ESTIMATION ENTROPY for NONLINEAR and SWITCHED SYSTEMS

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MOTIVATION and PROBLEM FORMULATION

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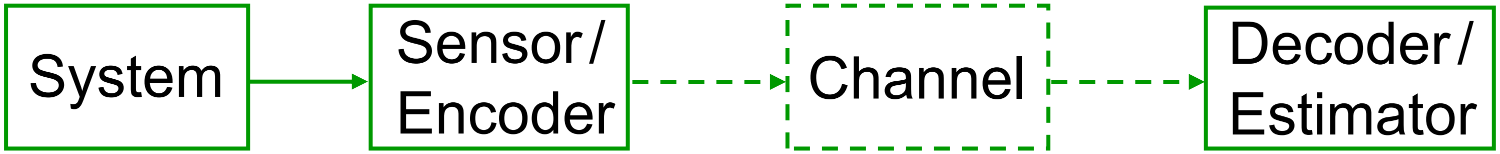

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Sensor/ Encoder

Channel

Decoder/ Estimator

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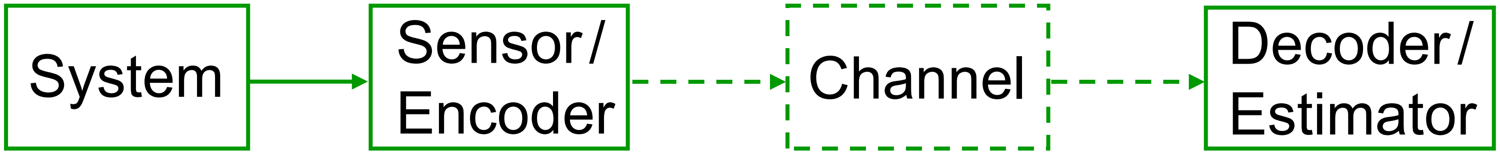

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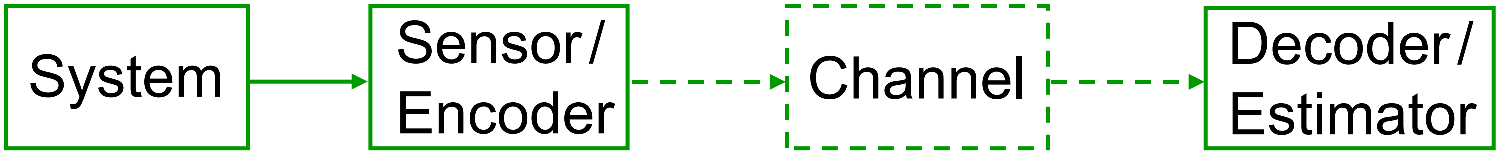
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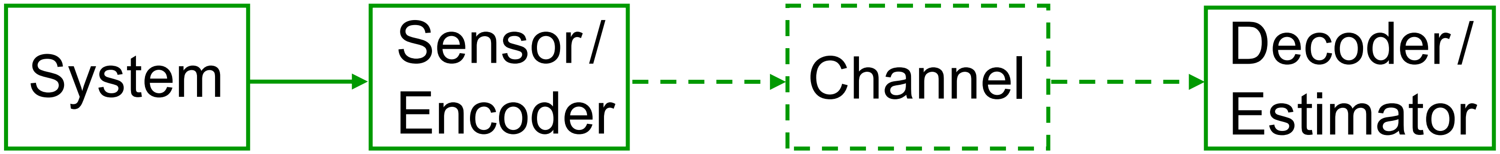

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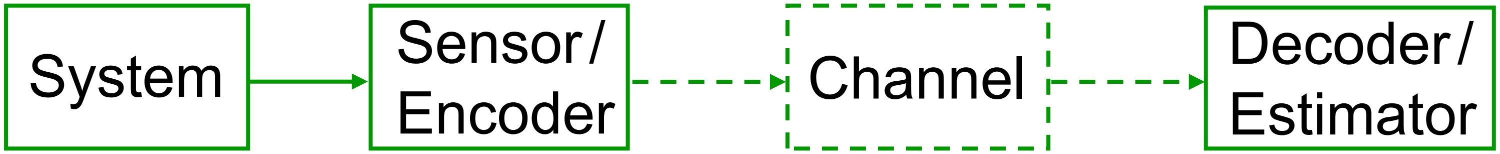

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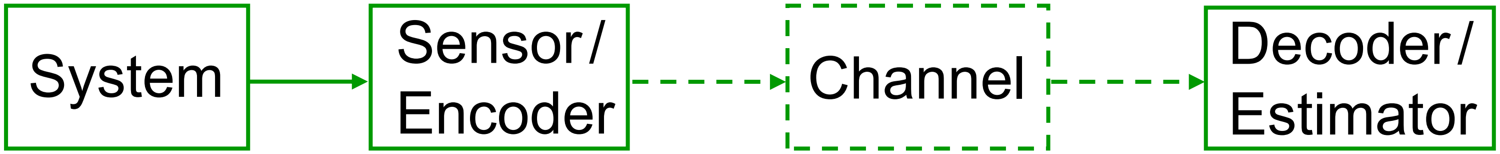

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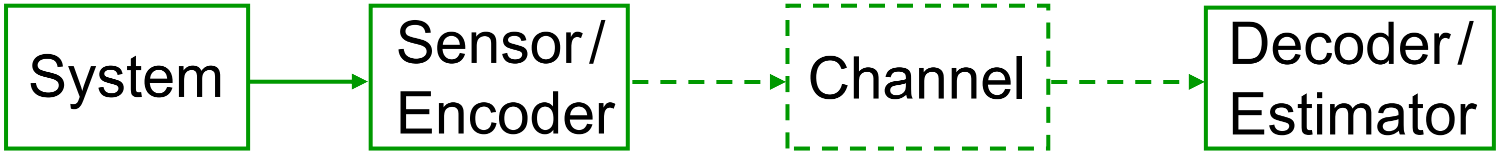

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(variant of previous entropy notions for control and estimation [Nair et al.; Colonius, Kawan; Leonov, Boichenko, Matveev, Savkin, Pogromsky])

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Intuition: $x_1, \dots, x_N$ are quantization points, $h_{\text{est}} =$ bit rate

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Divide by T, take $\limsup_{T \to \infty}$, then $\lim_{\varepsilon \to 0}$ – all become equal

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Take $\limsup_{T\to\infty} \frac{1}{T}\log$ of this to get $\boxed{h_{\text{est}} = a + \alpha}$

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the above LTV bound applied to its variational equation gives
$|\xi(x_1, t) - \xi(x_2, t)| \leq e^{Lt} |x_1 - x_2|$

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Note: we can instead take L to be the Lipschitz constant of f, which is more conservative but works if $\partial f/\partial x$ does not exist

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Taking $\frac{1}{T}\log$ gives $(L+\alpha)n$ ■

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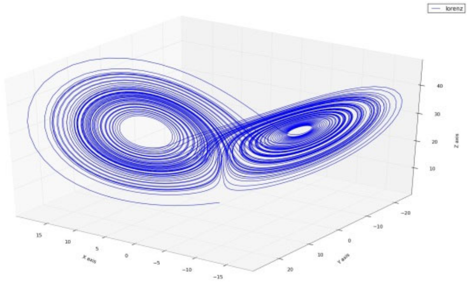
Similar argument gives lower bound for nonlinear system (cf. [Colonius]): $\boxed{h_{\text{est}} \geq \inf_x \text{tr}\, \partial f/\partial x(x) + \alpha n}$

EXAMPLE: LORENZ SYSTEM



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and so $h_{\text{est}} \leq 3(L + \alpha)$

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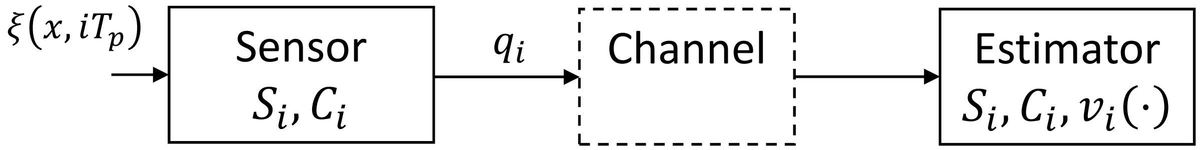
Bound in [L '21] gives $h_{\text{est}} \leq 3(\max\{\lambda_{\max}(A), 0\} + \alpha)$

ESTIMATION PROCEDURE



$\xi(x, iT_p)$

Sensor

S_i, C_i

q_i

Channel

Estimator

$S_i, C_i, v_i(\cdot)$

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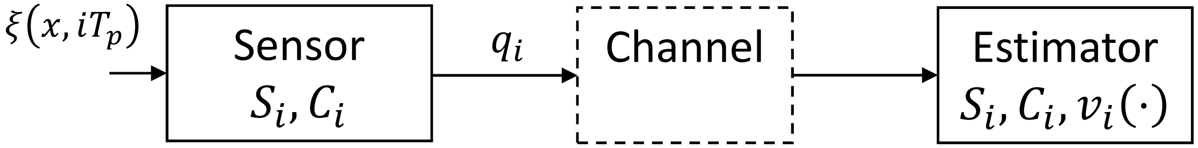

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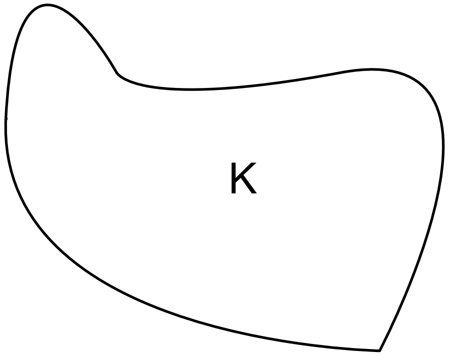

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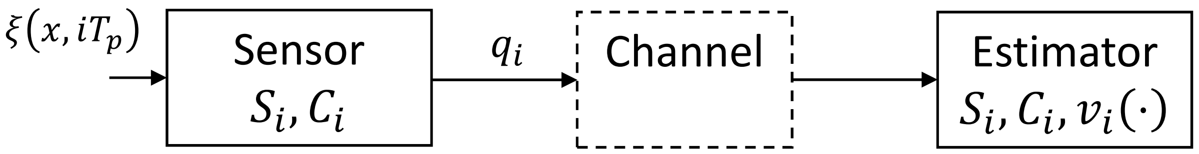

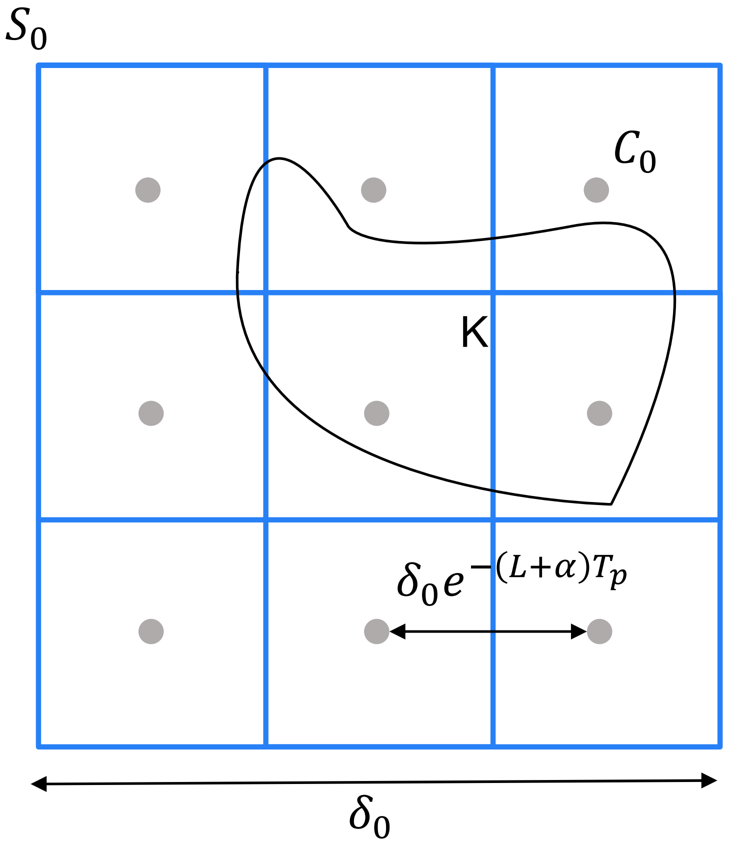

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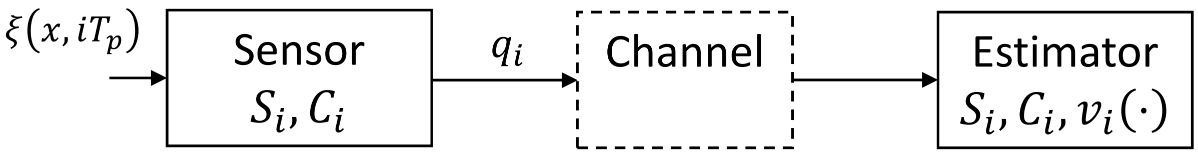

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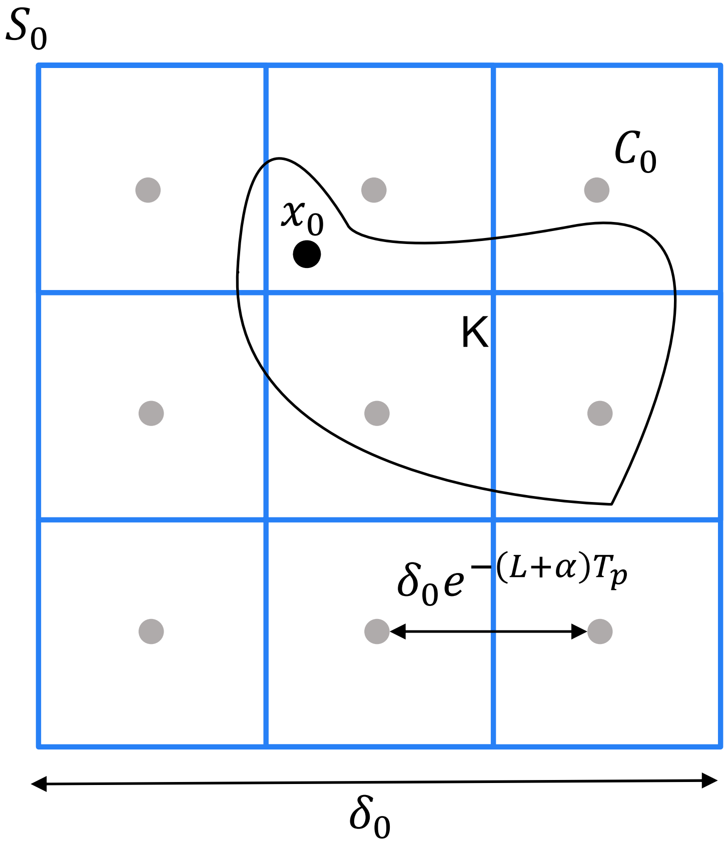

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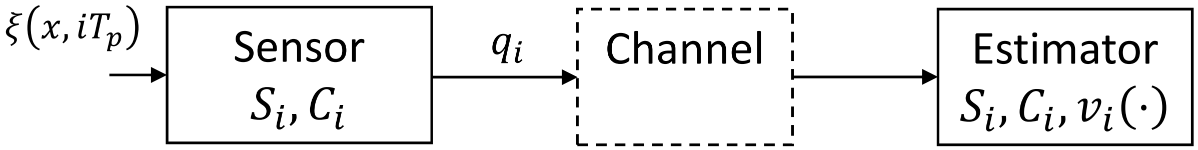

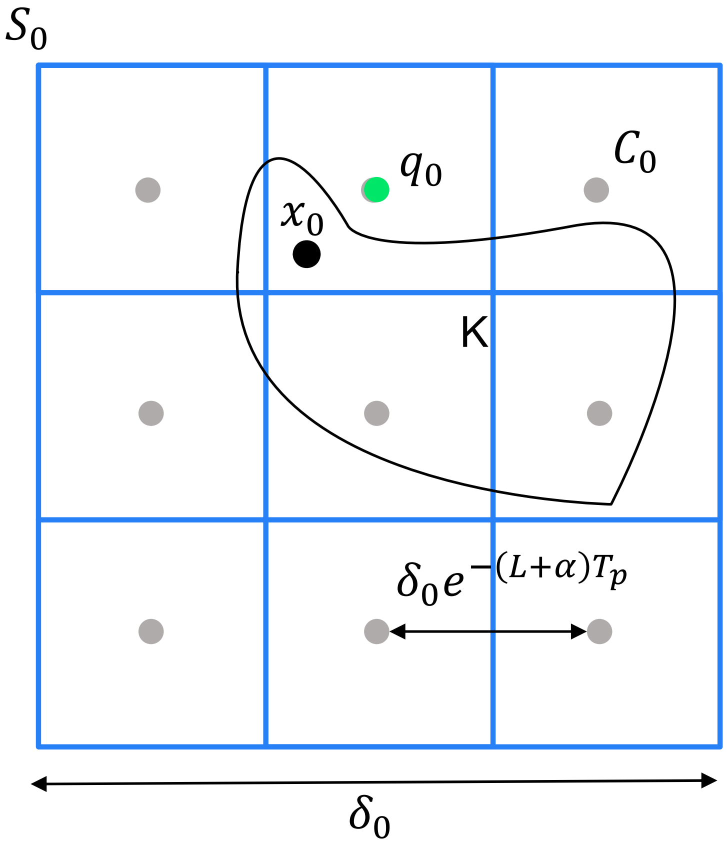

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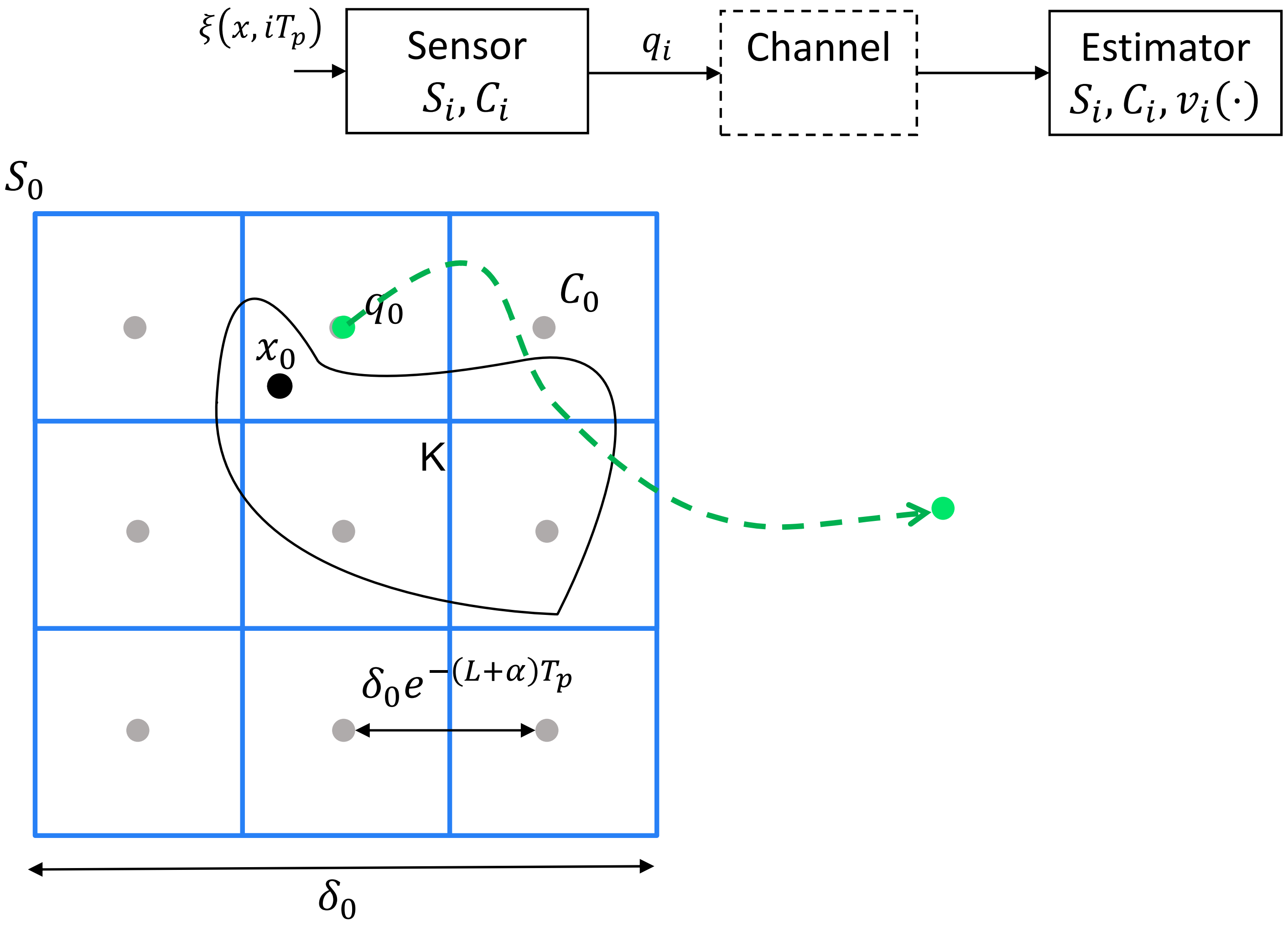

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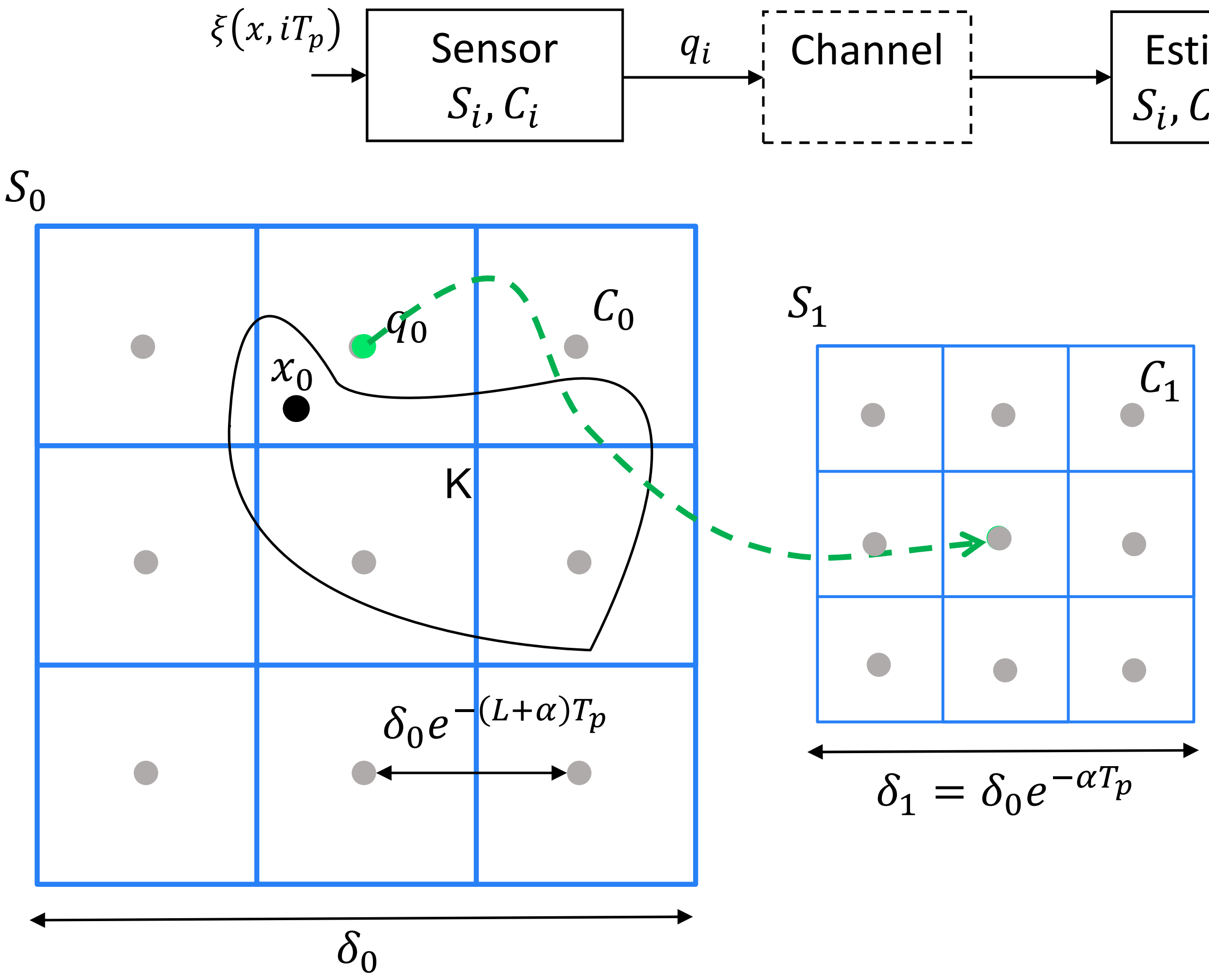

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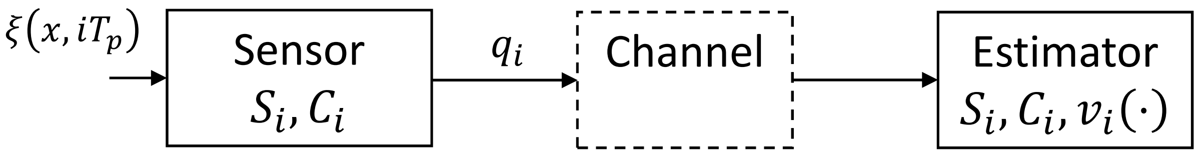

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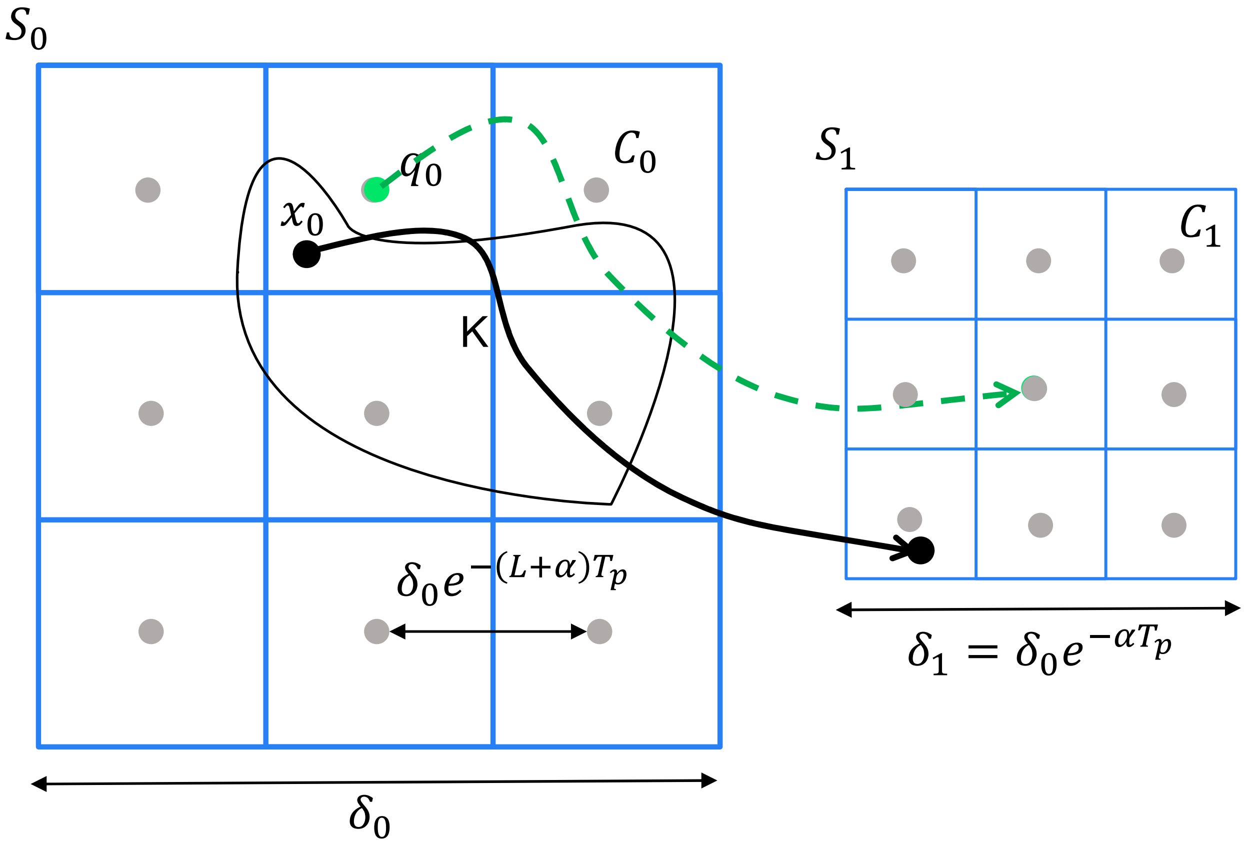

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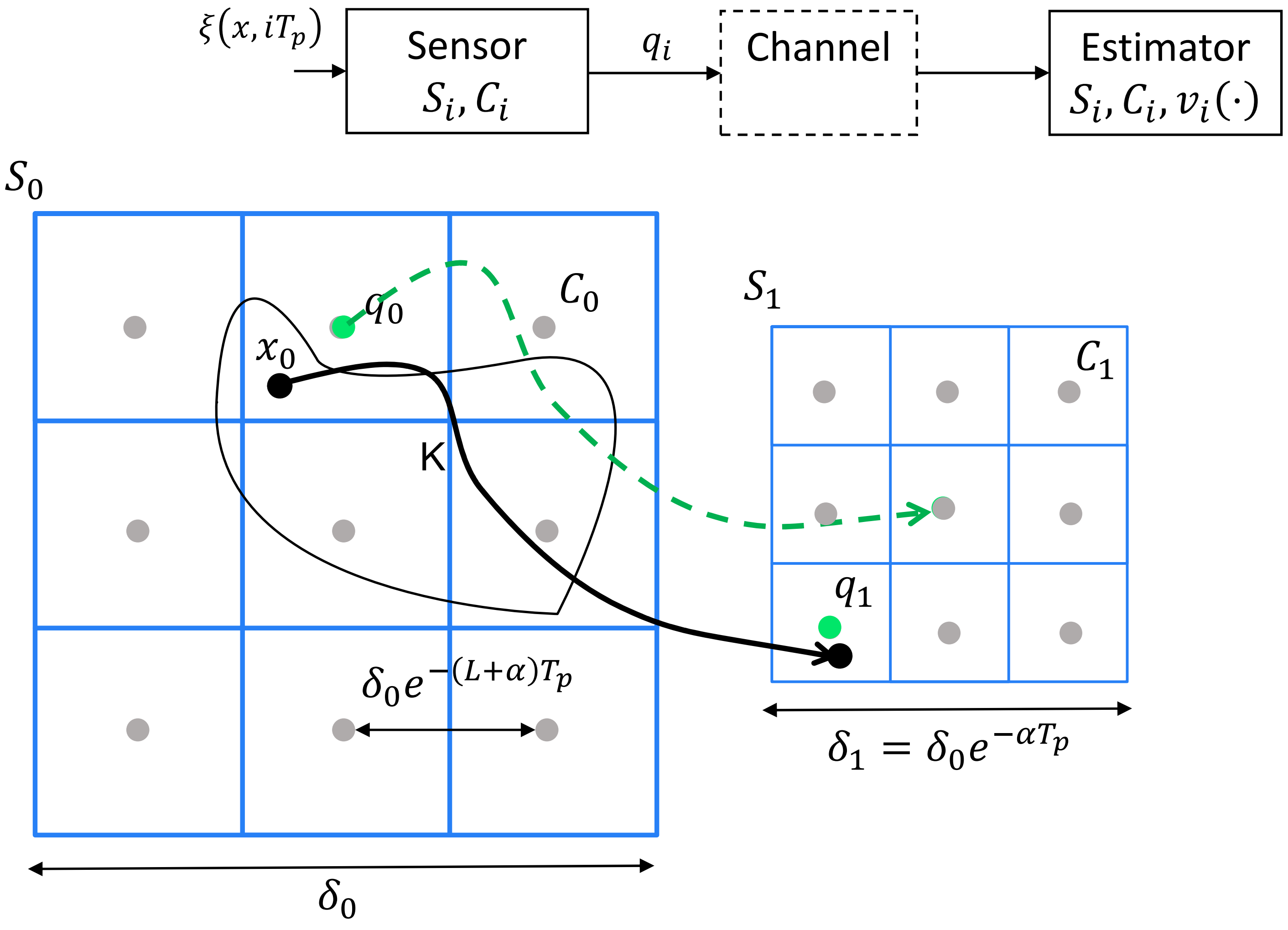

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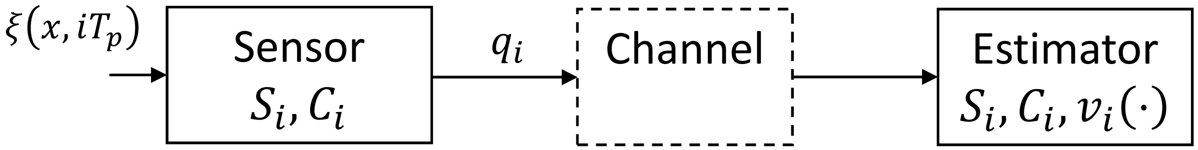

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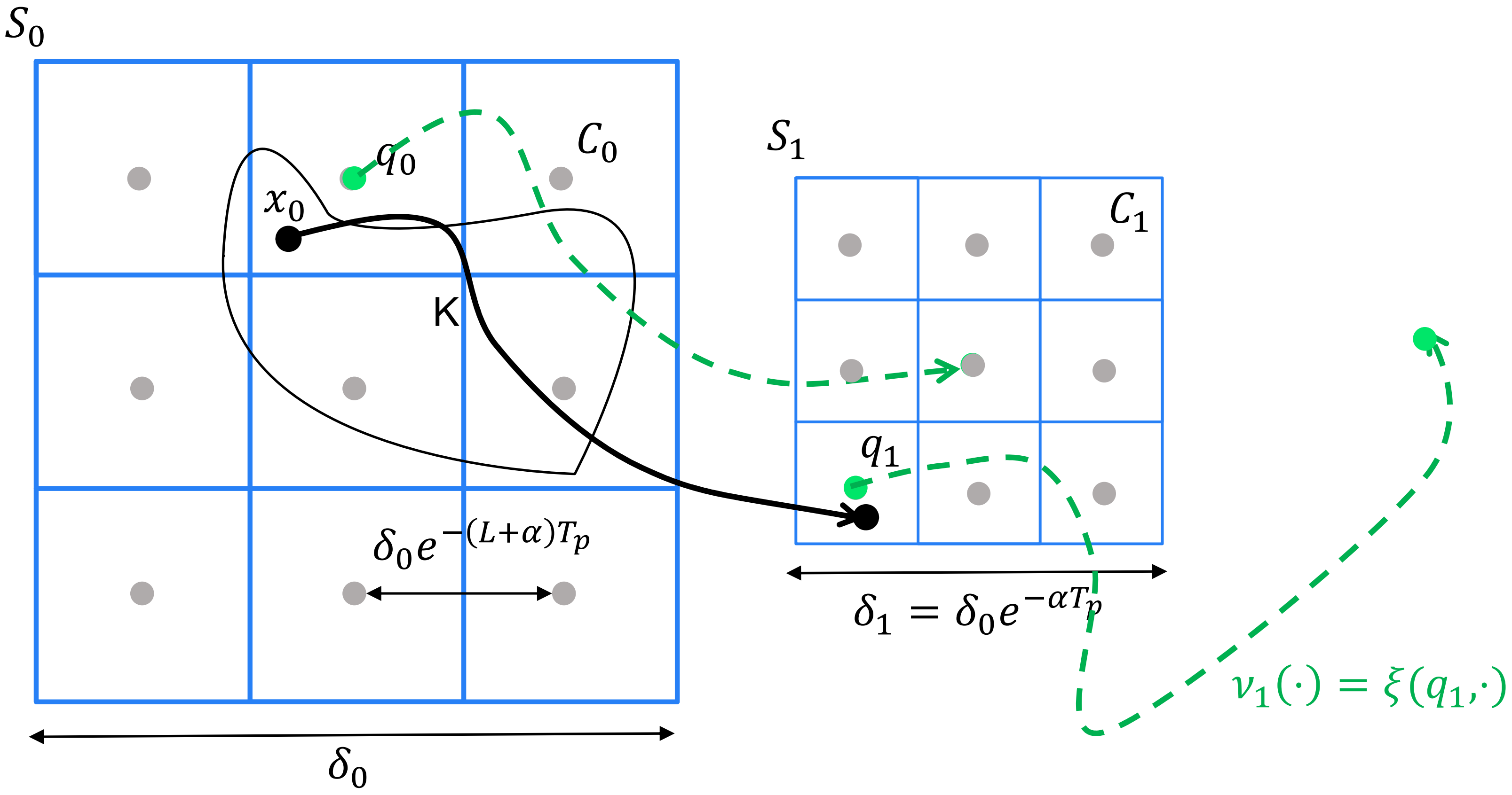

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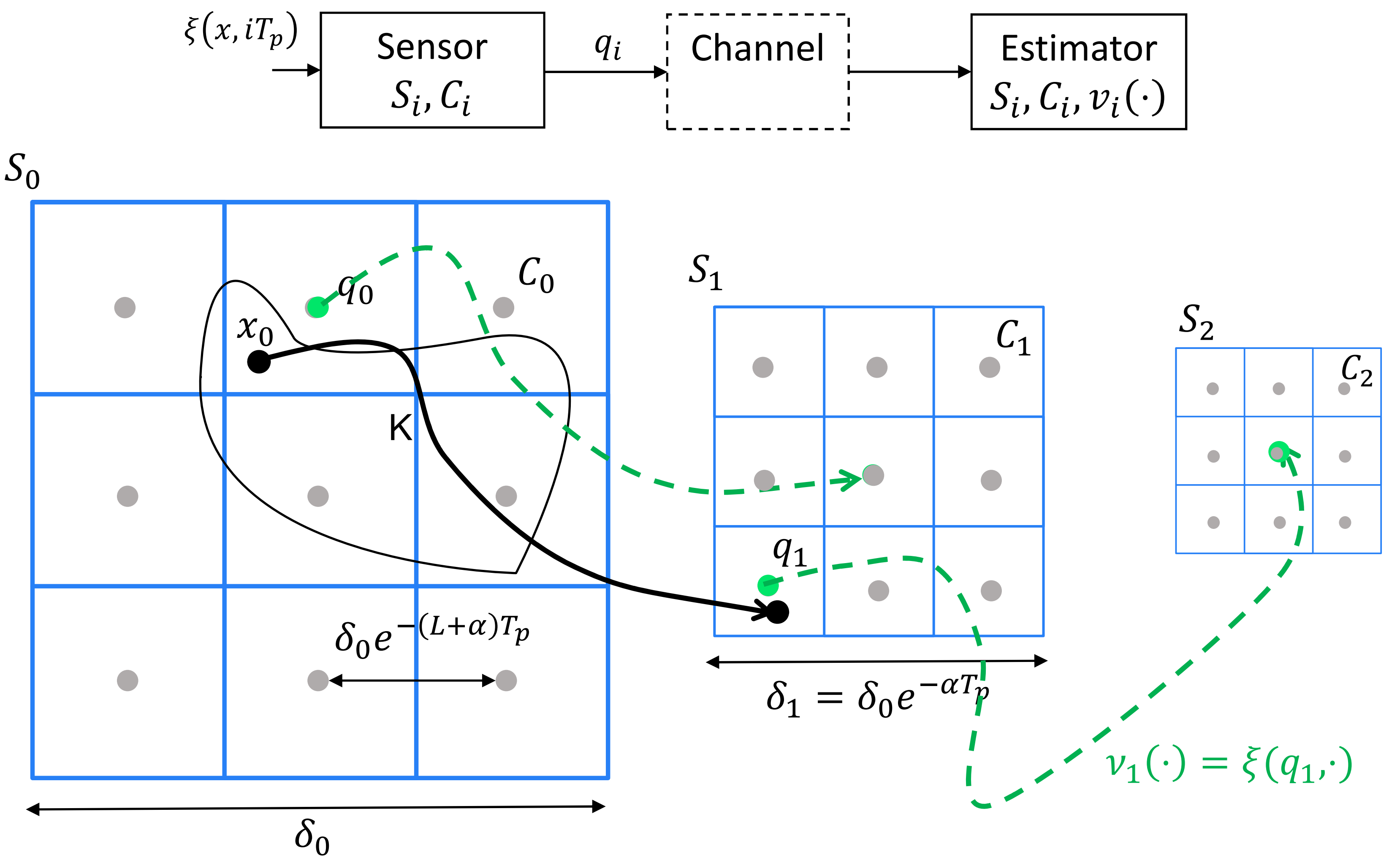

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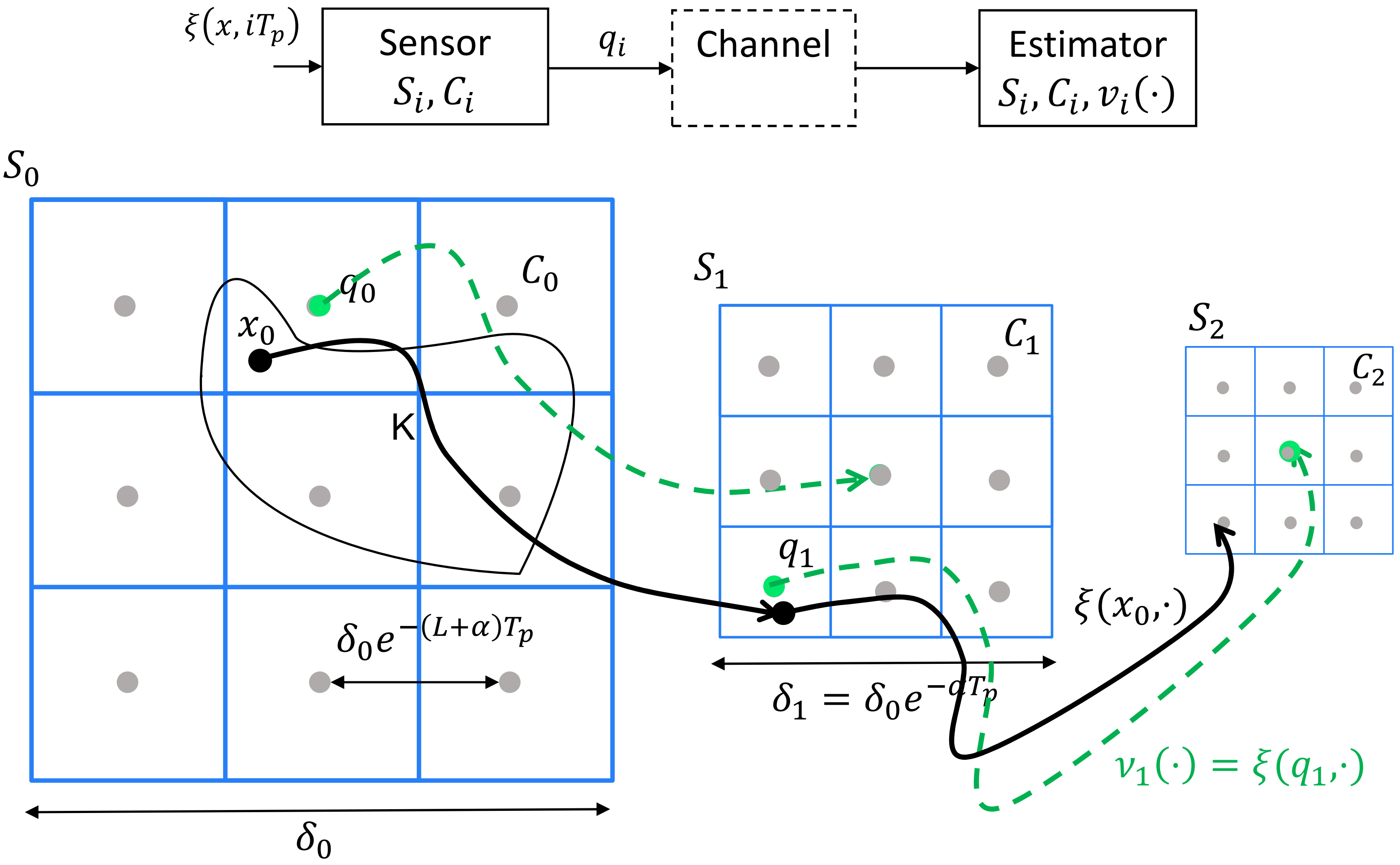

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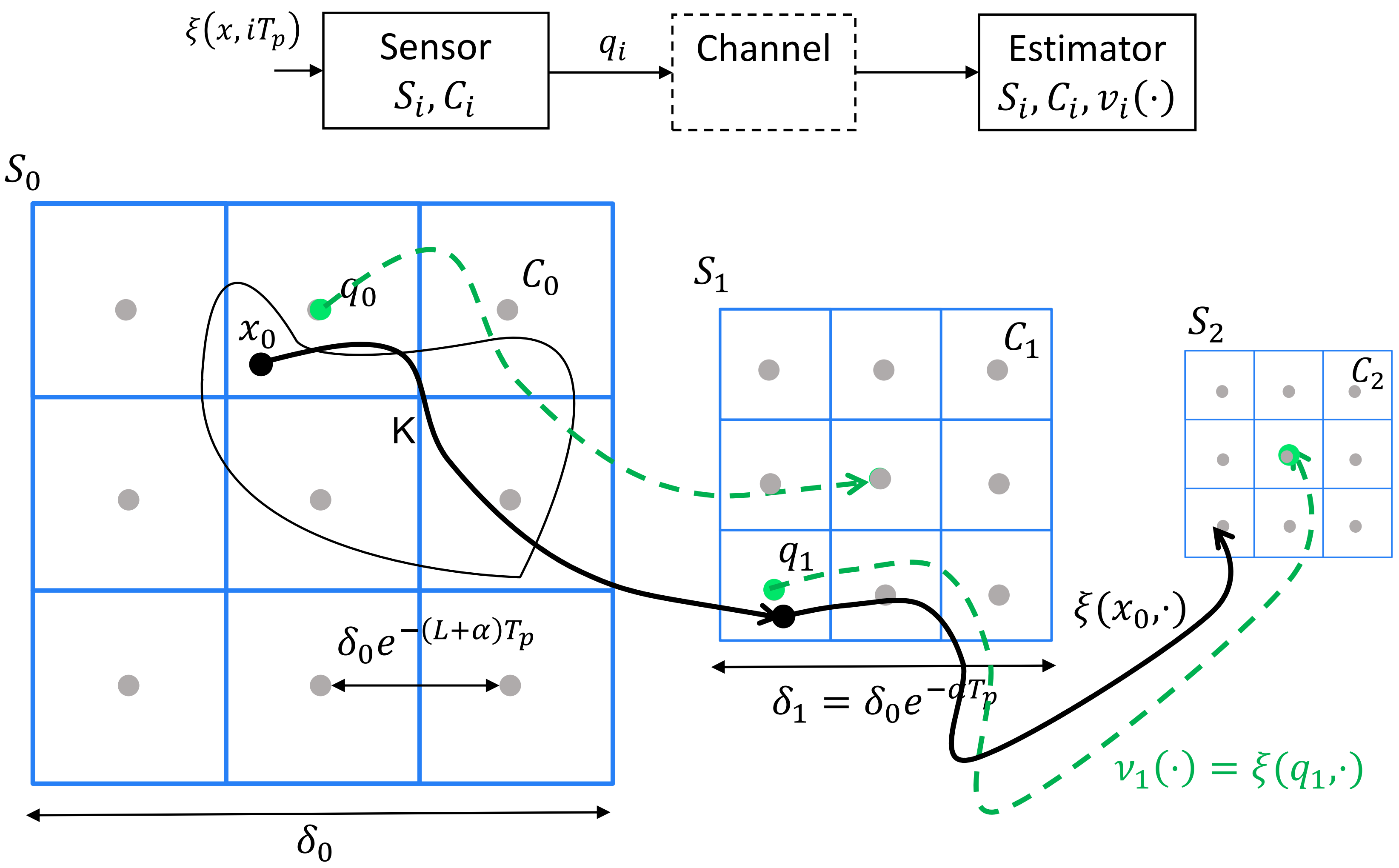

Properties: $\xi(x, iT_p) \in S_i \; \forall i$ and $\|\xi(x,t) - v(t)\|_\infty \leq \delta_0 e^{-\alpha t} \; \forall t$

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Procedure in [Savkin] operates at arbitrary bit rate > h_{est}, but does block coding using sequences from suitable spanning set – not constructive

MODEL DETECTION PROBLEM

Want to distinguish between two competing models

$$\dot{x} = f_i(x),\ i \in \{1,2\},\ x \in \mathbb{R}^n,\ x(0) \in K$$

using finite-data-rate state measurements (as above)

Need solutions of two models to be “sufficiently different”

$\xi_i(x,t)$ – solution of model i from x after time t

L_i – Lipschitz constant of f_i (can use matrix measure instead)

Call models (L,T)-separated if $\exists \varepsilon_{\min} > 0$ s.t. $\forall \varepsilon \leq \varepsilon_{\min}$:

$$|x_1 - x_2| \leq \varepsilon \ \Rightarrow \ |\xi_1(x_1,T) - \xi_2(x_2,T)| > \varepsilon e^{LT}$$

Sufficient condition: exponential separation holds over a compact set of states D if $f_1(x) \neq f_2(x)\ \ \forall x \in D$ (“generically true”)

MODEL DETECTION ALGORITHM

```
If $\xi(x_0, t) \notin S_{i-1}$ output "2"; break
Else  $q_i$:= Quantized measurement of $\xi\left(x_0, iT_p\right)$ w.r.t $C_{i-1}$
      $v(t)$:= $\xi_1\left(q_i, t - (i-1)T_p\right)$ for $t \in [(i-1)T_p, iT_p]$
      $\delta_i \coloneqq e^{-\alpha T_p}\delta_{i-1}$
      $S_i \coloneqq$ hypercube with center $v(iT_p)$ and radius $\delta_i$
      $C_i \coloneqq$ grid $S_i$ with size $e^{-(L_1+\alpha)T_p}\delta_i$
```

Theorem: Under (L_1, T_p)-separation, output "2" iff true model is f_2

If the true model is f_1: by correctness of estimation, actual state always stays in S_i, no output.

If the true model is f_2: since δ_i decays geometrically, it will eventually become smaller than $\varepsilon_{\min}$. By exponential separation, at next iteration the actual state will exit S_i.

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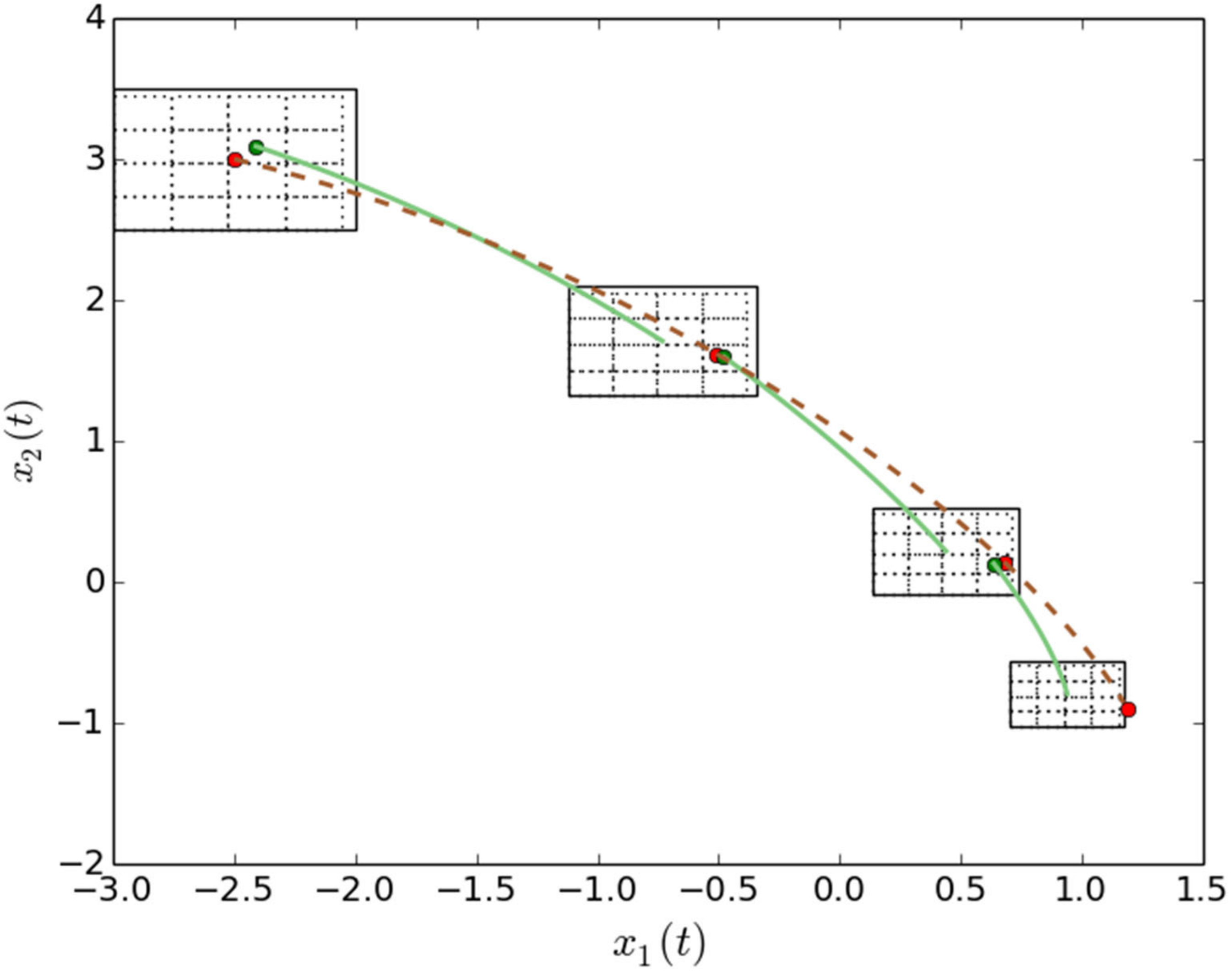

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where $L_p := \sup \mu\big(\partial f_p/\partial x(x)\big)$ and the sup is over x reachable from $\operatorname{conv}(K)$ when $\sigma = p$

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- How does entropy behave under input/output interconnections? [Kawan–Delvenne ’16, Matveev et al. ’19, Tomar–Zamani ’20, L ’21].