–Borkar (1996): approximation by piecewise-constant differential inclusions; unions of polyhedra


• Asarin–Dang–Maler (2000, 2002): linear dynamics; rectangular polyhedra; tool: \(\frac{d}{dt}\)

• Mitchell–Tomlin–et. al. (2000, 2003): nonlinear dynamics; level sets of value functions for HJB equations

• Kurzhanski–Varaiya (2002, 2005): affine open-loop dynamics; ellipsoids

• Chutinan–Krogh (2003): nonlinear dynamics; polyhedra; tool: \textit{CheckMate}

OUR APPROACH

Here we prefer a method that is:

• Analytical (off-line)

• Leads to an *a priori data-rate bound* for stabilization (may be more conservative than on-line methods)

• Works with linear dynamics and hypercubes

• Tailored to switched systems (time-dependent switching) but can be adopted/refined for hybrid systems
SLOW-SWITCHING and DATA-RATE ASSUMPTIONS

1) ∃ dwell time $\tau_d$ (lower bound on time between switches)

2) ∃ average dwell time (ADT) $\tau_a$ s.t.

   number of switches on $(s, t] \leq N_0 + \frac{t - s}{\tau_a} \quad \forall t > s \geq 0$

3) $\tau_a > \tau_d \geq \tau_s$ (sampling period)

   Implies: $\leq 1$ switch on each sampling interval $(t_k, t_{k+1}]$

   We’ll see how large $\tau_a$ should be for stability

Define $\Lambda_p := \| e^{A_p \tau_s} \|_\infty, \quad p \in \mathcal{P}$

4) $\Lambda_p < N \quad \forall p$

   (usual data-rate bound for individual modes)
ENCODING and CONTROL STRATEGY

Goal: generate, on the decoder/controller side, a sequence of points $x_k^* \in \mathbb{R}^n$ and numbers $E_k > 0$ s.t.

$$||x(t_k) - x_k^*|| \leq E_k \quad \forall k$$

(always $\infty$-norm)

Let $p := \sigma(t_k)$

Pick $K_p$ s.t. $A_p + B_p K_p$ is Hurwitz

Define state estimate $\hat{x}(\cdot)$ on $[t_k, t_{k+1})$ by

$$\dot{\hat{x}} = (A_p + B_p K_p) \hat{x}, \quad \hat{x}(t_k) = c_k$$

Define control $u(\cdot)$ on $[t_k, t_{k+1})$ by

$$u(t) = K_p \hat{x}(t)$$
GENERATING STATE BOUNDS

Choosing a sequence $E_0, E_1, E_2, \ldots$ that grows faster than system dynamics, for some $k_0$ we will have $\|x(t_{k_0})\| \leq E_{k_0}$

Inductively, assuming $\|x(t_k) - x^*_k\| \leq E_k$ we show how to find $x^*_{k+1}, E_{k+1}$ s.t. $\|x(t_{k+1}) - x^*_{k+1}\| \leq E_{k+1}$

**Case 1** (easy): sampling interval with no switch

$\sigma(t_k) = \sigma(t_{k+1}) = p \Rightarrow \sigma(t) = p \ \forall t \in [t_k, t_{k+1}]$

\[
\begin{align*}
\dot{x} &= A_p x + B_p u \\
\hat{x} &= A_p \hat{x} + B_p u \\
\hat{x}(t_k) &= c_k
\end{align*}
\]

Let $e := x - \hat{x}$

$\Rightarrow \dot{e} = A_p e$ \quad $\Rightarrow \|e(t_{k+1})\| \leq \|e^{A_p\tau_s}\| \cdot \|e(t_k)\| \leq \wedge_p E_k / N =: E_{k+1}$

$x^*_{k+1} := e(A_p + B_p K_p)\tau_s c_k$
Case 2 (harder): sampling interval with a switch

\[ \sigma(t_k) = p, \quad \sigma(t_{k+1}) = q \neq p \]

Before the switch: as on previous slide,

\[ \| x(t_k + t) - \hat{x}(t_k + t) \| \leq \| e^{Ap\bar{t}} \| E_k/N \]

but this is unknown \((\triangleq)\)

Instead, pick some \(t' \in [0, \tau_s]\) and use \(\hat{x}(t_k + t')\) as center

(triangle inequality)

Intermediate bound:

\[ \| x(t_k + \bar{t}) - \hat{x}(t_k + t') \| \leq D_{k+1}(\bar{t}) \]
GENERATING STATE BOUNDS

After the switch: on $[t_k + \bar{t}, t_{k+1}]$, closed-loop dynamics are

$$
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} =
\begin{pmatrix}
A_q & B_q K_p \\
0 & A_p + B_p K_p
\end{pmatrix}
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}, \text{ or } \dot{z} = \bar{A}_{pq} z
$$

Auxiliary system in $\mathbb{R}^{2n}$: $\dot{z} = \bar{A}_{pq} \hat{z}$, $\bar{z}(0) = \begin{pmatrix} \hat{x}(t_k + t') \\ \hat{x}(t_k + t') \end{pmatrix}$

(need to take maximum over $\bar{t}$ to obtain final bound)
STABILITY ANALYSIS: OUTLINE

1) sampling interval with no switch: $\sigma \equiv p$ on $[t_k, t_k+1]$

$$x_{k+1}^* = e^{(A_p+B_pK_p)\tau_s}c_k = e^{(A_p+B_pK_p)\tau_s}(x_k^*+(c_k-x_k^*))$$

This is exp. stable DT system w. input $\Delta_k := c_k - x_k^*$

$$\|\Delta_k\| \leq E_k(N-1)/N$$ data-rate assumption

and $E_{k+1} = E_k \land p/N < E_k \Rightarrow E_k \exp \rightarrow 0$

Thus, the overall “cascade” system is exp. stable

Lyapunov function: $V_p(x, E) := x^TP_px + \rho_p E^2$

satisfies $V_p(x_{k+1}^*, E_{k+1}) \leq \nu V_p(x_k^*, E_k)$, $\nu < 1$

2) if $[t_k, t_{k+1}]$ contains a switch from $p$ to $q$, then

$$V_q(x_{k+1}^*, E_{k+1}) \leq \mu V_p(x_k^*, E_k), \quad \mu > 1$$

If ADT satisfies $\tau_\alpha > (1+\log(\mu)/\log(1/\nu)) \tau_s$ then

$V_\sigma(t_k)(x_k^*, E_k) \exp \rightarrow 0$ as $k \rightarrow \infty$ $\Rightarrow$ same true for $x(t_k)$

Intersample behavior, Lyapunov stability – see paper
HYBRID SYSTEMS

Switching triggered by switching surfaces (guards) in state space

- Previous result applies if we can use relative location of switching surfaces to verify slow-switching hypotheses
- Can just run the algorithm and verify convergence on-line
- Can improve reachable set bounds

For example: \( \sigma(t_k) = p \), \( \sigma(t_{k+1}) = q \neq p \)

\[ \sigma = p \]

\[ \mathcal{E}_{k+1} \]

- State jumps – easy to incorporate
CONCLUSIONS and FUTURE WORK

Contributions:
• Stabilization of switched/hybrid systems with quantization
• Main step: computing over-approximations of reachable sets
• Data-rate bound is the usual one, maximized over modes

Extensions:
• Relaxing the slow-switching assumption
• Refining reachable set bounds
• Less frequent transmissions of discrete mode value

Challenges:
• External disturbances
• Modeling uncertainty
• Nonlinear dynamics
• Output feedback
Theoretical lower bound on $\tau_a$ is about 10 times larger.

\( \mathcal{P} = \{1, 2\} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -2 & 0 \end{pmatrix} \)

\( x_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & -1 \end{pmatrix} \)

\( E_0 = 0.5 \quad \tau_s = 0.5, \quad N = 5 \quad \) (data-rate assumption holds)

\( \tau_d = 1.05, \quad \tau_a = 7.55, \quad N_0 = 5 \)