On New Sufficient Conditions for Stability of Switched Linear Systems*

Daniel Liberzon
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, IL 61821, U.S.A.
Email: liberzon@uiuc.edu

Abstract

This work aims to connect two existing approaches to stability analysis of switched linear systems: stability conditions based on commutation relations between the subsystems and stability conditions of the slow-switching type. The proposed sufficient conditions for stability have an interpretation in terms of commutation relations; at the same time, they involve only elementary computations of matrix products and induced norms, and possess robustness to small perturbations of the subsystem matrices. These conditions are also related to slow switching, in the sense that they rely on the knowledge of how slow the switching should be to guarantee stability; however, they cover situations where the switching is actually not slow enough, by accounting for relations between the subsystems. Numerical examples are included for illustration.

1. Introduction

This paper deals with a question that has received a lot of attention in the last 15 years or so: when is a switched linear system—i.e., a system defined by a finite family of linear subsystems and a rule describing the switching between them—asymptotically stable for all admissible switching patterns? It is well known that a necessary and sufficient condition for stability under arbitrary switching is the existence of a common Lyapunov function for the family of subsystems; see, e.g., [1, Section 2.1]. A sufficient condition that is useful for computational purposes (but is no longer necessary) is the existence of a *quadratic* common Lyapunov function; such a function, if it exists, can be found by solving a system of LMIs or by using the method of [2]. As far as obtaining readily checkable analytic conditions for the existence of a quadratic common Lyapunov function, arguably the only available results for general switched linear systems are those formulated in terms of commutation relations between the subsystem matrices; see [3, 4, 5, 6, 7, 8] as well as [1, Section 2.2]. However, these sufficient conditions suffer from lack of robustness, in the sense that they are destroyed by arbitrarily small perturbations of the subsystem matrices [8, 1]. This is a serious limitation, because both stability itself and the existence of a quadratic common Lyapunov function are robust with respect to such small perturbations.

A closely related problem consists in classifying those switching patterns that make a given switched linear system stable. It is well known that a switched linear system is stable if all individual subsystems are stable and the switching is sufficiently slow (at least on the average) so as to allow the transient effects to dissipate after each switch. The desired slow-switching conditions can be explicitly derived from the subsystem matrices or from the corresponding Lyapunov functions; see [9, Lemma 2], [10], and [1, Section 3.2]. These results are robust with respect to perturbations of the system data, and certainly are more practically useful than the ones based on (fragile) commutation relations. However, slow-switching conditions tend to be conservative, and do not take into account possible commutation relations between the subsystems. (The conservatism of slow-switching conditions can be reduced if the switching depends on the state, as in [11], but here we assume no knowledge of such dependence.)

The present work is a preliminary attempt to bridge the two research directions mentioned above. The sufficient conditions for stability that we have in mind are inspired by the ones based on commutation relations, can be given an appropriate interpretation in terms of these earlier results, and in fact subsume some of them as spe-

*This work was supported by the NSF CSR program (Embedded and Hybrid systems area) under grant NSF CNS-0614993.

ISBN 978-963-311-369-1

cial cases. However, they are not explicitly formulated in terms of commutation relations and, by contrast, are inherently robust with respect to small perturbations of the system parameters. On the other hand, our approach is also related to slow-switching conditions, in the sense that it relies on the knowledge of how slow the switching should be in order to guarantee stability of the switched linear system. However, the proposed conditions cover situations where the switching is actually too fast for the slow-switching results to apply, by accounting for relations between the subsystems. We can thus hope to capture the two extreme cases—arbitrary switching between subsystems with nice commutation relations and sufficiently slow switching between arbitrary subsystems—as well as situations in between, all within a single framework. Our stability conditions are also easy to formulate and check, as they involve just elementary manipulations with matrix products and induced norm computations; we support this claim with some numerical examples.

2. Motivating example

Suppose that we are switching in discrete time between two subsystems

$$x(k+1) = Ax(k), \qquad x(k+1) = Bx(k) \tag{1}$$

where $x \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$. The (discrete-time, linear) *switched system* generated by these two subsystems is the system $x(k+1) = F(k)x(k)$ with $F(k) \in \{A, B\}$ for all k. We assume that the matrices A and B are nonsingular, and hence can be written as exponentials of some other (possibly complex-valued) matrices:

$$A = e^L, \qquad B = e^M. \tag{2}$$

One interpretation of this assumption is that our discrete-time subsystems (1) actually arise from sampling the continuous-time systems $\dot{x} = Lx$ and $\dot{x} = Mx$, with the sampling period normalized to 1. Therefore, our treatment also applies to continuous-time switched systems whose switching times are integer multiples of a fixed sampling period.

We assume—as is natural when seeking stability under arbitrary switching—that both A and B are Schur stable. This implies that there exists an integer $m \geq 1$ such that the induced norms of A^m and B^m (with respect to the Euclidean norm) satisfy

$$\|A^m\| \leq \rho_A < 1, \qquad \|B^m\| \leq \rho_B < 1. \tag{3}$$

For simplicity we work with the same m for both A and B, which corresponds to taking the maximum of two appropriate numbers m_A and m_B which are in general different. In fact, to fix ideas we concentrate in this section on the case when

$$m = 2. \tag{4}$$

Let us now make the following two simple observations.

FACT 1. The switched system generated by the two subsystems (1) satisfying (3) and (4) is asymptotically stable for 2-periodic switching.

Here, by 2-periodic switching we mean that switching takes place every two time steps, so that a typical solution is

$$x(k) = A^2B^2 \cdots A^2B^2x(0).$$

The claim is obvious because both A^2 and B^2 have norm less than 1, hence the matrix product is contracting to 0 as $k \to \infty$.

FACT 2. If the matrices A and B commute: $AB = BA$, then the switched system generated by the two subsystems (1) is asymptotically stable for arbitrary switching.

One way to show this well-known fact is to use (3), which as we noted holds for some m (not necessarily equal to 2) whenever A and B are Schur. Using commutativity of A and B, we can rewrite their arbitrary product in the form

$$A^mB^mA^mB^m \cdots$$

and this is clearly contracting to 0 as the number of terms grows to ∞.

Fact 1 imposes a lower bound on the switching period while Fact 2 does not. On the other hand, Fact 1 is robust to small perturbations of A and B while Fact 2 is not. We want to build a bridge between Fact 1 and Fact 2. To this end, let E be the matrix satisfying

$$ABAB = A^2EB^2. \tag{5}$$

A direct formula for this E is

$$E = A^{-1}BAB^{-1}. \tag{6}$$

Then we have:

FACT 3. If

$$\|E\| \leq 1 + \varepsilon \tag{7}$$

where ε is small enough so that

$$\rho_A(1+\varepsilon)\rho_B < 1 \tag{8}$$

then the switched system generated by the two subsystems (1) satisfying (3) and (4) is asymptotically stable for 1-periodic switching.

Indeed, solutions of the system are determined by matrix products of the form $ABABAB\cdots$ which we can split into products of $ABAB$. The result then easily follows from the submultiplicativity of the matrix induced norm and the formulas (3), (4), (5), (7), (8).

We can think of E as defining the "price" we pay for shuffling A and B in order to make the matrix product 2-periodic. When the switching is already 2-periodic, $E = I$ and we recover Fact 1. When A and B commute, $E = I$ again and we recover Fact 2. In this sense, Fact 3 is an improvement on both Fact 1 and Fact 2; it relies on A and B being "close" to commuting (i.e., $\|E\|$ being close to 1), where the knowledge that $m = 2$ and the switching is 1-periodic is used to define E and quantify the desired closeness.

Combining (2) with (6) and using the Baker-Campbell-Hausdorff formula $e^L e^M = e^{L+M+\frac{1}{2}[L,M]+\ldots}$, we can show that E is given by the following expression in terms of L and M:

$$E = e^{-L}e^{M}e^{L}e^{-M} = e^{-[L,M]-\frac{1}{2}[L,[M,L]]-\frac{1}{2}[M,[M,L]]+\cdots} \quad (9)$$

where $[L,M] := LM - ML$ is the standard Lie bracket. Thus we see that the bound (7), which requires E to be close to I, places an *indirect* bound on the Lie brackets between L and M.

Before finishing this example, we illustrate it using specific numerical values. Consider the following matrices:

$$A = \begin{pmatrix} 0.1 & -2 \\ 0.3 & 0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.2 & -1.5 \\ 0.3 & 0.2 \end{pmatrix}.$$

It is straightforward to check that (3) holds with $m = 2$ (but not with $m = 1$) and we have $\rho_A = 0.8032$, $\rho_B = 0.7856$. Computing the matrix E, we obtain

$$E = \begin{pmatrix} 1.3111 & -0.1255 \\ -0.0151 & 0.7641 \end{pmatrix}$$

and $\|E\| = 1.3215$, which yields $\rho_A(1+\varepsilon)\rho_B = 0.8339$. Therefore, the system is asymptotically stable under 1-periodic switching. These matrices can be viewed as perturbations of the commuting matrices $\left(\begin{smallmatrix} 0.1 & -2 \\ 0 & 0.1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0.2 & -1.5 \\ 0 & 0.2 \end{smallmatrix}\right)$ with the perturbation being given by the lower-left element 0.3. In this example, it turns out that the maximal allowable perturbation before the condition (8) fails is about 0.352. It is a known fact that when the subsystem matrices commute, there exists a quadratic common Lyapunov function which can be used to show stability and also characterize robustness to small perturbations (see [5] and [1, Section 2.2]). However, the argument we used here to verify stability is direct from the given data.

3. More general formulation

As in Section 2, we start with two subsystems (1) satisfying (2) and (3), but now m can be an arbitrary positive integer. We assume that the switching is ℓ-periodic, where ℓ is some positive integer. In other words, we allow switching that produces solutions of the form

$$x(k) = B^i A^\ell B^\ell \cdots A^\ell B^\ell A^j x(0)$$

where $0 \le i, j \le \ell$ and the roles of A and B can be interchanged. Clearly, for stability analysis it is enough to study matrix products of the above type with $i = j = 0$. We note that the ℓ-periodicity assumption does introduce a loss of generality. There is a general result that a suitable notion of asymptotic stability under periodic switching implies asymptotic stability under arbitrary switching; see [6, Theorem 2.3] and a detailed discussion in [12, Section 4.10]. Hence, the main gap is between ℓ-periodic switching and arbitrary periodic switching.

Since (3) guarantees that the switched system is asymptotically stable under ℓ-periodic switching when $\ell \ge m$, the case of interest is when $\ell < m$. Let r be the smallest integer such that

$$r\ell \ge m. \quad (10)$$

Define a matrix E by the formula

$$\underbrace{A^\ell B^\ell \cdots A^\ell B^\ell}_{r \text{ repetitions}} = A^{r\ell} E B^{r\ell} \quad (11)$$

(this can be solved for E since A and B are nonsingular). Note that $E = I$ when either $\ell = m$ or $AB = BA$. Intuitively, E captures both how far A and B are from commuting and how large m is compared to ℓ. The following result is a straightforward generalization of Fact 3 from Section 2.

Proposition 1 *Let the nonsingular matrices A and B satisfy* (3) *for some $m \ge 1$. For given ℓ and r related via* (10)*, assume that E defined by* (11) *satisfies* (7) *with ε small enough so that* (8) *holds. Then the switched system generated by the two subsystems* (1) *is asymptotically stable for ℓ-periodic switching.*

We think of the special case $\ell = 1$, $m = 2$ considered in Section 2 as the "elementary shuffling" case (cf. [6]), because in that case the definition of E can be more simply but equivalently written as

$$BA = AEB. \quad (12)$$

We know that in this case E can be expressed in terms of Lie brackets of L and M via the formula (9), where L and M come from (2). In principle, elementary shuffling can be used to generate an iterative formula for E in terms of Lie brackets of L and M for arbitrary values of ℓ and m. However, at present we are not aware of a general and tractable closed-form expression. There are of course some other special cases which can be easily treated. For example, consider the case when $\ell = 2$ and $m = 4$. Then (11) specializes to

$$A^2B^2A^2B^2 = A^4EB^4$$

hence

$$E = (A^{-1})^2B^2A^2(B^{-1})^2.$$

Using (2) and the Baker-Campbell-Hausdorff formula, we obtain the following close analogue of (9):

$$\begin{aligned} E &= e^{-2L}e^{2M}e^{2L}e^{-2M} \\ &= e^{-[2L,2M]-\frac{1}{2}[2L,[2M,2L]]-\frac{1}{2}[2M,[2M,2L]]+\cdots} \end{aligned} \tag{13}$$

We can always reduce ℓ-periodic switching to 1-periodic switching by working with the new matrices $\bar{A} := A^\ell$ and $\bar{B} := B^\ell$. If m is not an integer multiple of ℓ, then we also have to replace m in (3) with $r\ell$, but this does not change the result since we are already using $r\ell$ in (11). Thus, the only nontrivial generalization in this section compared to Section 2 consists of allowing an arbitrary m. However, we will see in the next section that keeping ℓ general allows more flexibility in obtaining sufficient conditions for stability.

4. Beyond commutativity: high-order Lie brackets

In the setting of Section 2, the matrix E is given by (9) and the condition (7) means that L and M are close to commuting. However, existing results on stability of switched linear systems under "nice" commutation relations go far beyond the commuting case. In particular, we know from [6] that the switched system is asymptotically stable for arbitrary switching if $[L,M] \neq 0$ but

$$[L,[L,M]] = [M,[L,M]] = 0. \tag{14}$$

When (14) holds, E in (9) need not be close to I but it commutes with both A and B; this is a simple consequence of the Baker-Campbell-Hausdorff formula. Let us consider a matrix product of length 6 arising in 1-periodic switching, and note that it can be written as follows:

$$A\underline{BA}BAB = AAEBBAB = AABBA\underline{EB} = AABBBA. \tag{15}$$

The first equality in (15) follows from the elementary shuffling formula (12) applied to the underlined product BA, the second equality follows from the fact that E commutes with A and B, and the last equality is obtained from (12) again, this time applied in the reverse direction to the underlined product AEB. Now, if we consider an infinite 1-periodic matrix product, split it into portions of length 6, and use (15), we obtain an equivalent 3-periodic product (because the last A of each portion given by the right-hand side of (15) gets appended to AA at the beginning of the next portion).

From the previous calculation we deduce the following interesting fact: If L and M are not close to commuting but satisfy (14), then the "price" of shuffling the terms in a 1-periodic product of A and B to get a 2-periodic product can be high (since E is not close to I), but on the other hand we can obtain a 3-periodic product "for free" (since the right-hand side of (15) no longer contains E). This implies, in particular, that the switched system is asymptotically stable under 1-periodic switching if its subsystem matrices A and B satisfy (2), (3) with $m = 3$, and (14). Of course, this conclusion is not very interesting because it is a special case of the result of [6] (which does not require $\ell = 1$ or $m = 3$). However, it suggests a way for us to capture situations where (14) is satisfied only approximately. Namely, let r, ℓ, m be as in Section 3 and define a matrix F by the formula

$$\underbrace{A^\ell B^\ell \cdots A^\ell B^\ell}_{r \text{ repetitions}} = A^{r\ell-1}FB^{r\ell}A. \tag{16}$$

In the next result, this F plays a role analogous to that of E in Proposition 1.

Proposition 2 *Let the nonsingular matrices A and B satisfy* (3) *for some $m \geq 1$. For given ℓ and r related via* (10)*, assume that F defined by* (16) *satisfies $\|F\| \leq 1+\varepsilon$ with ε small enough so that* (8) *holds. Then the switched system generated by the two subsystems* (1) *is asymptotically stable for ℓ-periodic switching.*

We showed earlier that $F = I$ if $\ell = 1$, $m = 3$, and (14) holds. Other combinations of values for ℓ and m satisfying $m = 3\ell$ can be treated by redefining the subsystem matrices to be A^ℓ and B^ℓ, as explained at the end of Section 3. More interestingly, we also have $F = I$ if $\ell = 2$, $m = 4$, and (14) holds. A verification of this fact proceeds along the lines of (15) but is a bit more tricky:

$$\begin{aligned} AAB\underline{BA}ABB &= AA\underline{BA}EBABB = AAAEBEBABB \\ &= AAABEB\underline{AEB}B = AAABEBBAB \\ &= AAABBBA\underline{EB} = AAABBBBA \end{aligned}$$

In the case $\ell = 1$, $m = 2$, the condition (14) does not guarantee that $F = I$, but we can still show stability by applying Proposition 2 with the more conservative value $m = 3$. These last two cases also demonstrate that it is not always advisable to pass from ℓ-periodic to 1-periodic switching by redefining the subsystem matrices, because this induces a loss of "granularity" that may be helpful in the context of Proposition 2. As we know, all the cases just mentioned are covered by the result of [6], but Proposition 2 provides robustness with respect to small perturbations which destroy the property (14).

We end this section with another numerical example. Consider the matrices

$$A = \begin{pmatrix} 1/4 & -1 & 1/2 \\ -\varepsilon & 1/4 & 1 \\ 0 & 0 & 1/4 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 1 & -1/2 \\ 0 & 1/2 & 0 \\ 0 & \varepsilon & 1/2 \end{pmatrix}$$

where $\varepsilon := 0.005$. The smallest value of m that makes (3) true is $m = 3$, and we have $\rho_A = 0.7075$, $\rho_B = 0.8533$. The matrix F from (16) with $\ell = 1$ and $m = 3$ is

$$F = \begin{pmatrix} 1.0039 & 0.1702 & -0.7808 \\ 0.0000 & 0.9992 & -0.0025 \\ 0.0000 & 0.0009 & 0.9969 \end{pmatrix}$$

and its induced norm is $\|F\| = 1.4771$. This gives $\rho_A(1+\varepsilon)\rho_B = 0.8917$, hence the switched system is asymptotically stable under 1-periodic switching. The intuition behind this example—which is not, however, required for the analysis—is that the matrices A and B are a small perturbation (of size 0.005) away from being upper-triangular and satisfying $[A,[A,B]] = [B,[A,B]] = 0$ as well as (14). The perturbation can be increased to about 0.0064 before the condition of Proposition 2 fails.

5. Discussion

Propositions 1 and 2 can be viewed as two special instances of a more general approach to developing new sufficient conditions for stability of switched linear systems. In a nutshell, this approach can be described as follows. Suppose that the subsystem matrices A and B satisfy (3) for some known m. Suppose also that the switching is constrained to be ℓ-periodic (or to some other known switching pattern). Consider the matrix products describing the solutions of the switched system; for the case of ℓ-periodic switching, a typical such product is $A^\ell B^\ell A^\ell B^\ell \cdots$. The objective then is to rewrite this product as another product, whose terms are A^m, B^m, and some auxiliary matrix such as E in (11) or F in (16). Since we know that the induced norms of A^m and B^m satisfy the bound (3), a sufficient condition for asymptotic stability of the switched system is that the induced norm of this auxiliary matrix does not exceed $1+\varepsilon$ where ε satisfies (8). Similar stability conditions can be formulated for the case of three or more subsystem matrices. Checking the conditions is a matter of performing elementary matrix calculations.

It must be noted that for a *given* periodic switching pattern, the above approach is not necessary. Indeed, it suffices to check that the matrix product corresponding to one period is Schur stable, which guarantees that over some number of periods we get a contraction. Thus the merit of the results reported here lies not in the stability conditions themselves but in their interpretation in terms of Lie brackets between the subsystem matrices A and B (or, more precisely, their matrix logarithms L and M), as well as their potential for generalization. To make the approach really useful, we need to demonstrate that it is capable of yielding conditions that guarantee stability for *all* periodic switching patterns (with a given "dwell time," i.e., minimal number of steps between switches). It is known that stability under ℓ-periodic switching does not imply stability under arbitrary periodic switching. On the other hand, sufficient conditions of the type given by Propositions 1 and 2 are stronger, and could in principle lead to a desired result. Currently, even for ℓ-periodic switching we do not have an "optimal" method for shuffling the matrices (especially when m is not an integer multiple of ℓ). A better understanding of the connection between the matrices E and F and the Lie brackets of L and M, besides being of theoretical interest, might also help render the approach more systematic.

We also currently investigating alternative approaches to capturing robustness in commutation relations. These will be reported in a future publication.

Acknowledgment. This work was inspired by stimulating discussions with Debasish Chatterjee. We also thank João Hespanha for useful comments that helped improve the presentation.

References

[1] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[2] D. Liberzon and R. Tempo. Switched systems, common Lyapunov functions, and gradient algorithms. *IEEE Trans. Automat. Control*, 49:990–994, 2004.

[3] S. A. Kutepov. Algebraic criterion for absolute stability. *Kibernet. i Vychisl. Tekhn.*, 54:52–55, 1982. In Russian.

[4] S. A. Kutepov. Absolute stability of bilinear systems with a compact Levi factor. *Kibernet. i Vychisl. Tekhn.*, 62:28–33,118, 1984. In Russian.

[5] K. S. Narendra and J. Balakrishnan. A common Lyapunov function for stable LTI systems with commuting A-matrices. *IEEE Trans. Automat. Control*, 39:2469–2471, 1994.

[6] L. Gurvits. Stability of discrete linear inclusion. *Linear Algebra Appl.*, 231:47–85, 1995.

[7] D. Liberzon, J. P. Hespanha, and A. S. Morse. Stability of switched systems: a Lie-algebraic condition. *Systems Control Lett.*, 37:117–122, 1999.

[8] A. A. Agrachev and D. Liberzon. Lie-algebraic stability criteria for switched systems. *SIAM J. Control Optim.*, 40:253–269, 2001.

[9] A. S. Morse. Supervisory control of families of linear set-point controllers, part 1: exact matching. *IEEE Trans. Automat. Control*, 41:1413–1431, 1996.

[10] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. 38th IEEE Conf. on Decision and Control*, pages 2655–2660, 1999.

[11] R. Langerak and J. W. Polderman. Tools for stability of switching linear systems: gain automata and delay compensation. In *Proc. 44th IEEE Conf. on Decision and Control*, pages 4867–4872, 2005.

[12] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49:545–592, 2007.